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q-GENERALIZATIONS OF THE PICARD AND GAUSS-WEIERSTRASS SINGULAR INTEGRALS

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Abstract. Introducing a higher order modulus of smoothness based on q-integers, in this paper first we obtain Jackson-type estimates in approximation by Jackson-type generalizations of the q-Picard and q-Gauss-Weierstrass singular integrals and give their global smoothness preservation property with respect to the uniform norm. Then, we study approximation and geometric properties of the complex variants for these q -singular integrals attached to analytic functions in compact disks. Finally, we prove approximation properties of these q-singular integrals attached to vector-valued functions.

1. INTRODUCTION

First we present some well known definitions and formulas for the $q-$ calculus used throughout the paper.

For $q > 0$, the q-real $[\lambda]_q$, where λ is any real number, is defined

$$[\lambda]_q := \begin{cases} \dfrac{1-q^\lambda}{1-q}, & q \neq 1 \\ \lambda, & q = 1 \end{cases} \qquad \text{and} \qquad [0]_q := 0.$$

If λ is an integer, i.e. $\lambda = n$ for some n, we write $[n]_q$ and call it $q-$integer. Also, the $q-$factorial is defined as

$$[n]_q! := \begin{cases} [n]_q [n-1]_q \cdots [1]_q, & n = 1, 2, ... \\ 1, & n = 0 \end{cases}.$$

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The $q-$binomial coefficients are given by

$$\left[\begin{array}{c} n \\ k \end{array}\right]_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}.$$

for integers $0 \leq k \leq n$, and as zero otherwise. Also, the $q-$binomial coefficients satisfy the following Pascal-type relation

$$\left[\begin{array}{c} n \\ k \end{array}\right]_q = q^{n-k}\left[\begin{array}{c} n-1 \\ k-1 \end{array}\right]_q + \left[\begin{array}{c} n-1 \\ k \end{array}\right]_q. \tag{1.1}$$

The $q-$extension of exponential function e^x is

$$E_q(x) := \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{(q;\, q)_n} x^n = (-x;\, q)_\infty\,, \tag{1.2}$$

where $(a;\, q)_n = \Pi_{k=0}^{n-1}\left(1-aq^k\right)$ and $(-x;\, q)_\infty = \Pi_{k=0}^{\infty}\left(1+xq^k\right)$.

Furthermore, the q-binomial expansion is defined as

$$\Pi_{k=0}^{n-1}\left(1+q^k x\right) = (-x;\, q)_n = \sum_{k=0}^{n} q^{k(k-1)/2}\left[\begin{array}{c} n \\ k \end{array}\right]_q x^k \tag{1.3}$$

More details on these can be found in [16] and [15].

The following two integrals will play an important role throughout the paper. For $0<q<1$, the first integral ,called the $q-$extension of Euler integral representation for the gamma function given in [13] and [2] that we use to define the $q-$Picard singular integral, is

$$c_q(x)\,\Gamma_q(x) = \frac{1-q}{\ln q^{-1}} q^{\frac{x(x-1)}{2}} \int_0^{\infty} \frac{t^{x-1}}{E_q\left((1-q)\,t\right)} dt\,, \quad \Re x>0 \tag{1.4}$$

where $\Gamma_q(x)$ is the $q-$gamma function defined by

$$\Gamma_q(x) = \frac{(q;\, q)_\infty}{(q^x;\, q)_\infty}(1-q)^{1-x}\,,\ \ 0<q<1$$

and $c_q(x)$ satisfies the following conditions: $c_q(x+1) = c_q(x)$, $c_q(n)=1$, $n = 0, 1, 2, \ldots$ and $\lim\limits_{q\to 1^-} c_q(x) = 1$.

When $x = n+1$ with n a non-negative integer, we obtain

$$\Gamma_q(n+1) = [n]_q!. \tag{1.5}$$

The second integral that we use to define the $q-$Gauss-Weierstrass singular integral is given in [14], by

$$\int_{-\infty}^{\infty} \frac{t^{2k}}{E_q\left(t^2\right)} dt = \pi \left(q^{1/2};\, q\right)_{1/2} q^{-\frac{k^2}{2}} \left(q^{1/2};\, q\right)_k, \quad k = 0,\, 1,\, 2... \tag{1.6}$$

where we have $(a;\, q)_\alpha = (a;\, q)_\infty / \left(aq^\alpha;\, q\right)_\infty$, for any $\alpha \in \mathbb{R}$.

In [9], the first author generalizes the Picard and Gauss-Weierstrass singular integrals, to the so-called q-Picard and q -Gauss-Weierstrass singular integrals. In this paper, first we introduce q-Jackson type generalizations of these q-Picard and q-Gauss-Weierstrass singular integrals and obtain Jackson type error estimate in approximation and global smoothness preservation properties with respect to a rth q-uniform moduli of smoothness.

These results generalize and improve some results for classical Picard and Gauss-Weierstrass singular integrals and their Jackson type generalization in [3], [4], [5] and [17].

Then, we consider the complex versions of these q-singular integrals and study their approximation and geometric properties in the unit disk. The last section deals with approximation properties of these q-singular integrals attached to vector-valued functions.

2. q-Jackson Type Generalization

First we give the q analogous of the rth-modulus of smoothness of f as it is defined in e.g. [17].

