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ON CHARACTERIZING THE REPRESENTATION FOR A REVERSED POINT MARTINGALE

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Abstract. In this paper, we obtain the representation for a reversed point martingale with respect to the reversed filtration generated by a point process. Besides, if a martingale can be expressed as a certain kind of stochastic integral with respect to some point martingale, then its reversed counterpart can also be expressed as a stochastic integral with respect to the corresponding reversed point martingale.

1. Introduction

When we estimate the common cumulative distribution function of some random variables $T, T_i, i = 1, 2, \ldots$, the empirical process $N_t^n := \sum_{i=1}^{n} 1_{\{T_i \le t\}}, t \in \Re^+$, plays a core role because it is the unique unbiased estimator. If $T_i, i = 1, 2, \ldots,$ are independent, positive, finite random variables, then the normalized N_t^n obeys Central Limit Theorem. The empirical processes, indexed by two parameters n and t, attract not only statisticians but probabilists. In the past studies, statisticians mainly focused on n but t. In fact, when we focus on the index t, the process N_t^n can be viewed as a point process which has given rise to many applications including survival analysis, mathematical finance, seismology, ..., etc. In order to model a point process, it is necessary to specify the information on which it is based. For counting processes, the collection of informations, which is called filtration, is usually characterized by the history of the experiment up to and including time t.

Recently, the seismological problems have been widely studied; especially, the forecast of a sensible earthquake. These problems are related to the models constructed by way of the point processes. In order to well analyze the persistency of an earthquake, tracing back according to the data collected several days before

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the present earthquake is important. Statistician might use the past data to fit a feasible model. As probabilists, we concentrate on the events that form the reversed filtration of a point process and characterize the representation for some kind of point martingales and its reversed correspondent.

Consider a homogeneous Poisson process $U_t, t \in [0,1]$. Let $\{W_1, W_2, \ldots\}$ be the waiting times of the occurrences, then $0 \leq W_1 < W_2 < \ldots$, a.s. Conditioned on the event $\{U_1 = n\}$, the joint density function of $(W_1, \ldots, W_n)$ is identical to that of $(Z_1, \ldots, Z_n)$ divided by $n!$, where $\{Z_1, Z_2, \ldots\}$ is a i.i.d. sequence of random variables with uniform distribution on $[0,1]$ (Theorem 5.6, Taylor and Karlin 1994.) Therefore, as a generalization, it is interesting to consider the case as the random variables $(T_i, i = 1, 2, \ldots)$ in $N_t^n := \sum_{i=1}^{n} 1_{\{T_i \leq t\}}$ are independent.

The problems on the time reversal of a stochastic process has been studied a lot; to name a few, Elliot and Anderson (1985), Haussman and Pardoux (1986), Pardoux(1986), and Meyer (1994) considered the time reversal of Markov diffusions; Elliot and Tsoi (1990) obtained some interesting results regarding the non-Markov point processes. Chou and Meyer's (1975) proposed an optional representation for general point martingales. Based on their works, we not only attempt to characterize the stochastic integral representation of the reversed point martingale in a more general setting but also to obtain the stochastic integral representation of a reversed point semimartingale. This paper is organized as follows: in section 2, we consider firstly the time-reverting of a point process with single jump; in section 3, we extend the results to a point process with independent, finitely many jumps. Since there is inherited difficulties in dealing with cases of not-independent jumps, we'll leave them alone presently for the time being. Throughout, all processes are defined on the common probability space $(\Omega, \mathcal{F}, P)$.

2. Reversing a Point Process with Single Jump

For any process $X = (X_t)_{t \in [0,1]}$ with $X_0 = 0$, we define

$$\begin{aligned} \tilde{X}_t &\equiv X_{(1-t)^-}, \quad \text{and} \\ X_t^R &\equiv \tilde{X}_t - X_1, \qquad \text{for } t \in [0,1). \end{aligned}$$

In this section, we focus on the single jump case. Set

$$N_t = 1_{\{S \leq t\}}, \quad t \in [0,1], \tag{1}$$

where S is a random variable satisfies $P(0 \leq S < 1) = 1$.

