TAIWANESE JOURNAL OF MATHEMATICS
Vol. 13, No. 1, pp. 317-338, February 2009
This paper is available online at http://www.tjm.nsysu.edu.tw/

SURFACES OF REVOLUTION WITH POINTWISE 1-TYPE GAUSS MAP IN MINKOWSKI 3-SPACE

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Abstract. We study the rational surfaces of revolution in Minkowski 3-space and characterize them with pointwise 1-type Gauss map. In this article, we give a complete classification of rational surfaces of revolution in Minkowski 3-space with pointwise 1-type Gauss map and provide new examples of cones in Minkowski 3-space.

1. Introduction

The notion of finite type immersion has been widely used in studying submanifolds of Euclidean and pseudo-Euclidean space ([2]). Also, such a notion can be extended to smooth maps on submanifolds. Among them the Gauss map is a very useful and extensively used to deal with submanifolds ([3]). The Gauss map G of some minimal or maximal surfaces including catenoid in Euclidean 3-space and the Enneper's surface of the second kind in Minkowski 3-space satisfy some partial differential equation similar to an eigenvalue problem that is not an actual eigenvalue problem. One of the present authors defined and used a notion of pointwise 1-type Gauss map to study certain surfaces in Euclidean or Minkowski space ([4, 5, 6, 7]). The Gauss map G on a submanifold M of pseudo-Euclidean space E^m_s of index s is said to be of pointwise 1-type if

$$\Delta G = F(G + C) \tag{1.1}$$

for some nonzero smooth function F on M and some constant vector C, where Δ denotes the Laplace operator defined on M. A pointwise 1-type Gauss map is called *proper* if the function F defined by (1.1) is non-constant. A non-proper pointwise

Received October 25, 2006, accepted May 19, 2007.
Communicated by Bang-Yen Chen.
2000 *Mathematics Subject Classification*: 35A25, 53B30.
Key words and phrases: Minkowski space, Surface of revolution, Gauss map, Hyperbolic cone, Right cone.

1-type Gauss map is just of 1-type in the usual sense ([2, 3, 5]). A submanifold with pointwise 1-type Gauss map is said to be of the first kind if the vector C in (1.1) is the zero vector. Otherwise, the pointwise 1-type Gauss map is said to be of the second kind ([4]).

In this article we study surfaces of revolution of the polynomial kind and the rational kind with pointwise 1-type Gauss map in Minkowski 3-space. We also provide new examples of surface of revolution in a Minkowski space.

2. Preliminaries

Let E_1^3 be a three-dimensional Minkowski space with the scalar product of index 1 given by $\langle\cdot,\cdot\rangle = -dx_0^2 + dx_1^2 + dx_2^2$, where (x_0, x_1, x_2) is a standard rectangular coordinate system of E_1^3. A vector x of E_1^3 is said to be space-like if $\langle x, x\rangle > 0$ or $x = 0$, time-like if $\langle x, x\rangle < 0$ and light-like or null if $\langle x, x\rangle = 0$ and $x \neq 0$. A time-like or light-like vector in E_1^3 is said to be causal.

Lemma 2.1. *For two vectors X and Y in E_1^3 the Lorentz cross product of X and Y is defined by*

$$X \times Y = (x_2y_1 - x_1y_2, x_2y_0 - x_0y_2, x_0y_1 - x_1y_0).$$

For the Lorentz vector space the next two lemmas are well known and useful.

Lemma 2.2. *There are no causal vectors in E_1^m orthogonal to a time-like vector.*

Lemma 2.3. *Two light-like vectors are orthogonal if and only if they are linearly dependent.*

Let I be an open interval and $\gamma : I \to \Pi$ a plane curve lying in a plane Π of E_1^3 and l a straight line in Π which does not intersect with the curve γ. A surface of revolution M with axis l in E_1^3 is defined to be invariant under the group of motions in E_1^3, which fixes each point of the line l (cf. [1]). From this we obtain four kinds of surface of revolution in E_1^3. If the axis l is space-like (resp. time-like), then there is a Lorentz transformation by which the axis l is transformed to the x_1-axis or x_2-axis (resp. x_0-axis). Hence, without loss of generality, we may consider as the axis of revolution with the x_2-axis (resp. the x_0-axis) if l is not null. If the axis is null, then we may assume that this axis is the line spanned by vector $(1, 1, 0)$ of the plane Ox_0x_1.

We now introduce three different types of surfaces of revolution in E_1^3.

Type I. The axis of revolution is a space-like line.

Without loss of generality, we may assume that the curve γ is lying in the x_1x_2-plane or in the x_0x_2-plane. In turn, the curve γ is parameterized by

$$\gamma(u) = (0, f(u), g(u))$$

or

$$\gamma(u) = (f(u), 0, g(u)), \tag{2.1}$$

where $f = f(u)$ is a smooth positive function and $g = g(u)$ is a smooth function on I. Hence, the surface of M can be defined by

$$x(u, v) = (f(u) \sinh v, f(u) \cosh v, g(u)) \tag{2.2}$$

or

$$x(u, v) = (f(u) \cosh v, f(u) \sinh v, g(u)). \tag{2.3}$$

Type II. The axis of revolution is a time-like line.

Without loss of generality we may assume that the curve γ lies in the x_0x_1-plane. Hence, its parametrization may be given by

$$\gamma(u) = (g(u), f(u), 0),$$

where $f = f(u)$ is a smooth positive function and $g = g(u)$ is a smooth function on I. Hence, the surface of revolution M revolving γ around the axis $0x_0$ may be given by

$$x(u, v) = (g(u), f(u) \cos v, f(u) \sin v). \tag{2.4}$$

Type III. The axis of revolution is a light-like line, or equivalently the line in the plane x_0x_1 spanned by the vector $(1, 1, 0)$.