Definition 1. For $f \in C(\mathbb{R})$, $r \in \mathbb{N}$ and $q \in (0,\, 1)$ we introduce the following rth order q-moduli of smoothness of f defined by

$$\omega_{r,q}(f;\ t) = sup\{\left|\Delta_{q,h}^r f(x)\right|; x, x + [r]_q h \ \in \mathbb{R}, 0 \leq h \leq t\},$$

where

$$\Delta_{q,h}^r f(x) = \sum_{k=0}^{r} (-1)^{r-k} q^{(r-k)(r-k-1)/2} \left[\begin{array}{c} r \\ k \end{array}\right]_q f\left(x + [k]_q h\right).$$

The modulus $\omega_{1,q}(f;\ t)$ is denoted by $\omega(f;\ t)$ as in classical case.

Note that for $q = 1$ one reduces to the classical rth order moduli of smoothness defined as in e.g. [17] and [4, Chapter 2].

Reasoning as in the classical case (see e.g. [1]), we easily get

Lemma 1. *For $f \in C(\mathbb{R})$ we have $\omega_{r,q}(f;\ \gamma t) \leq (\gamma+1)^r \omega_{r,q}(f;\ t)$.*

Definition 2. Let $f : \mathbb{R} \to \mathbb{R}$. For $\lambda > 0$, $r \in \mathbb{N}\bigcup\{0\}$ and $0 < q < 1$, the q-Jackson type generalization of q-Picard and q-Gauss-Weierstrass singular integrals of f are

$$P_{r\lambda}(f; q, x) \equiv P_{r\lambda}(f;\ x) :=$$
$$-\frac{(1-q)}{2[\lambda]_q \ln q^{-1}} \sum_{k=1}^{r+1} (-1)^k \frac{q^{(r-k+1)(r-k)/2}}{q^{(r+1)r/2}} \left[\begin{array}{c} r+1 \\ k \end{array}\right]_q \int_{-\infty}^{\infty} \frac{f\left(x+[k]_q t\right)}{E_q\left(\frac{(1-q)|t|}{[\lambda]_q}\right)} dt$$

and

$$W_{r\lambda}(f; q, x) \equiv W_{r\lambda}(f;\ x) := -\frac{1}{\pi\sqrt{[\lambda]_q}\left(q^{1/2};\ q\right)_{1/2}} \cdot$$
$$\sum_{k=1}^{r+1} (-1)^k \frac{q^{(r-k+1)(r-k)/2}}{q^{(r+1)r/2}} \left[\begin{array}{c} r+1 \\ k \end{array}\right]_q \int_{-\infty}^{\infty} \frac{f\left(x+[k]_q t\right)}{E_q\left(\frac{t^2}{[\lambda]_q}\right)} dt.$$

Note that for $q = 1$, the above definition one reduces to the classical Jackson-type generalization of Picard and Gauss-Weierstrass singular integrals of f defined in [17] and [4, Chapter 16], while for $r = 0$ we get the q singular integrals defined in [9].

Next we give approximation results with rates and global smoothness preservation properties.

Theorem 1. *If $f \in C(\mathbb{R})$, $r \in \mathbb{N}\bigcup\{0\}$ and $0 < q < 1$, then we have*

$$|f(x) - P_{r\lambda}(f; q, x)| \leq \omega_{r+1,q}\left(f;\ [\lambda]_q\right) \frac{1}{q^{(r+1)r/2}} \sum_{k=0}^{r+1} \binom{r+1}{k} \frac{[k]_q!}{q^{\frac{k(k+1)}{2}}}$$

and

$$\left|f(x) - W_{(2r-1)\lambda}(f; q, x)\right| \leq \omega_{2r,q}\left(f;\ \sqrt{[\lambda]_q}\right) 2^{2r-1} \left(1 + q^{-\frac{r^2}{2}} \left(q^{1/2};\ q\right)_r\right).$$

Proof. Since $\frac{(1-q)}{2[\lambda]_q \ln q^{-1}} \int\limits_{-\infty}^{\infty} \frac{1}{E_q\left(\frac{(1-q)|t|}{[\lambda]_q}\right)} dt = 1$, we can write

$$|f(x) - P_{r\lambda}(f; q, x)| \leq \frac{(1-q)}{2[\lambda]_q \ln q^{-1}} \frac{1}{q^{(r+1)r/2}} \int_{-\infty}^{\infty} \frac{\omega_{r+1,q}(f;\ |t|)}{E_q\left(\frac{(1-q)|t|}{[\lambda]_q}\right)} dt.$$

By the properties of the modulus of smoothness of a function given in Lemma 1, (1.4) and (1.5), we get

$$\begin{aligned}
&\left|f(x) - P_{r\lambda}(f; q, x)\right| \\
&\leq \omega_{r+1,q}\left(f;\ [\lambda]_q\right) \frac{(1-q)}{[\lambda]_q \ln q^{-1}} \frac{1}{q^{(r+1)r/2}} \int_0^{\infty} \frac{\left(1 + t/[\lambda]_q\right)^{r+1}}{E_q\left(\frac{(1-q)t}{[\lambda]_q}\right)} dt \\
&= \omega_{r+1,q}\left(f;\ [\lambda]_q\right) \frac{1}{q^{(r+1)r/2}} \sum_{k=0}^{r+1} \binom{r+1}{k} \frac{[k]_q!}{q^{\frac{k(k+1)}{2}}}.
\end{aligned}$$