The natural filtration $\{\tilde{\mathcal{F}}_t, t \in [0,1)\}$ generated by the reversed process $\tilde{N}_t$, $t \in [0,1)$, is characterized in the following lemma.

Lemma 1. *The natural filtration* $\{\tilde{\mathcal{F}}_t, t \in [0,1)\}$ *generated by* $\tilde{N}_t$, $t \in [0,1)$ *is the collection of all sets of the form*

$$A = \{S \in B\}, \tag{2}$$

where B *is any Borel measurable subset of* $[0,1)$ *satisfying either* $[0,1-t) \subset B$ *or* $B \cap [0,1-t) = \emptyset$.

Remark. The reason that $t = 1$ is not included in the index of the reversed process is due to the restriction $P(0 \le S < 1) = 1$.

Proof. Take $B = [0,1-t)$. Since $1_{\{S<1-t\}} = \tilde{N}_t$, we deduce that $\tilde{N}_t$ is adapted to $\tilde{\mathcal{F}}_t$, where $\tilde{\mathcal{F}}_t$ consists of all sets A described in the statement of the lemma. Note that, for each $t \in [0,1)$, $\tilde{\mathcal{F}}_t$ is an increasing family and it suffices to verify the right continuity of $\tilde{\mathcal{F}}_t$.

For $A \in \bigcap_{u^n \downarrow\downarrow t; u^n \in Q} \tilde{\mathcal{F}}_{u^n}$, by definition, for any $u^n > t, u^n \in Q$, we can find a Borel measurable set B_{u^n} such that $A = \{S \in B_{u^n}\}$, where either $[0,1-u^n) \subset B_{u^n}$ or $[0,1-u^n) \cap B_{u^n} = \emptyset$. Setting $B = \lim_{u^n \downarrow\downarrow t\,; u^n \in Q} \inf B_{u^n}$, we have $A = \{S \in B\}$. Next, we show that $[0,1-t) \subset B$ whenever $B \cap [0,1-t) \neq \emptyset$. By the assumption $B \cap [0,1-t) \neq \emptyset$, there must be a point $1-u \in B \cap [0,1-t)$.

For all real r, $0 \le 1-u < 1-r < 1-t$, $B \cap [0,1-r) \neq \emptyset$. From the definition of B, there are two positive integers M^1 and M^2 such that the following conditions hold:

(1) for all $n \ge M^1$, $B_{u^n} \cap [0,1-r) \neq \emptyset$,
(2) for all $n \ge M^2$, $1-r < 1-u^n < 1-t$.

Clearly,

$$\emptyset \neq B_{u^n} \cap [0,1-r) \subset B_{u^n} \cap [0,1-u^n), \tag{3}$$

therefore $[0,1-u^n) \subset B_{u^n}$ for all $n \ge \max\{M^1, M^2\}$. Taking limit in n, we have

$$[0,1-t) \subset \bigcup_{m=1}^{\infty} \bigcap_{n \ge m} B_{u^n} = B,$$

which completes the proof of Lemma 1.

The following result is a simple application of Lemma 1.

Proposition 1. *Let* S *be a positive random variable with survival function* $F(t) = P(S > t)$ *and* $P(0 \le S < 1) = 1$. *Then, for any finite Borel measurable* *function* H *on* $[0,1)$ *with* $-\int |H(u)|dF(u) < \infty$,

$$
\begin{aligned}
&E[H(S)|\tilde{\mathcal{F}}_t] \\
(4)\qquad &= \frac{1_{\{S<1-t\}}}{1-\tilde{F}_t}\int_{[0,1-t)} -H(u)dF(u) + 1_{\{S\geq 1-t\}}H(S) \\
(5)\qquad &= \frac{1_{\{S<1-t\}}}{1-\tilde{F}_t}\int_{(t,1]} -H(1-u)d\tilde{F}(u) + 1_{\{S\geq 1-t\}}H(S),
\end{aligned}
$$

for $t \in [0,1)$.