Since the surface M is non-degenerate, we can assume that the curve γ lies in the x_0x_1-plane and its parametrization is given by

$$\gamma(u) = (f(u), g(u), 0),$$

where $f = f(u)$ is a smooth positive function and $g = g(u)$ is a smooth function on I such that $h(u) = f(u) - g(u) \neq 0$. Then, the surface of revolution M may be parameterized by

$$x(u, v) = \big(f(u) + \frac{v^2}{2}h(u), g(u) + \frac{v^2}{2}h(u), h(u)v\big). \tag{2.5}$$

A surface of revolution is called a *polynomial kind* if the function $f(u)$ and $g(u)$ are both some polynomials and a *rational kind* if the functions $f(u)$ and $g(u)$ are both some rational functions.

Now, let us consider the Gauss map G on a surface M in E^3_1. The map $G : M \longrightarrow Q^2(\varepsilon) \subset E^3_1$ which maps each point of M into the parallel displacement of the unit normal vector to M at the point to the origin is called the Gauss map of surface M, where $\varepsilon(=\pm 1)$ denotes the sign of the vector field G and $Q^2(\varepsilon)$ is a $2-$dimensional space form as follows;

$$Q^2(\varepsilon) = \begin{cases} S^2_1(1) & \text{in } E^3_1 \text{ if } \varepsilon = 1; \\ H^2(-1) & \text{in } E^3_1 \text{ if } \varepsilon = -1. \end{cases}$$

For the matrix $g = (g_{ij})$ consisting of the components of the Riemannian metric on M, we denote by $g^{-1} = (g^{ij})$ is the inverse matrix of the matrix (g_{ij}). Then, the Laplacian operator Δ on M is given by

$$\Delta = -\frac{1}{\sqrt{|\mathcal{G}|}} \sum_{i,j} \frac{\partial}{\partial x_i} (\sqrt{|\mathcal{G}|} g^{ij} \frac{\partial}{\partial x_j}),$$

where $\mathcal{G} = \det g$.

We need the following lemma for later use.

Lemma 2.4. *Let M be a surface of revolution with pointwise 1-type Gauss map of the second kind. Then, the function F defined in (1.1) depends only on the parameter of the profile curve and the vector C in (1.1) is parallel to the axis of the surface of revolution.*

Proof. We now separate the cases of proof according to the character of the profile curves and the axes.

Case 1. Suppose M is a surface of revolution of type I in E^3_1 parameterized by (2.2) for some smooth function f and g. We may assume that the curve γ given by (2.1) is of unit speed. By a straightforward computation, we obtain

$$G = (g'(u) \sinh v, g'(u) \cosh v, -f'(u))$$

and the Laplacian ΔG of the Gauss map G satisfies

$$\Delta G = -\frac{1}{f}((f'g'' + fg''' - \frac{g'}{f}) \sinh v, (f'g'' + fg''' - \frac{g'}{f}) \cosh v, -f'f'' - ff''').$$

If M has pointwise 1-type Gauss map of the second kind, then (1.1) holds for some nonzero function F and some nonzero vector C. Since $F \neq 0$, a direct argument gives the first two components of C must be zero and

$$F(u,v)g' = -\frac{1}{f}(f'g'' + fg''' - \frac{g'}{f}),$$
$$F(u,v)(-f' + c) = \frac{1}{f}(f'f'' + ff'''),$$

where $C = (0,0,c)$, $c \neq 0$. Since $f'(u)$ and $g'(u)$ are not both zero, the function F is independent of v. And if M is a surface of revolution of type I in E^3_1 parameterized by (2.3), we also obtain the same result.

Case 2. Suppose that M is a surface of revolution of type II in E^3_1 parameterized by (2.4) for some smooth function f and g. We may assume that

$$f'^2 - g'^2 = \pm 1$$

since the profile curve γ is of unit speed. Suppose that

$$f'^2(u) - g'^2(u) = 1, \forall u \in I.$$

Then, the Gauss map G is easily obtained by

$$G = (-f', -g'\cos v, -g'\sin v)$$

and its Laplacian ΔG is given as

$$\Delta G = -\frac{1}{f}(-f'f'' - ff''', (-g'''f - g''f' + \frac{g'}{f})\cos v, (-g'''f - g''f' + \frac{g'}{f})\sin v).$$

We now suppose that M has pointwise 1-type Gauss map of the second kind. Then, (1.1) holds for some nonzero function F and some nonzero vector C. Then, we easily see that the last two components of C must be zero and

$$\frac{1}{f}(g'''f + g''f' - \frac{g'}{f}) = F(u,v)(-g'(u)),$$
$$\frac{f'f'' + ff'''}{f} = F(u,v)(-f'(u) + c),$$

where $C = (c,0,0)$, $c \neq 0$. Since $f'(u)$ and $g'(u)$ are not both zero, the function F is independent of v. For $f'^2 - g'^2 = -1$, by the similar discussion developed as above we can get the same result.

Case 3. Let M be a surface of revolution of type III, which is obtained by revolving a smooth curve of $\gamma(u)$ around a light-like axis. Without loss of generality, we may choose the axis which is defined by the origin and the vector $(1,1,0)$. Then, the parametrization x of M is given by

$$x(u,v) = \big(f(u) + \frac{v^2}{2}h(u), g(u) + \frac{v^2}{2}h(u), h(u)v\big), \tag{2.6}$$

where $h(u) = f(u) - g(u) \neq 0$. Since M is nondegenerate, $-f'(u)^2 + g'(u)^2$ never vanishes and thus $h'(u) = f'(u) - g'(u) \neq 0$ everywhere. We may take the parameter in such a way that

$$h(u) = -2u.$$

Let $k(u) = f(u) + u$. Then, the functions f and g in the definition of the profile curve γ look like

$$f(u) = k(u) - u, \qquad g(u) = k(u) + u.$$

So, the parametrization of M becomes

$$x(u,v) = (k(u) - u - uv^2, k(u) + u - uv^2, -2uv). \tag{2.7}$$

Then, we get $\langle x_u, x_u\rangle = 4k'(u)$, $\langle x_u, x_v\rangle = 0$ and $\langle x_v, x_v\rangle = 4u^2$. Since the induced metric $\langle\cdot,\cdot\rangle$ on M is nondegenerate, $k'(u)u$ never vanishes. For $u > 0$, $k'(u) > 0$, the Gauss map G can be obtained as