Theorem 2. *Let $f \in C(\mathbb{R})$, with $\omega_{r,q}(f;\ \delta) < \infty$ for $r \in \mathbb{N}\bigcup\{0\}$, $q \in (0,\ 1)$ and any $\delta > 0$. We have*

$$\omega_{r,q}(P_{r\lambda} f;\ \delta) \leq q^{-(r+1)r/2}\left((-1,\ q)_{r+1} - 1\right)\omega_{r,q}(f;\ \delta)$$

and

$$\omega_{r,q}(W_{r\lambda} f;\ \delta) \leq q^{-(r+1)r/2}\left((-1,\ q)_{r+1} - 1\right)\omega_{r,q}(f;\ \delta).$$

Proof. We have for each $0 \leq h \leq \delta$

$$\Delta_{q,h}^{r}(P_{r\lambda} f)(x) = -\frac{(1-q)}{2[\lambda]_q \ln q^{-1}}.$$

$$\sum_{k=1}^{r+1} (-1)^{r-k+1} \frac{q^{(r-k+1)(r-k)/2}}{q^{(r+1)r/2}} \begin{bmatrix} r+1 \\ k \end{bmatrix}_q \int_{-\infty}^{\infty} \frac{\Delta_{q,h}^{r} f\left(x + [k]_q t\right)}{E_q\left(\frac{(1-q)|t|}{[\lambda]_q}\right)} dt.$$

By (1.3), we have desired result. The proof in the case of $W_{r\lambda}(f; x)$ is similar.

3. Complex Q-Picard and Q-Gauss-weierstrass Integrals

In this section we extend the results in the case of classical complex Picard and Gauss-Weierstrass singular integrals proved in [6], [7], to their q-analogues.

Let us consider the open disk of radius $R > 0$, $D_R = \{z \in \mathbb{C};\ |z| < R\}$, $A(D_R) = \{f\colon \overline{D_R} \to \mathbb{C};\ f$ is analytic on D_R, continuous on $\overline{D_R}\}$ and $A^*(D_R) = \{f \in A(D_R);\ f(0) = 0,\ f'(0) = 1\}$. Therefore, if $f \in A^*(D_R)$ then we have $f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k$ for all $z \in D_R$.

For $f \in A(D_R)$, $\lambda \in \mathbb{R}$, $\lambda > 0$, $0 < q < 1$, $r \in \mathbb{N} \bigcup \{0\}$ and $z \in \overline{D_R}$, let us define the q-complex singular integrals

$$P_{r\lambda}(f; q, z) \equiv P_{r\lambda}(f; z) :=$$
$$-\frac{(1-q)}{2[\lambda]_q \ln q^{-1}} \sum_{k=1}^{r+1} (-1)^k \frac{q^{(r-k+1)(r-k)/2}}{q^{(r+1)r/2}} \begin{bmatrix} r+1 \\ k \end{bmatrix}_q \int_{-\infty}^{\infty} \frac{f\left(ze^{i[k]_q t}\right)}{E_q\left(\frac{(1-q)|t|}{[\lambda]_q}\right)} dt$$

and

$$W_{r\lambda}(f; q, z) \equiv W_{r\lambda}(f; z) := -\frac{1}{\pi\sqrt{[\lambda]_q}\left(q^{1/2}; q\right)_{1/2}} \cdot$$
$$\sum_{k=1}^{r+1} (-1)^k \frac{q^{(r-k+1)(r-k)/2}}{q^{(r+1)r/2}} \begin{bmatrix} r+1 \\ k \end{bmatrix}_q \int_{-\infty}^{\infty} \frac{f\left(ze^{i[k]_q t}\right)}{E_q\left(\frac{t^2}{[\lambda]_q}\right)} dt.$$

called as the complex q- Jackson type generalization of the q-Picard and q-Gauss-Weierstrass singular integrals, respectively. For $r = 0$ we denote these singular integrals by $P_\lambda(f; q, z) \equiv P_\lambda(f; z)$ and $W_\lambda(f; q, z) \equiv W_\lambda(f; z)$, respectively.

First we present the approximation properties.

Theorem 3. *Let $f \in A^*(D_R)$, i.e. $f(z) = \sum_{k=0}^{\infty} a_k z^k, z \in D_R$ with $a_0 = 0, a_1 = 1$ and $\lambda > 0$, $0 < q < 1$. We have :*

(i) *$P_\lambda(f; q, z) := P_\lambda(f; z)$ is continuous in $\overline{D_R}$, analytic in D_R so that*

$$P_\lambda(f; z) = \sum_{k=0}^{\infty} a_k c_k(\lambda, q) z^k, z \in D_R, P_\lambda(f; 0) = 0 \text{ and}$$

$$c_k(\lambda, q) = \frac{(1-q)}{[\lambda]_q \ln q^{-1}} \int_0^{\infty} \frac{cos(ku)}{E_q\left(\frac{(1-q)u}{[\lambda]_q}\right)} du, k = 0, 1, ...$$

Also, there exists $\widehat{q} \in (0, 1)$ such that for all $q \in (\widehat{q},\ 1)$ we have $c_1(\lambda, q) > 0$ and if we choose q_λ such that $0 < q_\lambda < 1$ and $q_\lambda \to 1$ as $\lambda \to 0$, then we have $lim_{\lambda \to 0} c_1(\lambda, q_\lambda) = 1$;

(ii) $|P_\lambda(f; z) - f(z)| \le (R+1)(1+\frac{1}{q})\omega_1(f; [\lambda]_q)_{\overline{D_R}}$, *for all $z \in \overline{D_R}$, where*

$$\omega_1(f; \delta)_{\overline{D_R}} = \sup\{|f(z_1) - f(z_2)|; z_1, z_2 \in \overline{D_R}, |z_1 - z_2| \le \delta\}.$$