Proof. The second equality is the consequence of change of variables. It is sufficient to prove the first equality. Suppose that $A \in \tilde{\mathcal{F}}_t$. By Lemma 1, A can be expressed as $A = \{S \in B\}$ for which B is described in the statement of Lemma 1. Simple calculation leads to

$$E\left[\Big(\frac{1_{\{S<1-t\}}}{1-\tilde{F}(t)}\int_{[0,1-t)} -H(u)dF(u) + 1_{\{S\geq 1-t\}}H(S)\Big)1_A\right] = E(H(S)1_A).$$

Remark. Since $N_1 = 1$ a.s., the natural filtration $(\mathcal{F}^R_t)_{t\in[0,1)}$ generated by $N^R_t = \tilde{N}_t - N_1$, $t \in [0,1)$, is the same as $(\tilde{\mathcal{F}}_t)_{t\in[0,1)}$. As a matter of fact, $E(H(S)|\mathcal{F}^R_t) = E(H(S)|\tilde{\mathcal{F}}_t)$.

For clarity and simplicity, we define

$$\tilde{M}^H_t = E(H(S)|\tilde{\mathcal{F}}_t), \quad t \in [0,1) \tag{6}$$

for any finite Borel measurable function H as that stated in Proposition 1.

Let $(q_t)_{t\in[0,1]}$, be the point martingale related to $(N_t)_{t\in[0,1]}$. Chou and Meyer (1975) obtained a representation for q_t,

$$q_t = 1_{\{S\leq t\}}\Big(1-\phi(S)\Big) + 1_{\{S>t\}}(-\phi(t)), \quad t \in [0,1], \tag{7}$$

where $\phi(t) = -\int_{(0,t]} \frac{dF(u)}{F(u^-)}$.

The reversed $q^R_t = \tilde{q}_t - q_1$ can be written as

$$q^R_t = 1_{\{S<1-t\}}\tilde{\phi}(t) + 1_{\{S\geq 1-t\}}(\phi(S)-1) - \tilde{\phi}(t), t \in [0,1). \tag{8}$$

Integration-by-part formula leads to the following equality,

$$\phi\Big(1-(t+\varepsilon)\Big)F\Big(1-(t+\varepsilon)\Big) = \int_{[0,1-(t+\varepsilon)]} \phi(u)dF(u) + \int_{[0,1-(t+\varepsilon)]} F(u^-)d\phi(u);$$

letting $\varepsilon \downarrow 0$, we immediately have

$$\tilde{\phi}(t)\tilde{F}(t) = \int_{[0,1-t)} (\phi(u) - 1) dF(u). \tag{9}$$

We may rewrite $\tilde{\phi}(t)$ as

$$\tilde{\phi}(t) = \frac{\tilde{\phi}(t) - \tilde{\phi}(t)\tilde{F}(t)}{1 - \tilde{F}(t)} = \frac{\tilde{\phi}(t)}{1 - \tilde{F}(t)} - \frac{\int_{[0,1-t)}(\phi(u) - 1)dF(u)}{1 - \tilde{F}(t)}. \tag{10}$$

From the above observations, we deduce the semimartingale representation for $(q_t^R)_{t\in[0,1)}$.

Theorem 1. *The reversed $(q_t^R)_{t\in[0,1)}$ has the following semimartingale representation with respect to the reversed filtration $(\mathcal{F}_t^R)_{t\in[0,1)}$:*

$$q_t^R = \tilde{M}_t^{\phi-1} + 1_{\{S<1-t\}} \frac{\tilde{\phi}(t)}{1 - \tilde{F}(t)} - \tilde{\phi}(t). \tag{11}$$

Let $(\mathcal{F}_t)_{t\in[0,1]}$ be the natural filtration generated by the point process $(N_t)_{t\in[0,1]}$. From Chou and Meyer (1975), we might obtain a stochastic integral representation for any $\mathcal{F}_t$-local martingale $(M_t)_{t\in[0,1]}$,

$$M_t = \int_{(0,t]} h_u dq_u, \tag{12}$$

where $h_t = H(t) - \frac{1}{F(t)} \int_{(0,t]} H(u)dF(u)$ and $H(t)$ is a finite Borel measurable function such that

$$M_t = \frac{1_{\{S>t\}}}{F(t)} \int_{(0,t]} -H(u)dF(u) + 1_{\{S\leq t\}} H(S). \tag{13}$$

If h is a function of finite variation, so is H, and vice versa. In this case, the reversed M_t^R is a stochastic integral of h_{1-t} with respect to q_t^R, for $t \in [0, 1)$.