$$G = \frac{1}{2\sqrt{k'(u)}}(k'(u) + v^2 + 1, k'(u) + v^2 - 1, 2v).$$

For a function φ on M, its Laplacian $\Delta\varphi$ is computed by

$$\Delta\varphi = -\frac{1}{4u\sqrt{k'(u)}}\Big(\frac{2k'(u) - uk''(u)}{2k'(u)^{3/2}}\varphi_u + \frac{u}{\sqrt{k'(u)}}\varphi_{uu} + \frac{\sqrt{k'(u)}}{u}\varphi_{vv}\Big).$$

Let us compute G_u, G_{uu} and G_{vv} to get ΔG. Then, we have

$$G_u = -\frac{k''(u)}{4k'(u)^{3/2}}(v^2+1, v^2-1, 2v) + \frac{k''(u)}{4\sqrt{k'(u)}}(1,1,0),$$

$$G_{uu} = -\frac{2k'(u)^2k'''(u) - 3k'(u)k''(u)^2}{8k'(u)^{5/2}}(v^2+1, v^2-1, 2v) + \frac{2k'(u)k'''(u) - k''(u)^2}{8k'(u)^{3/2}}(1,1,0),$$

$$G_{vv} = \frac{1}{\sqrt{k'(u)}}(1,1,0).$$

Suppose that the Gauss map G is of pointwise 1-type of the second kind. Let $(\Delta G)_i$ be the $i-$th component of ΔG. Then, we have

$$(\Delta G)_1 = F(u,v)\Big(\frac{k'(u) + v^2 + 1}{2\sqrt{k'(u)}} + c_1\Big), \tag{2.8}$$

(2.9) $$(\Delta G)_2 = F(u,v)(\frac{k'(u)+v^2-1}{2\sqrt{k'(u)}} + c_2),$$

(2.10) $$(\Delta G)_3 = F(u,v)(\frac{v}{\sqrt{k'(u)}} + c_3),$$

where $C = (c_1, c_2, c_3)$ and $(\Delta G)_i$ is the $i-$th component of ΔG $(i = 1,2,3)$. Subtracting (2.9) from (2.8), we get

(2.11) $$\frac{(2k' - uk'')k''}{16uk'^{7/2}} + \frac{2k'^2k''' - 3k'k''^2}{16k'^{9/2}} = F(u,v)(\frac{1}{\sqrt{k'}} + c_1 - c_2).$$

For simplicity, we put

$$A(u) = \frac{(2k' - uk'')k''}{16uk'^{7/2}} + \frac{2k'^2k''' - 3k'k''^2}{16k'^{9/2}}.$$

(2.10) and (2.11) imply

$$F(u,v)(\frac{v}{\sqrt{k'(u)}} + c_3) = vA(u),$$

$$F(u,v)(\frac{1}{\sqrt{k'(u)}} + c_1 - c_2) = A(u).$$

Therefore, the function F depends only on u and $c_1 = c_2$ and $c_3 = 0$. This means that the given constant vector C is parallel to the axis of revolution. It completes the proof. ■

3. Examples

In this section, we provide some examples of surfaces of revolution with pointwise 1-type Gauss map of the first kind and the second kind in Minkowski 3-space.

Example 3.1. (Hyperbolic cylinder). Consider a hyperbolic cylinder parameterized by

$$x(u,v) = (a\sinh v, a\cosh v, u)$$

for some constant $a > 0$. Then its Gauss map G is given by

$$G = (\sinh v, \cosh v, 0).$$

Hence, the Laplacian ΔG of the Gauss map G satisfies

$$\Delta G = \frac{1}{a^2}G,$$

so that the hyperbolic cylinder has pointwise 1-type Gauss map of the first kind. Indeed, it is of 1-type in the usual sense.

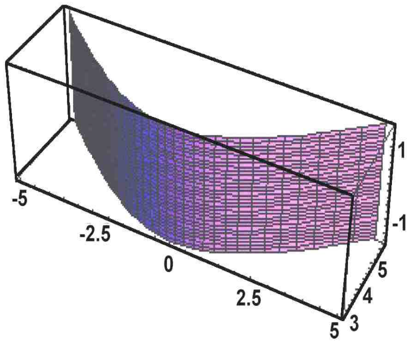

Fig. 1. Hyperbolic cylinder.

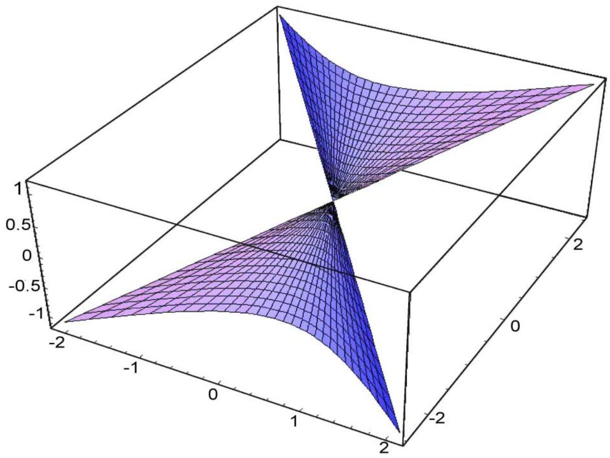

Fig. 2. Hyperbolic cone.

Example 3.2. (Right cone). A right cone is parameterized by

$$x(u,v) = (au, u\cos v, u\sin v)$$

for $u > 0$ and some constant $a > 1$. Then, the Gauss map G and its Laplacian ΔG are respectively given by

$$G = \frac{-1}{\sqrt{a^2-1}}(1, a\cos u, a\sin u),$$

$$\Delta G = \frac{1}{u^2}\big(G + (\frac{1}{\sqrt{a^2-1}}, 0, 0)\big).$$

Thus, the right cone has pointwise 1-type Gauss map of the second kind.