Proof. (i) Let $z_0, z_n \in \overline{D_R}$ be with $\lim_{n\to\infty} z_n = z_0$. Since $|e^{iu}| = 1$, we get

$$|P_\lambda(f; z_n) - P_\lambda(f; z_0)| \le$$

$$\frac{(1-q)}{2\,[\lambda]_q \ln q^{-1}} \int_{-\infty}^{+\infty} |f(z_n e^{iu}) - f(z_0 e^{iu})| \cdot \frac{1}{E_q\left(\frac{(1-q)|u|}{[\lambda]_q}\right)}\, du$$

$$\leq \frac{(1-q)}{2\,[\lambda]_q \ln q^{-1}} \int_{-\infty}^{+\infty} \omega_1(f;|z_n - z_0|)_{\overline{D_R}} \cdot \frac{1}{E_q\left(\frac{(1-q)|u|}{[\lambda]_q}\right)}\, du = \omega_1(f;|z_n - z_0|)_{\overline{D_R}}.$$

Passing to limit with $n \to \infty$, it follows that $P_\lambda(f;z)$ is continuous at $z_0 \in \overline{D_R}$, since f is continuous on $\overline{D_R}$. It remains to prove that $P_\lambda(f;z)$ is analytic in D_R. For $f \in A^*(D_R)$, we can write $f(z) = \sum\limits_{k=0}^{\infty} a_k z^k$, $z \in D_R$. For fixed $z \in D_R$, we get $f(ze^{iu}) = \sum\limits_{k=0}^{\infty} a_k e^{iku} z^k$ and since $|a_k e^{iku}| = |a_k|$, for all $u \in \mathbb{R}$ and the series $\sum\limits_{k=0}^{\infty} a_k z^k$ is absolutely convergent, it follows that the series $\sum\limits_{k=0}^{\infty} a_k e^{iku} z^k$ is uniformly convergent with respect to $u \in \mathbb{R}$. This immediately implies that the series can be integrated term by term, i.e.

$$P_\lambda(f;z) = \frac{(1-q)}{2\,[\lambda]_q \ln q^{-1}} \sum_{k=0}^{\infty} a_k z^k \left(\int_{-\infty}^{\infty} e^{iku} \cdot \frac{1}{E_q\left(\frac{(1-q)|u|}{[\lambda]_q}\right)}\, du \right)$$

$$= \sum_{k=0}^{\infty} a_k c_k(\lambda, q) z^k, \text{ where } c_k(\lambda,q) = \frac{(1-q)}{[\lambda]_q \ln q^{-1}} \int\limits_0^{\infty} \frac{cos(ku)}{E_q\left(\frac{(1-q)u}{[\lambda]_q}\right)} du.$$

Since $a_0 = 0$, we get $P_\lambda(f;0) = 0$.

Then we have

$$c_1(\lambda,q) = \frac{(1-q)}{[\lambda]_q \ln q^{-1}} \int_0^{\infty} \frac{cos(u)}{E_q\left(\frac{(1-q)u}{[\lambda]_q}\right)} du = \frac{(1-q)}{\ln q^{-1}} \int_0^{\infty} \frac{cos([\lambda]_q\, u)}{E_q\left((1-q)\, u\right)} du.$$

Now, if we choose $q_\lambda \to 1$ as $\lambda \to 0$, then we get $[\lambda]_{q_\lambda} \to 0$ (see [9]). Since $\underset{q \to 1^-}{lim} E_q\left((1-q)\, t\right) = e^t$ (see [16, p. 9, (1.3.16)]) and $\lim_{q\to 1^-} [\lambda]_q = \lambda$, by Lebesgue's Dominated Convergence theorem, we obtain

$$lim_{\lambda \to 0} c_1(\lambda, q_\lambda) = \int\limits_0^{\infty} e^{-t} du = 1 \text{ and}$$

$$\lim_{q\to 1^-} c_1(\lambda,q) = \int_0^{\infty} \frac{cos(\lambda u)}{e^u} du > (\text{ by e.g. } [6, p.4]) > 0.$$

Thus, there exists $\widehat{q} \in (0,1)$ such that for all $q \in (\widehat{q},\ 1)$ we have $c_1(\lambda,q) > 0$.

(ii) By the Maximum Modulus Principle, it suffices to take $|z| = R$. Since $|e^{iu} - 1| \leq 2|sin\frac{u}{2}| \leq |u|$ for all $u \in \mathbb{R}$, we easily get

$$\begin{aligned}
&|P_\lambda(f;z) - f(z)| \\
&\leq \frac{(1-q)}{2\,[\lambda]_q \ln q^{-1}} \int_{-\infty}^{\infty} \omega_1(f;|ze^{iu}-z|)_{\overline{D_R}} \cdot \frac{1}{E_q\left(\frac{(1-q)|u|}{[\lambda]_q}\right)}\,du \\
&\leq \frac{(1-q)}{2\,[\lambda]_q \ln q^{-1}} \int_{-\infty}^{\infty} \omega_1(f;R|u|)_{\overline{D_R}} \cdot \frac{1}{E_q\left(\frac{(1-q)|u|}{[\lambda]_q}\right)}\,du \\
&\leq \omega_1(f;[\lambda]_q)_{\overline{D_R}}(R+1)\frac{(1-q)}{2\,[\lambda]_q \ln q^{-1}} \int_{-\infty}^{\infty} \left(1+\frac{|u|}{[\lambda]_q}\right) \cdot \frac{1}{E_q\left(\frac{(1-q)|u|}{[\lambda]_q}\right)}\,du \\
&\leq (\text{ by [9]}) \leq (R+1)\left(1+\frac{1}{q}\right)\omega_1(f;[\lambda]_q)_{\overline{D_R}}.
\end{aligned}$$
■

Theorem 4.