Theorem 2. *Suppose that $(M_t)_{t\in[0,1]}$ is an $\mathcal{F}_t$-local martingale with the expression of* (12). *The reversed M_t^R, $t \in [0, 1)$, can be represented in terms of a stochastic integral as below:*

$$M_t^R = \int_{(0,t]} h_{1-u} dq_u^R. \tag{14}$$

Proof. Set $X_t = \int_{(0,t]} h_{u-} dq_u$, then

$$M_t = X_t + \sum_{s \le t} \Delta h_s \Delta q_s. \tag{15}$$

As an analogy to the proof of the Theorem 3.3 in Jacod and Protter (1988), we can deduce that

$$X_t^R + \{[h,q]\}_t^R = \int_{(0,t]} h_{1-u} dq_u^R. \tag{16}$$

Combining (15) and (16), we obtain

$$M_t^R = -\{[h,q]^c\}_t^R + \int_{(0,t]} h_{1-u} dq_u^R = \int_{(0,t]} h_{1-u} dq_u^R,$$

where $[h,q]^c$ denotes the continuous part of the quadratic variational process $[h,q]$.

3. Reversing a Point Process with Multiple Jumps

In this section, we are dealing with a point process with multiple jumps. The proof of results for reversing a point process with multiple jumps is, however, more complex. Define $N_t^n = \sum_{i=1}^{n} 1_{\{T_i \le t\}}$, where $(T_i)_{i=1,2,\dots}$ is a sequence of independent positive random variables with $P(0 \le T_i < 1) = 1$ for all $i = 1, 2, \dots$. The reversed filtration $\tilde{\mathcal{F}}_t^n$ is given in the following lemma.

Lemma 2. *The natural filtration* $(\tilde{\mathcal{F}}_t^n)_{t \in [0,1)}$ *generated by* $(\tilde{N}_t^n)_{t \in [0,1)}$ *is characterized in the following way: For each* $t \in [0,1)$, $\tilde{\mathcal{F}}_t^n$ *assembles all sets of the form*

$$A_t^n = \bigcap_{i=1}^{n} \{T_i \in B_i\}, \tag{17}$$

where B_i *satisfies either* $[0, 1-t) \subset B_i$ *or* $B_i \cap [0, 1-t) = \emptyset$ *for* $i = 1, 2 \dots, n$.

Proof. By induction, one can see that:

(i) for each $n \ge 1$, $\tilde{\mathcal{F}}_t^n \subset \tilde{\mathcal{F}}_t^{n+1}$, for all $t \in [0,1)$, and

(ii) for all $0 \le s < t < 1$, $\tilde{\mathcal{F}}_s^n \subset \mathcal{F}_t^n$, for each $n \ge 1$.

It is clear that $\tilde{N}_t^n = \sum_{i=1}^{n} 1_{\{T_i<1-t\}}$ is adapted to $\tilde{\mathcal{F}}_t^n$.

Similar to the proof of Lemma 1, it remains to show that $(\tilde{\mathcal{F}}_t^n)_{t\in[0,1)}$ is right continuous.

Regarding the proof of part (2), we may assume that for $n = 2, 3, \ldots, k, \tilde{\mathcal{F}}_t^n$ is right continuous.

Let $A_t^{k+1} \in \bigcap_{u^i>t;u^i\in Q} \tilde{\mathcal{F}}_{u^i}^{k+1}$. It remains to show that $A_t^{k+1} \in \tilde{\mathcal{F}}_t^{k+1}$.

For each $u^i > t$, there exist Borel measurable sets $B_j^{u^i}$, $j = 1, 2, \ldots, k+1$ as described in the statement of Lemma 2, such that

$$A_t^j = \bigcap_{\ell=1}^{j} \{T_\ell \in B_\ell^{u^i}\} = A_t^{j-1} \cap \{T_j \in B_j^{u^i}\},$$

for $j = 2, 3, \ldots, k+1$.