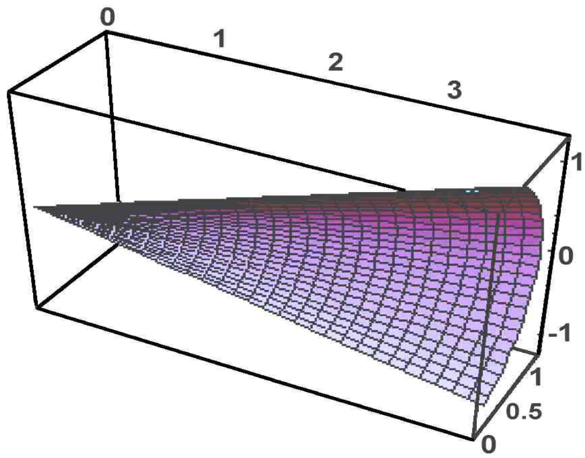

Fig. 3. Right cone.

Example 3.3. (Hyperbolic cone). Consider the hyperbolic cone which is parameterized by

$$x(u,v)=(u\sinh v, u\cosh v, au),\ \ a\neq 0.$$

Then the Gauss map G is given by

$$G=\frac{1}{\sqrt{a^2+1}}\big(a\sinh v, a\cosh v, -1\big).$$

Hence, the Laplacian ΔG of the Gauss map G satisfies

$$\Delta G=\frac{1}{u^2}\big(G+(0,0,\frac{1}{\sqrt{a^2+1}})\big).$$

This implies that the hyperbolic cone has pointwise 1-type Gauss map of the second kind.

Example 3.4. (Enneper's surface of second kind ([8])). The Enneper's surface of second kind is parameterized by

$$x(u,v)=a(\frac{1}{3}u^3-u-uv^2,\frac{1}{3}u^3+u-uv^2,-2uv),\ \ a\neq 0.$$

Then the Gauss map G and its Laplacian ΔG are respectively given by

$$G=\frac{1}{2u}\big(u^2+v^2+1, u^2+v^2-1, 2v\big)$$

and

$$\Delta G = -\frac{1}{2a^2u^4}G$$

for $u > 0$. Therefore, the Enneper's surface of second kind has pointwise 1-type Gauss map of the first kind.

4. Revolution in Minkowski Space with Pointwise 1-Type Gauss Map of the First Kind

Theorem 4.1. *Let M be a surface of revolution in a three-dimensional Minkowski space. Then, the mean curvature is constant if and only if M has pointwise 1-type Gauss map of the first kind.*

Proof. Suppose that a surface of revolution has pointwise 1-type Gauss map of the first kind.

Case 1. M is a surface of revolution of type I parameterized by

$$x(u,v) = (f(u)\sinh v, f(u)\cosh v, g(u))$$

for some smooth functions $f(u)$ and $g(u)$ as is given in (2.2). Then, Lemma 2.4 implies the following system of differential equations

$$\begin{aligned} F(u)g' &= -\frac{1}{f}(f'g'' + fg''' - \frac{g'}{f}), \\ F(u)f' &= -\frac{1}{f}(f'f'' + ff'''). \end{aligned} \tag{4.1}$$

Since

$$f'(u)^2 + g(u)'^2 = 1,$$

we may put

$$f'(u) = \cos t(u),\ \ g'(u) = \sin t(u).$$

for some function $t = t(u)$. Then, (4.1) yields

$$-\frac{\sin t\cos t}{f^2} + \frac{t'\cos t}{f} + t'' = 0,$$

which implies that $\frac{\sin t}{f} + t'$ is a constant.

On the other hand, the mean curvature H of M is obtained by

$$\begin{aligned} H &= \frac{-f^2(f''g' - f'g'') + fg'}{-2f^2} \\ &= -\frac{1}{2}(\frac{\sin t}{f} + t'). \end{aligned}$$

Thus, M has constant mean curvature. And, if M is a surface of revolution of type I parameterized by (2.3), then by similar discussion as above we can get same result.

Case 2. M is a surface of revolution of type II parameterized by

$$x(u,v) = (g(u), f(u)\cos v, f(u)\sin v),$$

where $f'^2 - g'^2 = 1$. If we adapt Lemma 2.4 again, the following system of differential equations is derived :

$$\begin{aligned} F(u)g' &= -\frac{1}{f}(f'g'' + fg''' - \frac{g'}{f}), \\ F(u)f' &= -\frac{1}{f}(f'f'' + ff'''). \end{aligned}$$

By the similar discussion as is developed in Case 1 we get

$$t'' + \frac{t'\cosh t}{f} - \frac{\sinh t \cosh t}{f^2} = 0$$

which implies that $t' + \frac{\sinh t}{f}$ is a constant and the mean curvature H is a constant.

Also, we obtain the same result by the similar argument as above in case of $f'^2 - g'^2 = -1$.

Case 3. M is a surface of revolution of type III parameterized by

$$x(u,v) = \big(f(u) + \frac{v^2}{2}h(u), g(u) + \frac{v^2}{2}h(u), h(u)v\big),$$

where $h(u) = f(u) - g(u) \neq 0$. By a straightforward computation, the following system of differential equations are obtained:

$$\begin{aligned} &\frac{1}{h}(h'g'' + hg''' - \frac{1}{2}v^2h'h'' - \frac{1}{2}v^2hh''' - \frac{h'}{h}) = F(u)(-g' + \frac{1}{2}v^2h'), \\ &\frac{1}{h}(h'f'' + hf''' - \frac{1}{2}v^2h'h'' - \frac{1}{2}v^2hh''' - \frac{h'}{h}) = F(u)(-f' + \frac{1}{2}v^2h'), \\ &\frac{1}{h}(vh'h'' + vhh''') = -F(u)vh', \end{aligned}$$

which are reduced to

$$h'(g'f'' - f'g'') + h(g'f''' - f'g''') + \frac{h'^2}{h} = 0. \tag{4.2}$$

Since we may assume

$$f'^2(u) - g'^2(u) = -1,$$

there exists a smooth function $t = t(u)$ such that

$$f'(u) = \sinh t(u), \; g'(u) = \cosh t(u).$$

Then, (4.2) yields

$$t'' + \frac{h't'}{h} + \frac{h'^2}{h^2} = 0$$

which means that $(t' - \frac{h'}{h})$ is a constant.