(i) *If* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *is analytic in* D_R, *then for all* $\lambda > 0$, $0 < q < 1$, $W_\lambda(f;q,z) := W_\lambda(f;z)$ *is analytic in* D_R *and we have in* D_R

$$W_\lambda(f;z) = \sum_{k=0}^{\infty} a_k d_k(\lambda,q) z^k,$$

where

$$d_k(\lambda,q) = \frac{2}{\pi\sqrt{[\lambda]_q}\,\left(q^{1/2};\,q\right)_{1/2}} \int_0^{\infty} \frac{cos(ku)}{E_q\left(\frac{u^2}{[\lambda]_q}\right)}\,du.$$

Also, there exists $\widehat{q} \in (0,1)$ *such that for all* $q \in (\widehat{q},\ 1)$ *we have* $d_1(\lambda,q) > 0$ *and if we choose* q_λ *such that* $0 < q_\lambda < 1$ *and* $q_\lambda \to 1$ *as* $\lambda \to 0$, *then we have* $lim_{\lambda\to 0} d_1(\lambda,q_\lambda) = 1$.
In addition, if f *is continuous on* $\overline{D_R}$ *then* $W_\lambda(f;z)$ *is continuous on* $\overline{D_R}$.

(ii) $|W_\lambda(f;z) - f(z)| \leq (R+1)\left(1+\sqrt{q^{-1/2}(1-q^{1/2})}\right)\omega_1\left(f;\sqrt{[\lambda]_q}\right)_{\overline{D_R}}$,
for all $z \in \overline{D_R}$.

Proof.

(i) Reasoning as for the $P_\lambda(f)$ operator, we easily deduce

$$W_\lambda(f;z) = \frac{1}{\pi\sqrt{[\lambda]_q}\,\left(q^{1/2};\,q\right)_{1/2}} \int_{-\infty}^{+\infty} \sum_{k=0}^{\infty} a_k z^k e^{iuk} \cdot \frac{1}{E_q\left(\frac{u^2}{[\lambda]_q}\right)}\,du$$

$$=\sum_{k=0}^{\infty} a_k d_k(\lambda,q) z^k, \text{ where } d_k(\lambda,q)=\frac{2}{\pi\sqrt{[\lambda]_q}\left(q^{1/2};\,q\right)_{1/2}}\int_0^{+\infty}\frac{cos(ku)}{E_q\left(\frac{u^2}{[\lambda]_q}\right)}.$$

Similar results with those for $c_1(\lambda,q)$ (in Theorem 3), can be obtained for $d_1(\lambda,q)$ too. Indeed, if we choose q_λ such that $0<q_\lambda<1$ and $q_\lambda\to 1$ as $\lambda\to 0$, then from Lebesgue's Dominated Convergence theorem, we get

$$\lim_{\lambda\to 0} d_1(\lambda,q_\lambda)=lim_{\lambda\to 0}\frac{2}{\pi\sqrt{[\lambda]_q}\left(q^{1/2};\,q\right)_{1/2}}\int_0^{\infty}\frac{cos(u)}{E_q\left(\frac{u^2}{[\lambda]_q}\right)}\,du$$

$$=\lim_{\lambda\to 0}\frac{2}{\pi\left(q^{1/2};\,q\right)_{1/2}}\int_0^{\infty}\frac{cos(\sqrt{[\lambda]_q}u)}{E_q\left(u^2\right)}\,du=(\text{ see e.g. } [2,p.132])=1$$

Similarly we can see that $\lim_{q\to 1^-} d_1(\lambda,q)>0$, which implies that there exists $\widehat{q}\in(0,1)$ such that for all $q\in(\widehat{q},\ 1)$ we have $d_1(\lambda,q)>0$.

The proof of continuity of $W_\lambda(f;z)$ is similar to that for $P_\lambda(f;z)$.

(ii) Reasoning as in the case of $P_\lambda(f;z)$, we can write

$$|W_\lambda(f;z)-f(z)|$$
$$\leq\frac{1}{\pi\sqrt{[\lambda]_q}\left(q^{1/2};\,q\right)_{1/2}}\int_{-\infty}^{+\infty}|f(ze^{-iu})-f(z)|\frac{1}{E_q\left(\frac{u^2}{[\lambda]_q}\right)}\,du$$
$$\leq\omega_1(f;\sqrt{[\lambda]_q})_{\overline{D_R}}(R+1)\frac{1}{\pi\sqrt{[\lambda]_q}\left(q^{1/2};\,q\right)_{1/2}}$$
$$\int_{-\infty}^{+\infty}\left(1+\frac{|u|}{\sqrt{[\lambda]_q}}\right)\frac{1}{E_q\left(\frac{u^2}{[\lambda]_q}\right)}\,du$$
$$\leq(\text{ see } [9])\leq(R+1)\left(1+\sqrt{q^{-1/2}(1-q^{1/2})}\right)\omega_1\left(f;\sqrt{[\lambda]_q}\right)_{\overline{D_R}}.\qquad\blacksquare$$