Set $B = \lim_{u^i\downarrow\downarrow t;\, u^i\in Q} \inf B_1^{u^i} \times \ldots \times B_k^{u^i} \times B_{k+1}^{u^i}$. Then

$$\begin{aligned} A_t^{k+1} &= \lim_{u^i\downarrow\downarrow t;\, u^i\in Q} \inf \left\{(T_1, \ldots, T_k) \in B_1^{u^i} \times \ldots \times B_k^{u^i}\right\} \\ &\quad \lim_{u^i\downarrow\downarrow t;\, u^i\in Q} \inf \{T_{k+1} \in B_{k+1}^{u^i}\} \\ &= A_t^k \cap \lim_{u^i\downarrow\downarrow t;\, u^i\in Q} \inf \{T_{k+1} \in B_{k+1}^{u^i}\}. \end{aligned} \tag{18}$$

By assumption, $A_t^k \in \tilde{\mathcal{F}}_t^k \subset \tilde{\mathcal{F}}_t^{k+1}$, and it suffices to show that

$$\lim_{u^i\downarrow\downarrow t;\, u^i\in Q} \inf \{T_{k+1} \in B_{k+1}^{u^i}\} \in \tilde{\mathcal{F}}_t^{k+1}. \tag{19}$$

Define $\lim_{u^i\downarrow\downarrow t;\, u^i\in Q} \inf B_{k+1}^{u^i} = B_{k+1}$, then

$$\lim_{u^i\downarrow\downarrow t;\, u^i\in Q} \inf \{T_{k+1} \in B_{k+1}^{u^i}\} = \{T_{k+1} \in B_{k+1}\}.$$

Assume that $B_{k+1} \cap [0, 1-t) \neq \emptyset$. Similar to the proof of Lemma 1, it is to show that $[0, 1-t) \subset B_{k+1}$, which implies (19). Consequently, $A_t^{k+1} \in \tilde{\mathcal{F}}_t^{k+1}$, and this completes the proof.

For each $i = 1, 2, \ldots, n$, $(q_i(t))_{t\in[0,1]}$ is defined to be the fundamental martingale related to the point process $1_{\{T_i\leq t\}}$ with respect to the natural filtration. In addition, we assume that each T_i, $i = 1, 2, \ldots, n$, has survival function $F_i(t)$.

Proposition 2. *If $(\mathcal{F}_t^n)_{[0,1]}$ is the natural filtration generated by $N_t^n = \sum_{j=1}^{n} 1_{\{T_i\leq t\}}$, then each $(q_i(t))_{t\in[0,1]}$, $i = 1, 2, \ldots$, is a martingale with respect to*

$(\mathcal{F}_t^n)_{[0,1]}$. *Moreover, if we define*

$$Q_t^n = \sum_{j=1}^{n} q_t^i, \tag{20}$$

then Q_t^n *is nothing more than the fundamental martingale with respect to* $(\mathcal{F}\,{}_t^n)_{[0,1]}$.

Proof. Analogous to the proof of Lemma 2, it is to show that $(\mathcal{F}_t^n)_{t\in[0,1]}$ assembles all sets of the form

$$A_t^n = \{T_1 \in B_1, \ldots, T_n\},$$

where each B_i is a Borel measurable subset of $[0,1]$, which satisfies either $B_i \cap (t,1] = \emptyset$ or $(t,1] \subset B_i$, for $i = 1,2,\ldots,n$. For $0 \le s < t \le 1$, the proof of Proposition 2 is tantamount to the work of verifying that

$$E[q_i(t)|\mathcal{F}_s^n] = q_i(s) \quad a.s., \tag{21}$$

for each $i = 1,2,\ldots,n$.

Let $A_s^n \in \mathcal{F}$, $A_s^n = \{T_1 \in B_1, \ldots, T_n \in B_n\}$, where $B_i \cap (s,1] = \emptyset$ or $(s,1] \subset B_i$. As a fact, it remains to show that

$$E\Big[(q_i(t) - q_i(s))1_{A_s^n}\Big] = 0, \tag{22}$$

which is a direct consequence of the independency of $(T_i)_{i=1,2,\ldots,n}$.