On the other hand, the mean curvature H of M is obtained by

$$\begin{aligned} H &= \frac{h^2(f''g' - f'g'') - hh'}{2h^2} \\ &= \frac{1}{2}(f''g' - f'g'' - \frac{h'}{h}) \\ &= \frac{1}{2}(t' - \frac{h'}{h}), \end{aligned}$$

where $h(u) = f(u) - g(u) \neq 0$. Therefore, the mean curvature H is a constant. In case of $f'^2 - g'^2 = 1$, by the similar computation as above we obtain the same result. The converse is straightforward. ∎

For simplicity, from now on, we call a surface of revolution of rational kind as a *rational surface of revolution*.

Theorem 4.2. (Characterization). *A rational surface of revolution of type I has pointwise* 1*-type Gauss map of the first kind if and only if it is an open part of a plane or a hyperbolic cylinder. A rational surface of revolution of type II has pointwise 1-type Gauss map of the first kind if and only if it is an open part of a plane or a circular cylinder. A rational surface of revolution of type III has pointwise 1-type Gauss map of the first kind if and only if it is an open part of an Enneper's surface of second kind, a de Sitter space or an anti-de Sitter space up to rigid motion.*

Proof. Suppose that M is a rational surface of revolution of type I. Then, one of its parametrizations is given by (2.2). If the function f is a constant, then the surface is a hyperbolic cylinder. When f is a not constant, we may put $f(u)$=u without loss of generality. Then, M can be parameterized by

$$x(u, v) = (u \sinh v, u \cosh v, g(u)). \tag{4.3}$$

In this case, the surface of revolution M has constant mean curvature if and only if $g = g(u)$ is a solution of the following differential equation

$$g'' + \frac{g'}{u}(1 + g'^2) + 2\alpha(1 + g'^2)^{\frac{3}{2}} = 0 \tag{4.4}$$

for some constant α. If we make the following change of variable $g' = \sinh y$, then (4.4) becomes

$$y' + \frac{1}{u}\sinh y \cosh y + 2\alpha \cosh^2 y = 0.$$

After we make another change of variable $y = \tanh^{-1}\omega$, we get

$$y' = \frac{\omega'}{1-\omega^2},\ \sinh y = \frac{\omega}{\sqrt{1-\omega^2}},\ \cosh y = \frac{1}{\sqrt{1-\omega^2}}.$$

Thus we get

$$u\omega'(u) + \omega + 2\alpha u = 0.$$

Solving the above equation yields $\omega(u) = (a - \alpha u^2)/u$ for some constant a. Hence

$$g'(u) = \frac{a-\alpha u^2}{\sqrt{u^2-(a-\alpha u^2)^2}},$$

where a is a constant. Therefore $g(u)$ is given by

$$g(u) = \int \frac{a-\alpha u^2}{\sqrt{u^2-(a-\alpha u^2)^2}}du.$$

If $a = \alpha = 0$, g is a constant. In this case, the surface is an open part of a plane. If $\alpha = 0$ and $a \neq 0$, then obtain $g(u)$=$a\cosh^{-1}(u/a) + c_1$ for some constant c_1. In this case, the surface is certainly not of rational kind. If $a = 0$ and $\alpha \neq 0$, then $g(u) = \sqrt{\alpha^{-2} - u^2} + c_2$. In this case, the surface is up to rigid motion a de Sitter space which is also not of rational kind. If $a \neq 0$, $\alpha \neq 0$, then that $g(u)$ can be expressed in terms of elliptic functions and $g(u)$ is not a rational function of u. The converse is easy to verify.

On the other hand, if M is parameterized by (2.3), by similar computation as above we can get the similar result.

Now, we consider the case that M is a surface of revolution of type II given by (2.4). If the function f is a constant, then the surface is a circular cylinder. Suppose f is not constant. By putting $f(u) = u$, the surface of revolution M can be parameterized by

$$x(u, v) = (g(u), u\cos v, u\sin v).$$

Suppose that $g'^2 > 1$. In this case, the surface of revolution has constant mean curvature if and only if $g = g(u)$ is a solution of the following differential equation:

$$g'' - \frac{g'}{u}(g'^2 - 1) + 2\alpha(g'^2 - 1)^{\frac{3}{2}} = 0 \tag{4.5}$$

for some constant α. If we make the following change of variable by $g' = \cosh y$, then (4.5) becomes

$$y' - \frac{1}{u}\sinh y \cosh y + 2\alpha \sinh^2 y = 0.$$

By another change of variable by $y = \coth^{-1}\omega$, we get

$$y' = \frac{\omega'}{1-\omega^2},\ \sinh y = \frac{1}{\sqrt{\omega^2-1}},\ \cosh y = \frac{\omega}{\sqrt{\omega^2-1}}.$$

Thus we have

$$u\omega'(u) + \omega - 2\alpha u = 0.$$

Solving above equation yields $\omega(u) = (a+\alpha u^2)/u$ for some constant a. Hence

$$g'(u) = \cosh(\coth^{-1}(\frac{a+\alpha u^2}{u})) = \frac{a+\alpha u^2}{\sqrt{(a+\alpha u^2)^2 - u^2}},$$

where a is a constant. Therefore $g(u)$ is given by

$$g(u) = \int \frac{a+\alpha u^2}{\sqrt{(a+\alpha u^2)^2-u^2}}du.$$

If $a = \alpha = 0$, g is a constant. In this case, the surface is an open part of a plane. If $\alpha = 0$ and $a \neq 0$, then we obtain $g(u)$=$-a\cosh^{-1}(u/a) + c_3$ for some constant c_3. In this case, the surface is a catenoid which is not of rational kind. If $a = 0$ and $\alpha \neq 0$, then $g(u) = \sqrt{u^2 - \alpha^{-2}} + c_4$. In this case, the surface is also not of rational kind, either. If $a\alpha \neq 0$, then that $g(u)$ can be expressed in terms of elliptic functions and $g(u)$ is not a rational function of u. The proof of converse is easy. In case of $g'^2 < 1$. we get the similar result.