Theorem 5. *For $R>0$, $z\in\overline{D_R}$, $\lambda\in(0,1]$, $0<q<1$ and $r\in\mathbb{N}$, we have*

$$|P_{r\lambda}(f;z)-f(z)|\leq\frac{1}{q^{(r+1)r/2}}\sum_{k=0}^{r+1}\binom{r+1}{k}\frac{[k]_q!}{q^{\frac{k(k+1)}{2}}}\omega_{r+1,q}\left(f;[\lambda]_q\right)_{\partial D_R},$$

$$|W_{(2r-1)\lambda}(f;z)-f(z)|\leq 2^{2r-1}\left(1+q^{-\frac{r^2}{2}}\left(q^{1/2};\ q\right)_r\right)\omega_{2r,q}\left(f;\sqrt{[\lambda]_q}\right)_{\partial D_R},$$

where

$$\omega_{r,q}(f;\delta)_{\partial D_R}=\sup\left\{|\Delta_u^r f(Re^{ix})|;\,|x|\leq\pi,|u|\leq\delta\right\}.$$

Proof. Let $z \in \overline{D_R}$, $|z| = R$ be fixed. Because of the Maximum Modulus Principle, it suffices to estimate $|P_{r\lambda}(f; z) - f(z)|$, for this $|z| = R$, $z = Re^{ix}$. Reasoning now exactly as in the proof of Theorem 3, we get

$$f(z) - P_{r\lambda}(f;z) = \frac{(1-q)}{2\left[\lambda\right]_q \ln q^{-1}} \frac{(-1)^{r+1}}{q^{(r+1)r/2}} \int_{-\infty}^{\infty} \frac{\Delta_{q,t}^{r+1} f\left(Re^{ix}\right)}{E_q\left(\frac{(1-q)|t|}{[\lambda]_q}\right)} dt,$$

which implies

$$\begin{aligned} |f(z) - P_{r\lambda}(f;z)| &\leq \frac{(1-q)}{2\left[\lambda\right]_q \ln q^{-1}} \frac{1}{q^{(r+1)r/2}} \int_{-\infty}^{\infty} \frac{\omega_{r+1,q}\left(f;\ |t|\right)_{\partial D_R}}{E_q\left(\frac{(1-q)|t|}{[\lambda]_q}\right)} dt \\ &\leq \omega_{r+1,q}\left(f;\ [\lambda]_q\right)_{\partial D_R} \frac{1}{q^{(r+1)r/2}} \sum_{k=0}^{r+1} \binom{r+1}{k} \frac{[k]_q!}{q^{\frac{k(k+1)}{2}}}. \end{aligned}$$

The proof in the case of $W_{(2r-1)\lambda}(f;z)$ is similar. ∎

The geometric properties are consequences of Theorems 3 and 4 and are expressed by the following.

Theorem 6. *Let us suppose that $G \subset \mathbb{C}$ is open, such that $\overline{D_1} \subset G$ and $f : G \to \mathbb{C}$ is analytic in G. Denote by $(B_\lambda(f)(z))_{\lambda>0}$ any from $(P_\lambda(f;q,z))_{\lambda>0}$, $(W_\lambda(f;q,z))_{\lambda>0}$, where we choose $q := q_\lambda$ such that $0 < q_\lambda < 1$ and $q_\lambda \to 1$ as $\lambda \to 0$.*

(i) *If f is univalent in $\overline{D_1}$, then there exists $\lambda_0 > 0$ sufficiently small (depending on f), such that for all $\lambda \in (0, \lambda_0)$, $B_\lambda(f)(z)$ are univalent in $\overline{D_1}$.*

(ii) *Let $\gamma \in (-\pi/2, \pi/2)$. If $f(0) = f'(0) - 1 = 0$ (and $f(z) \neq 0$, for all $z \in \overline{D_1} \setminus \{0\}$ in the case of spirallikeness of order γ) and f is starlike (convex, spirallike of order γ, respectively) in $\overline{D_1}$, that is for all $z \in \overline{D_1}$*

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \left(Re\left(\frac{zf''(z)}{f'(z)}\right) + 1 > 0, Re\left(e^{i\gamma}\frac{zf'(z)}{f(z)}\right) > 0, resp.\right),$$

then there exists $\lambda_0 > 0$ sufficiently small (depending on f, and on f and γ in the case of spirallikeness), such that for all $\lambda \in (0, \lambda_0)$, $B_\lambda(f)(z)$ are starlike (convex, spirallike of order γ, respectively) in $\overline{D_1}$.

If $f(0) = f'(0) - 1 = 0$ (and $f(z) \neq 0$, for all $z \in D_1 \setminus \{0\}$ in the case of spirallikeness of order γ) and f is starlike (convex, spirallike of order γ, respectively) only in D_1 (that is the corresponding inequalities hold only in

D_1), then for any disk of radius $0 < \rho < 1$ and center 0 denoted by D_ρ, there exists $\lambda_0 > 0$ sufficiently small (depending on f and D_ρ, and in addition on γ for spirallikeness), such that for all $\lambda \in (0, \lambda_0)$, $B_\lambda(f)(z)$ are starlike (convex, spirallike of order γ, respectively) in $\overline{D_\rho}$ (that is, the corresponding inequalities hold in $\overline{D_\rho}$).

Proof. (i) Reasoning as in [9, Theorem 2.3], we get uniform convergence (as $\lambda \to 0$) in Theorems 3 and 4, which together with a well-known results concerning sequences of analytic functions converging locally uniformly to an univalent function (see e.g. [20], p. 130, Theorem 4.1.17) implies the univalence of $B_\lambda(f)(z)$ for sufficiently small λ.