By Lemma 2, analogous to Theorem 1, there is a parallel result of representation for reversed point martingale with multiple jumps.

Theorem 3. *The reversed* $(Q^n)_t^R$, $t \in [0,1)$, *is an* $\tilde{\mathcal{F}}_t^n$*-semimartingale with the representation*

$$(Q^n)_t^R = \sum_{i=1}^{n} q_i^R(t) = \sum_{i=1}^{n} \tilde{M}_t^{\phi_i - 1} + \sum_{i=1}^{n} W_i(t), \tag{23}$$

where $\phi_i(t) = \int_{(0,t]} \frac{-dF_i(u)}{\bar{F}(u^-)}$, $\tilde{M}_t^{\phi_i+1}$ *is defined by* (6), *and* $W_i(t) = 1_{\{T_i<1-t\}}\frac{\tilde{\phi}_i(t)}{1-\tilde{F}(t)}$ $-\tilde{\phi}_i(t)$, *for* $i = 1,2,\ldots,n$ *and* $t \in [0,1)$. *Especially,* $\sum_{i=1}^{n} \tilde{M}_t^{\phi_i-1}$ *is an* $(\tilde{\mathcal{F}}_t^n)$*-martingale for* $t \in [0,1)$.

Proof. Similar to the proof of Theorem 1, each reversed $q_i^R(t)$ admits an expression of the following form:

$$q_i^R(t) = \tilde{M}_t^{\phi_i-1} + 1_{\{S<1-t\}}\frac{\tilde{\phi}_i(t)}{1-\tilde{F}(t)} - \tilde{\phi}_i(t),$$

for $i = 1, \ldots, n$ and $t \in [0, 1)$. It remains to show that each $\tilde{M}^{\phi-1}, i = 1, \ldots, n,$ is a $\tilde{\mathcal{F}}_t^n$-martingale, which is identical to show that for any $0 \le s < t < 1$,

$$\int_{\{T_1 \in B_1, \ldots, T_n \in B_n\}} (\tilde{M}_t^{\phi_i - 1} - \tilde{M}_s^{\phi_i - 1}) dP = 0, \tag{24}$$

where B_i, $i = 1, 2, \ldots, n$, is a Borel measurable set such that either $[0, 1-s) \subset B_i \neq \emptyset$ or $[0, 1-s) \subset B_i$. (24) can be obtained by independency of $(T_i)_{i=1,2,\ldots,n}$.

Let $(M_t^{n,H})_{t \in [0,1]}$ be an $(\mathcal{F}_t^n)_{t \in [0,1]}$-martingale with the representation

$$M_t^{n,H} = \sum_{i=1}^{n} \Big\{ \frac{1_{\{T_i \le t\}}}{F_i(t)} \int_{(0,t]} -H_i(u) dF_i(u) + 1_{T > t} H_i(T_i) \Big\}, \text{ and} \tag{25}$$

$$h(t) = \sum_{i=1}^{n} (H_i(t) - \frac{1}{F_i(t)} \int_{(0,t]} H_i(u) dF_i(u)), \tag{26}$$

where $H = (H_1, \ldots, H_n)$ is a vector of functions of finite variation on $[0, 1]$ with

$$\sum_{i=1}^{n} - \int |H_i(u)| dF_i(u) < \infty.$$

Again by Chou and Meyer (1975), $M_t^{n,H}, t \in [0, 1]$, can be expressed as

$$M^{n,H} = \int_{(0,t]} h(u) dQ_u^n. \tag{27}$$

With (27), Theorem 2 can be easily extended to the case of point processes with finitely many jumps.

Theorem 4. *Let $(M_t^{n,H})_{t \in [0,1]}$ be a $(\mathcal{F}_t^n)_{t \in [0,1]}$-martingale with the expression of* (27). *Then the reversed $(M^{n,H})_t^R$ is an $(\tilde{\mathcal{F}}_t^n)$-semimartingale with the following representation*

$$(M^{n,H})_t^R = \int_{(0,t]} h(1-u) d(Q^n)_u^R, \quad \textit{for } t \in [0, 1). \tag{28}$$

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