Finally, we consider the case that M is a rational surface of type III parameterized in the form of (2.5). We use the parametrization of M described in Lemma 2.4:

$$x(u,v) = (k(u) - u - uv^2, k(u) + u - uv^2, -2uv). \tag{4.6}$$

A straightforward computation implies that the mean curvature is constant if and only if

$$k''(u) - \frac{2}{u}k'(u) = 4\alpha(k'(u))^{3/2} \tag{4.7}$$

for some constant α if $k'(u) > 0$ and $u > 0$. (4.7) is a Bernoulli's differential equation and can be solved as

$$k'(u) = \frac{u^2}{(-\alpha u^2 + a)^2}$$

for some constant a. If $\alpha = 0$, $k(u) = \frac{1}{3a^2}u^3 + b$ for some constant b. In this case, M is part of Enneper's surface of second kind (see [8]). If $a = 0$, $k(u) = -\frac{1}{\alpha^2}\frac{1}{u}+b$ for some constant b. In this case,

$$\langle x(u,v) - B, x(u,v) - B\rangle = -\frac{4}{\alpha^2},$$

where $B = (b, b, 0)$ and thus M is part of an anti-de Sitter space up to rigid motion. Similarly, if $k'(u)u < 0$, we obtain that M is part of Ennerper's surface of second kind or a de Sitter space. If $\alpha a \neq 0$, the function $k(u)$ cannot be expressed as a rational function. The converse is obvious. ■

5. Surfaces of Revolution in Minkowski Space with Pointwise 1-Type Gauss Map of the Second Kind

For a surfaces of revolution of type I or type II, we may assume $f(u) = u$ without loss of generality. Then, the surface of revolution of type I or type II in E_1^3 is parameterized by

$$x(u,v) = (u\sinh v, u\cosh v, g(u)) \tag{5.1}$$

or

$$x(u,v) = (u\cosh v, u\sinh v, g(u)) \tag{5.2}$$

and

$$x(u,v) = (g(u), u\cos v, u\sin v). \tag{5.3}$$

It is enough for us to consider (5.1) in the case of a surface of revolution M of type I.
We now prove

Theorem 5.1. *Let M be a polynomial surface of revolution of type I or type II. Then, it has pointwise 1-type Gauss map of the second kind if and only if it is open portion of a hyperbolic cone or a right cone.*

Proof.

Case 1. Let M be a surface of revolution given by (5.1) for some smooth function $g(u)$. Then the Gauss map G is obtained as

$$G = \frac{1}{\sqrt{1+g'^2}}(g'\sinh v, g'\cosh v, -1).$$

Applying the Laplacian operator Δ to G, we get

$$\Delta G = \frac{-1}{u\sqrt{1+g'^2}}\Big\{\Big(\frac{(g''+ug''')(1+g'^2)-4ug'g''^2}{(1+g'^2)^3} - \frac{g'}{u}\Big)\sinh v,\ \Big(\frac{(g''+ug''')(1+g'^2)-4ug'g''^2}{(1+g'^2)^3} - \frac{g'}{u}\Big)\cosh v,\ \frac{(g'g''+ug''^2+ug'g''')(1+g'^2)-4ug'^2g''^2}{(1+g'^2)^3}\Big\}.$$

Suppose M has pointwise 1-type Gauss map of the second kind. Then, we get

(5.4) $$\frac{1}{u}g''(1+g'^2)+g'''(1+g'^2)-4g'g''^2-\frac{g'}{u^2}(1+g'^2)^3 = -F(u,v)g'(1+g'^2)^3,$$

(5.5) $$\begin{aligned}&\frac{1}{u}(1+g'^2)g''g'+g''^2+g'g'''(1+g'^2)-3g'^2g''^2\\ &= F(u,v)(1+g'^2)^3(1-c\sqrt{1+g'^2}),\end{aligned}$$

where $C=(0,0,c)$, $c\neq 0$. Equations (5.4) and (5.5) imply

(5.6) $$\begin{aligned}&g''(1+g'^2)^2u+g'''(1+g'^2)^2u^2-3g'g''^2(1+g'^2)u^2-g'(1+g'^2)^3\\ &= c\sqrt{1+g'^2}\{g''(1+g'^2)u+g'''(1+g'^2)u^2-4g'g''^2u^2-g'(1+g'^2)^3\}.\end{aligned}$$

Let us rewrite equation (5.6) as

(5.7) $$P(u) = c\sqrt{1+g'^2(u)}Q(u),$$

where

$$P(u) = g''(1+g'^2)^2u+g'''(1+g'^2)^2u^2-3g'g''^2(1+g'^2)u^2-g'(1+g'^2)^3,$$

$$Q(u) = g''(1+g'^2)u+g'''(1+g'^2)u^2-4g'g''^2u^2-g'(1+g'^2)^3.$$

Denote by deg $g(u)$ the degree of $g(u)$. If deg $g(u)\geq 2$, it is impossible by comparing the degree of $P(u)$ and $Q(u)$. Consequently, deg $g(u)=1$. Thus, $g'(u)=a$ for some constant $a\neq 0$. Therefore, $c=\frac{1}{\sqrt{1+a^2}}$. Hence, the parametrization of M is reduced to

$$x(u,v) = (u\sinh v, u\cosh v, au),\ a\neq 0,\ a\in R,$$

that is, the surface of revolution M is part of a hyperbolic cone. Next, let M be a surface of revolution given by (5.2) for some smooth function $g(u)$. Then, by a similar discussion as above we can obtain the similar result.