For the proof of the conclusions in (ii), let us make some general useful considerations. By Theorems 3 and 4 (reasoning again as in [9, Theorem 2.3]), it follows that for $\lambda \to 0$, we have $B_\lambda(f)(z) \to f(z)$, uniformly in any compact disk included in G. By the well-known Weierstrass' result (see e.g. [20], p. 18, Theorem 1.1.6), this implies that $B'_\lambda(f)(z) \to f'(z)$ and $B''_\lambda(f)(z) \to f''(z)$, uniformly in any compact disk in G and therefore in $\overline{D_1}$ too, when $\lambda \to 0$. In all what follows, denote $P_\lambda(f)(z) = \frac{B_\lambda(f)(z)}{b_1(\lambda, q_\lambda)}$, where $b_1(\lambda, q_\lambda) > 0$ (for λ sufficiently small) is the coefficient of z in the Taylor series representing the analytic function $B_\lambda(f)(z)$.

If $f(0) = f'(0) - 1 = 0$, then we get $P_\lambda(f)(0) = \frac{f(0)}{b_1(\lambda, q_\lambda)} = 0$ and $P'_\lambda(f)(0) = \frac{B'_\lambda(f)(0)}{b_1(\lambda, q_\lambda)} = 1$. Also, if $f(0) = 0$ and $f'(0) = 1$, then $b_1(\lambda, q_\lambda)$ converges to $f'(0) = 1$ as $\lambda \to 0$, which obviously implies that for $\lambda \to 0$, we have $P_\lambda(f)(z) \to f(z)$, $P'_\lambda(f)(z) \to f'(z)$ and $P''_\lambda(f)(z) \to f''(z)$, uniformly in $\overline{D_1}$.

(ii) Suppose first that f is starlike in $\overline{D_1}$. By hypothesis we get $|f(z)| > 0$ for all $z \in \overline{D_1}$ with $z \neq 0$, which from the univalence of f in D_1, implies that we can write $f(z) = zg(z)$, with $g(z) \neq 0$, for all $z \in \overline{D_1}$, where g is analytic in D_1 and continuous in $\overline{D_1}$.

Write $P_\lambda(f)(z)$ in the form $P_\lambda(f)(z) = zQ_\lambda(f)(z)$. For $|z| = 1$ we have

$$|f(z) - P_\lambda(f)(z)| = |z| \cdot |g(z) - Q_\lambda(f)(z)| = |g(z) - Q_\lambda(f)(z)|,$$

which by the uniform convergence in $\overline{D_1}$ of $P_\lambda(f)$ to f and by the maximum modulus principle, implies the uniform convergence in $\overline{D_1}$ of $Q_\lambda(f)(z)$ to $g(z)$, as $\lambda \to 0$.

Since g is continuous in $\overline{D_1}$ and $|g(z)| > 0$ for all $z \in \overline{D_1}$, there exist an index $\lambda_0 > 0$ and $a > 0$ depending on g, such that $|Q_\lambda(f)(z)| > a > 0$, for all $z \in \overline{D_1}$ and all $\lambda \in (0, \lambda_0)$. Also, for all $|z| = 1$, we have

$$\begin{aligned} |f'(z) - P'_\lambda(f)(z)| &= |z[g'(z) - Q'_\lambda(f)(z)] + [g(z) - Q_\lambda(f)(z)]| \\ &\geq \big|\ |z| \cdot |g'(z) - Q'_\lambda(f)(z)| - |g(z) - Q_\lambda(f)(z)|\ \big| \\ &= \big|\ |g'(z) - Q'_\lambda(f)(z)| - |g(z) - Q_\lambda(f)(z)|\ \big|, \end{aligned}$$

which from the maximum modulus principle, the uniform convergence of $P'_\lambda(f)$ to f' and of $Q_\lambda(f)$ to g, evidently implies the uniform convergence of $Q'_\lambda(f)$ to g', as $\lambda \to 0$. Then, for $|z| = 1$, we get

$$\begin{aligned}\frac{zP'_\lambda(f)(z)}{P_\lambda(f)} &= \frac{z[zQ'_\lambda(f)(z) + Q_\lambda(f)(z)]}{zQ_\lambda(f)(z)} \\ &= \frac{zQ'_\lambda(f)(z) + Q_\lambda(f)(z)}{Q_\lambda(f)(z)} \to \frac{zg'(z) + g(z)}{g(z)} = \frac{f'(z)}{g(z)} = \frac{zf'(z)}{f(z)},\end{aligned}$$

which again from the maximum modulus principle, implies

$$\frac{zP'_\lambda(f)(z)}{P_\lambda(f)} \to \frac{zf'(z)}{f(z)}, \text{ uniformly in } \overline{D_1}.$$

Since $Re\left(\frac{zf'(z)}{f(z)}\right)$ is continuous in $\overline{D_1}$, there exists $\alpha \in (0, 1)$, such that

$$Re\left(\frac{zf'(z)}{f(z)}\right) \geq \alpha, \text{ for all } z \in \overline{D_1}.$$

Therefore

$$Re\left[\frac{zP'_\lambda(f)(z)}{P_\lambda(f)(z)}\right] \to Re\left[\frac{zf'(z)}{f(z)}\right] \geq \alpha > 0$$

uniformly on $\overline{D_1}$, i.e. for any $0 < \beta < \alpha$, there is $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$ we have

$$Re\left[\frac{zP'_\lambda(f)(z)}{P_\lambda(f)(z)}\right] > \beta > 0, \text{ for all } z \in \overline{D_1}.$$

Since $P_\lambda(f)(z)$ differs from $B_\lambda(f)(z)$ only by a constant, this proves the starlikeness in $\overline{D_1}$.