Case 2. Let M be a surface of revolution parametrized by (5.3) for some smooth function $g(u)$.

First, we consider the case: $g'^2 > 1$.

Then, the Gauss map of M is given by

$$G = \frac{-1}{\sqrt{g'^2 - 1}}(1, g' \cos v, g' \sin v)$$

and the Laplacian of the Gauss map ΔG is computed as

$$\begin{aligned}\Delta G = \frac{1}{u\sqrt{g'^2-1}}\Big(& \frac{(g'g'' + ug''^2 + ug'g''')(g'^2-1) - 4ug'^2g''^2}{(g'^2-1)^3}, \\ & (\frac{(g'' + ug''')(g'^2-1) - 4ug'g''^2}{(g'^2-1)^3} - \frac{g'}{u}) \cos v, \\ & (\frac{(g'' + ug''')(g'^2-1) - 4ug'g''^2}{(g'^2-1)^3} - \frac{g'}{u}) \sin v\Big).\end{aligned}$$

Similarly to computation as above Case 1, we obtain

$$\begin{aligned} & g''(g'^2-1)^2u + g'''(g'^2-1)^2u^2 - 3g'g''^2(g'^2-1)u^2 + g'(g'^2-1)^3 \\ & = c\sqrt{g'^2-1}\{g''(g'^2-1)u - g'''(g'^2-1)u^2 + 4g'g''^2u^2 + g'(g'^2-1)^3\}.\end{aligned} \tag{5.8}$$

We may put (5.8) as

$$A(u) = c\sqrt{g'^2-1}B(u), \tag{5.9}$$

where

$$A(u) = g''(g'^2-1)^2u + g'''(g'^2-1)^2u^2 - 3g'g''^2(g'^2-1)u^2 + g'(g'^2-1)^3,$$

$$B(u) = g''(g'^2-1)u - g'''(g'^2-1)u^2 + 4g'g''^2u^2 + g'(g'^2-1)^3.$$

If deg $g(u) \geq 2$, we get a contradiction by comparing the degree of $A(u)$ and $B(u)$. Thus, deg $g(u) = 1$ and $g'(u) = a$ for some constant $a \neq 0$, $|a| \neq 1$. Thus, it gives $c = \frac{1}{\sqrt{a^2-1}}$. Therefore, the parametrization of M reduces to

$$x(u, v) = (au, u\cos v, u \sin v),\ a > 1 \text{ or } a < -1,$$

that is, the surface of revolution M is part of a right cone.

In case of $g'^2 < 1$, we can get a similar result. It completes the proof. ∎

Next, we prove

Theorem 5.2. *There do not exist rational surfaces of revolution of type I or type II except polynomial surfaces with pointwise* 1*-type Gauss map of the second kind.*

Proof. Suppose that M is a rational surface of revolution, that is, $g(u)$ is a rational function in u. The function $g(u)$ and $g'(u)$ are both rational functions in u. If $g'(u)$ is not a constant, we may put $g'(u) = r(u)/q(u)$, where $r(u)$ and $q(u)$ do not have a common factor of degree ≥ 1. Let deg $q(u)$=m.
In order to prove the theorem, we split the proof into two cases.

Case 1. M is of type I.
From (5.7) we know that $\sqrt{1+g'^2(u)}$ is also a rational function. Hence, if $g'(u)$ is non-constant, then there exists a polynomial $p(u)$ satisfying $q^2(u)+r^2(u) = p^2(u)$, where $q(u), r(u)$ and $p(u)$ are relatively prime. We put

$$
\begin{aligned}
&P_1(u) = g''(u)(1+g'^2(u))^2u,\ P_2(u) = g'''(u)(1+g'^2(u))^2u^2,\\
&P_3(u) = g'(u)g''^2(u)(1+g'^2(u))u^2,\ P_4(u) = g'(u)(1+g'^2(u))^3,\\
&Q_1(u) = g''(u)(1+g'^2(u))u,\ Q_2(u) = g'''(u)(1+g'^2(u))u^2,\\
&Q_3(u) = g'(u)g''^2(u)u^2,\ Q_4(u) = P_4(u).
\end{aligned} \tag{5.10}
$$

Then, $P_1, ..., P_4, Q_1, ...Q_4$ are rational functions, too.

Suppose that $m \geq 1$. Then, for each $i = 1, \cdots, 4$, we see that $q^7(u)P_i(u)$ is a polynomial. Similarly, we see that for each $i = 1, 2, 3$, $q^6(u)Q_i(u)$ is a polynomial. But, $q^6(u)Q_4(u)$ is given by

$$q^6(u)Q_4(u) = \frac{r(u)p^6(u)}{q(u)}. \tag{5.11}$$

From (5.7) we get

$$P(u) = c\frac{p(u)}{q(u)}Q(u). \tag{5.12}$$

Therefore, we see that $q^6(u)Q_4(u)$ is a polynomial. This is a contradiction because $p(u), q(u), r(u)$ are relatively prime. Hence, $m = 0$, that is, $g(u)$ is a polynomial.