If f is supposed to be starlike only in D_1, the proof is identical, with the only difference that instead of $\overline{D_1}$, we reason for $\overline{D_\rho}$.

The proofs in the cases when f is convex or spirallike of order γ are similar and follows from the following uniform convergences (on $\overline{D_1}$ or on $\overline{D_\rho}$)

$$Re\left[\frac{zP''_\lambda(f)(z)}{P'_\lambda(f)(z)}\right] + 1 \to Re\left[\frac{zf''(z)}{f'(z)}\right] + 1.$$

and

$$Re\left[e^{i\gamma}\frac{zP'_{n\lambda}(f)(z)}{P_\lambda(f)(z)}\right] \to Re\left[e^{i\gamma}\frac{zf'(z)}{f(z)}\right],$$

The proof is complete. ∎

Remark 1. By using Theorem 5 and reasoning as above, it is not difficult to prove that the geometric properties in Theorem 6 remain valid for $P_{r\lambda}(f;z)$ and $W_{r\lambda}(f;z)$ too.

4. q-Singular Integrals Attached to Vector Valued Functions

In this section we extend some of the above results to vector-valued functions. Note that the case of classical singular integrals attached to vector valued functions was considered in [7].

If $(X, \|\cdot\|)$ is a complex Banach space and $R > 0$, let us denote by $A(D_R; X)$ the space of all functions $f\colon \overline{D_R} \to X$, which are continuous in $\overline{D_R}$ and holomorphic in D_R. Recall that according to e.g. [19], p. 97), any $f \in A(D_R; X)$ has the Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n, \quad z \in D_R,$$

where the series converges uniformly on any compact subset of D_R.

We will use the following well-known result in Functional Analysis.

Theorem 7. *Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{R}$ of $\mathbb{C}$ and denote by X^* the conjugate of X. Then $\|x\| = \sup\{|x^*(x)|;\ x^* \in X^*,\ |||x^*||| \leq 1\}$, for all $x \in X$, where $|||\cdot|||$ represents the usual norm in the dual space X^*.*

Now we are in position to prove our result. We present

Theorem 8. *Let $f \in A(D_R; X)$, $(X, \|\cdot\|)$ a complex normed space. If for $\lambda > 0$, $0 < q < 1$, we consider the operators*

$$P_\lambda(f; q, z) \equiv P_\lambda(f; z) := \frac{(1-q)}{2[\lambda]_q \ln q^{-1}} \int_{-\infty}^{\infty} \frac{f\left(ze^{it}\right)}{E_q\left(\frac{(1-q)|t|}{[\lambda]_q}\right)} dt,$$

$$W_\lambda(f; q, z) \equiv W_\lambda(f; z) := \frac{1}{\pi\sqrt{[\lambda]_q}\left(q^{1/2}; q\right)_{1/2}} \int_{-\infty}^{\infty} \frac{f\left(ze^{it}\right)}{E_q\left(\frac{t^2}{[\lambda]_q}\right)} dt,$$

then we have

$$||P_\lambda(f;z) - f(z)|| \leq (R+1)(1+\frac{1}{q})\omega_1(f;[\lambda]_q)_{\overline{D_R}},$$

$$||W_\lambda(f;z) - f(z)|| \leq (R+1)\left(1+\sqrt{q^{-1/2}(1-q^{1/2})}\right)\omega_1\left(f;\sqrt{[\lambda]_q}\right)_{\overline{D_R}},$$

for all $z \in \overline{D_R}$, *where* $\omega_1(f;\delta)_{\overline{D_R}} = \sup\{||f(z_1)-f(z_2)||; z_1, z_2 \in \overline{D_R}, |z_1-z_2| \leq \delta\}$.

Proof. Let $x^* \in B_1$ and define $g(z) = x^*[f(z)]$, $g: \overline{D_R} \to \mathbb{C}$. By Theorem 3 we have $|P_\lambda(g;z) - g(z)| \leq 2(1+\frac{1}{q})\omega_1(g;[\lambda]_q)_{\overline{D_R}}$, for all $z \in \overline{D_R}$, where

$$\omega_1(g;\delta)_{\overline{D_R}} = \sup\{|x^*[f(z_1)-f(z_2)]|; z_1, z_2 \in \overline{D_R}, |z_1-z_2| \leq \delta\}$$

$$\leq \sup\{||f(z_1)-f(z_2)||; z_1, z_2 \in \overline{D_R}, |z_1-z_2| \leq \delta\} = \omega_1(f;\delta)_{\overline{D_R}}.$$

Therefore, we obtain $|x^*[P_\lambda(f;z)-f(z)]| \leq 2(1+\frac{1}{q})\omega_1(f;[\lambda]_q)_{\overline{D_R}}$, for all $x^* \in B_1$, and passing here to supremum, according to Theorem 7 it follows the required estimate. The proof in the case of $W_\lambda(f;z)$ is similar. ∎

Remark 2. By using the method in the proof of Theorem 8, analogous results can easily be proved for $P_{r\lambda}(f;z)$ and $W_{r\lambda}(f;z)$.

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