Case 2. M is of type II.
The function $\sqrt{g'^2 - 1}$ in (5.9) is also a rational function. So, if $g'(u)$ is non-constant, then there exists a polynomial $p(u)$ satisfying $r(u)^2 - q(u)^2 = p^2(u)$, where $q(u)$, $r(u)$ and $p(u)$ are relatively prime. It only makes sense in case of

degree $r(u) \geq$ degree $q(u)$. Similarly as above, $m \geq 1$ derives a contradiction. Thus, $m = 0$, that is, $g(u)$ is a polynomial. ■

Finally, we consider the case of surface of revolution M of type III in E^3_1. The parametrization x of M is assumed to be

$$x(u, v) = (k(u) - u - uv^2, k(u) + u - uv^2, -2uv). \tag{5.13}$$

which is given in (4.6). Let us prove the following

Theorem 5.3. *There exists no rational surface of revolution of type III in a Minkowski 3-space with pointwise* 1*-type Gauss map of the second kind.*

Proof. Let M be a surface of revolution of type III. Suppose the Gauss map G is of pointwise 1-type of the second kind, that is,

$$\Delta G = F(G + C) \tag{5.14}$$

for some nonzero smooth function F and a nonzero constant vector C. By Lemma 2.4, the function F depends on u only and the vector C is parallel to the axis of revolution such that $C = (c, c, 0)$ for some nonzero constant c. From (2.11), we get

$$F(u) = \frac{2k'(u) - uk''(u)}{16uk'(u)^3}k''(u) + \frac{2k'(u)^2k'''(u) - 3k'(u)k''(u)^2}{16k'(u)^4}. \tag{5.15}$$

Put (5.15) into (2.8) with $c_1 = c_2 = c$, we obtain

$$\begin{aligned} &\sqrt{k'(u)}\{2u^2k'(u)k'''(u) - 3u^2k''(u)^2 + 2uk'(u)k''(u) + 4k'(u)^2\} \\ &+ 2cu\{k'(u)k'''(u) - 2uk''(u)^2 + uk'(u)k''(u)\} = 0. \end{aligned} \tag{5.16}$$

Since $k(u)$ is a rational function, so is $Q(u) = \sqrt{k'(u)}$ because of (5.16). If we rearrange (5.16) with respect to Q, we obtain

$$\begin{aligned} &u^2Q(u)^2Q''(u) - 2u^2Q(u)Q'(u)^2 + uQ(u)^2Q'(u) + Q(u)^3 \\ &= -c\{u^2Q(u)Q''(u) - 3u^2Q'(u)^2 + uQ(u)Q'(u)\}. \end{aligned} \tag{5.17}$$

From now on, we regard the rational function Q as a complex meromorphic function. Let $Q(z) = q(z)/p(z)$, where p and q are relatively prime polynomials.

First, we show that $q(z) = az^m$ for some constant a and a nonnegative integer m. Suppose $q(z_0) = 0$ for $z_0 \neq 0$. Then, $Q(z_0) = 0$ and

$$Q(z) = \sum_{n=k}^{\infty} a_n(z - z_0)^n$$

for some $k \geq 1$ and $a_k \neq 0$. Since $z = z_0 + (z - z_0)$, we get $z^2 = z_0^2 + 2z_0(z - z_0) + (z - z_0)^2$. If we compare the lowest degrees of both sides of (5.17) after putting z in (5.17) instead of u, we see that the lowest degree of the left hand side of (5.17) is $3k - 2$ and that of the right hand side is $2k - 2$. Therefore, the coefficient of term of degree $2k - 2$ in the right hand side is zero, that is,

$$0 = c(z_0^2 a_k^2 k(k-1) - 3z_0^2 k^2 a_k^2) = -cz_0^2 a_k^2(2k^2 + k),$$

which is a contradiction. Therefore,

$$Q(z) = \frac{az^m}{p(z)}$$

for some constant a and a nonnegative integer m.

Suppose $m \geq 1$. Let $p(z) = z^k + a_1 z^{k-1} + \cdots + a_k$. Since $(p, q) = 1$, $a_k \neq 0$. The series expansion of $Q(z)$ at $z = 0$ looks like

$$Q(z) = az^m + a_1 z^{m+1} + a_2 z^{m+2} + \cdots .$$

Then, the lowest degree of the left hand side of (5.17) is $3m$ and that of the right hand side is $2m$. Since $m \geq 1$, the coefficient of term with degree $2m$ must be zero, that is,

$$0 = -c\{m(m-1)a^2 - 3m^2 a^2 + ma^2\} = 2ca^2 m^2,$$

a contradiction. Thus, $m = 0$ and Q has the form

$$Q(z) = \frac{a}{p(z)}.$$

Finally, suppose that $\deg p = k \geq 1$. Then for some complex numbers $\alpha_1, \alpha_2, \cdots, \alpha_k$, $p(z)$ can be written as $p(z) = (z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_k)$. Since

$$\frac{1}{z - \alpha_1} = \frac{1}{z}\left(\frac{1}{1 - \alpha_1/z}\right) = \frac{1}{z} + \frac{\alpha_1}{z^2} + \frac{\alpha_1^2}{z^3} + \cdots, \quad (|z| > |\alpha_1|)$$

the meromorphic function $Q(z)$ has the form

$$Q(z) = \frac{a}{z^k} + \frac{a_1}{z^{k+1}} + \frac{a_2}{z^{k+2}} + \cdots \qquad (|z| > r) \tag{5.18}$$

for some $r > 0$. Putting (5.18) into (5.17) and comparing the degrees of terms in the both sides, the lowest degree of terms in $1/z$ of the left hand side is $3k$ and that of the right hand side is $2k$. Therefore, the coefficient in the term with degree $2k$ in $1/z$ must be zero, in other words,

$$0 = -ca^2\{k(k+1) - 3k^2 - k\} = 2ca^2 k^2,$$

which is a contradiction.

Consequently, if $c \neq 0$, the rational solutions of the equation (5.17) are constant functions and thus $Q(z) = 0$ by (5.17). Thus, k' vanishes. It contradicts that M is nondegenerate. ■

Combining Theorem 5.1, 5.2 and Theorem 5.3, we have

Theorem 5.4. (Characterization). *Let M be a rational surface of revolution. Then, M has pointwise 1-type Gauss map of the second kind in E^3_1 if and only if M is part of either a right cone or a hyperbolic cone.*

Acknowledgment

The authors would like to express their deep thanks to the referee who provided valuable suggestions to improve the paper and they also thank Professor J. Baek who gave the crucial remarks to revise it.

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