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BANACH ALGEBRAS RELATED TO THE ELEMENTS OF THE UNIT BALL OF A BANACH ALGEBRA

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Abstract. Suppose A is a Banach algebra and ϵ is in A with $\|\epsilon\| \leq 1$. In this note we aim to study the algebraic properties of the Banach algebra A_ϵ, where the product on A_ϵ is given by $a \odot b = a\epsilon b$, for $a, b \in A$. In particular we study the Arens regularity, amenability and derivations on A_ϵ. Also we prove that if A has an involution then A_ϵ has the same involution just when $\epsilon = 1$ or -1.

1. Introduction

Let A be a Banach algebra and ϵ be an element in the closed unit ball of A. A new product $\odot$ is defined on A by

$$a \odot b = a \ \epsilon \ b \qquad \text{for all a,b} \in A$$

A with this product is a Banach algebra which we denote it by A_ϵ. We aim to study the algebraic properties of A_ϵ such as when A_ϵ has a unit, when an element of A_ϵ is invertible and so on. The necessary and sufficient conditions for the existence of involution on A_ϵ is investigated. In particular, when is A_ϵ a C^*-algebra. Derivations on A_ϵ , the Arens regularity of A_ϵ and amenability of A_ϵ are also examined.

2. The Elementary Properties of A_ϵ

Definition 2.1. Let A be a Banach algebra and ϵ an element of it's closed unit ball i.e. $\|\epsilon\| \leq 1$. We define the new product $\odot$ on a A by

$$a \odot b = a \ \epsilon \ b \qquad \text{for all a,b} \in A.$$

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One can easily check that A with this product is an algebra which we denote it by A_ϵ.

Proposition 2.2. *With the above assumptions A_ϵ is a Banach algebra.*

Proof. is immediate. ■

In the next proposition the algebraic properties of A_ϵ are investigated.

Proposition 2.3. *If A is a Banach algebra. Then*

(i) *A_ϵ is unital if and only if A is unital and ϵ is invertible.*

(ii) *If A_ϵ is unital, then for any $a \in A$, $Sp_{A_\epsilon}(a) = Sp_A(a\epsilon)$. Where Sp_{A_ϵ} and Sp_A stand for the spectrum relative to A_ϵ and A respectively.*

(iii) *If A_ϵ is unital then $Inv(A_\epsilon) = Inv(A)$. Where Inv denotes the set of all invertible elements.*

(iv) *If ϵ_1 and ϵ_2 are in the closed unit ball of A, then $(A_{\epsilon_1})_{\epsilon_2} = A_{\epsilon_1\epsilon_2\epsilon_1}$. In particular, if ϵ is invertible then $(A_\epsilon)_{\epsilon^{-2}} = A$.*

Proof.

(i) Let A_ϵ be unital and 1_ϵ be the identity of A_ϵ. Then for any $a \in A$,

$$a \odot 1_\epsilon = 1_\epsilon \odot a = a.$$

Consequently $a(\epsilon 1_\epsilon) = (1_\epsilon \epsilon)a = a$. But $1_\epsilon \epsilon = (1_\epsilon \epsilon)(\epsilon 1_\epsilon) = \epsilon 1_\epsilon$. So $\epsilon 1_\epsilon$ is the unit of A and $\epsilon^{-1} = 1_\epsilon$.

For the converse, one can easily check that if ϵ is invertible, then ϵ^{-1} is the unit of A_ϵ.

(ii) Let A_ϵ be unital and $\lambda \in P_A(a)$. Then there exists $b \in A$ such that

$$1_\epsilon = \epsilon^{-1} = (\lambda\epsilon^{-1} - a) \odot b = (\lambda\epsilon^{-1} - a)\epsilon b = (\lambda - a\epsilon)b.$$

So that $1 = (\lambda - a\epsilon)b\epsilon$. This means that $\lambda - a\epsilon$ is left invertible in A. Similarly $(\lambda - a\epsilon)$ has a right inverse in A. Therefore $\lambda \in Sp_A(a\epsilon)$. In other words, we have $Sp_{A_\epsilon}(a) \subseteq Sp_A(a\epsilon)$.
In a similar way, we can see $Sp_A(a\epsilon) \subseteq Sp_{A_\epsilon}(a)$.

(iii) Let $a \in Inv(A)$. Then there is $b \in A$ such that

$$ab = ba = 1.$$

Therefore $a\epsilon(\epsilon^{-1}b\epsilon^{-1}) = (\epsilon^{-1}b\epsilon^{-1})\epsilon a = \epsilon^{-1}$.
This means that

$$a \odot (\epsilon^{-1}b\epsilon^{-1}) = (\epsilon^{-1}b\epsilon^{-1}) \odot a = \epsilon^{-1}.$$

Consequently, $a \in Inv(A_\epsilon)$ i.e. $Inv(A) \subseteq Inv(A_\epsilon)$. The reverse inclusion holds similarly.

(iv) Proof is immediate. ∎

In the next proposition we study the relation between the multiplicative linear functionals on A and A_ϵ.

Proposition 2.4.

(i) *If ϕ is a multiplicative linear functional on A, then $\psi = \phi(\epsilon)\phi$ is a multiplicative linear functional on A_ϵ.*

(ii) *If A_ϵ is unital, and ψ is a multiplicative linear functional on A_ϵ, then $\phi(a) = \psi(\epsilon^{-1}a)$ is a multiplicative linear functional on A_ϵ.*

Proof. (i) Let $a, b \in A$. Then

$$\psi(a \odot b) = \psi(a\epsilon b) = \phi(\epsilon)\phi(a)\phi(\epsilon)\phi(b) = \psi(a)\psi(b).$$

The proof of (ii) is clear by the identity $(A_\epsilon)_{\epsilon^{-2}} = A$ and (i), also one can verify it directly. ∎

Corollary 2.5.

(i) *If A_ϵ is unital, then the mapping $\phi \mapsto \psi$ between the set of all multiplicative linear functionals on A and A_ϵ is a one-to-one correspondence.*

(ii) *$Ker\phi = Ker\psi$ and in particular $\bigcap M = \bigcap M_\epsilon$. Where M and M_ϵ run over the maximal ideal spaces of A and A_ϵ respectively.*

3. Involution on A_ϵ

In this section the involutive Banach algebras are considered. Especially the necessary conditions for ϵ that A_ϵ is an involutive Banach algebra or a C^*-algebra, is investigated.

Proposition 3.1. *Let A be an involutive Banach algebra with involution $*$. Then*

(i) *If ϵ is self-adjoint, then A_ϵ is a $*$-involutive Banach algebra.*

(ii) *If A is unital or has a bounded approximate identity and $*$ is an involution on A_ϵ, then ϵ is self-adjoint.*

In particular, any C^*-algebra has a bounded approximate identity and so (i) and (ii) is valid.

Proof.

(i) is immediate.

(ii) Let $\{e_\alpha\}_{\alpha\in I}$ be a bounded approximate identity for A. Then by the continuity of $*$, $\{e_\alpha^*\}$ is also a bounded approximate identity for A. On the other hand, since $*$ is an involution for A_ϵ we have:

$$(e_\alpha^* \odot e_\alpha)^* = e_\alpha^* \odot e_\alpha$$

and it is easy to see that $\lim_\alpha e_\alpha^* \epsilon e_\alpha = \epsilon$. Now,

$$\begin{aligned}\epsilon^* &= \lim_\alpha (e_\alpha^* \epsilon^* e_\alpha) = \lim_\alpha (e_\alpha^* \epsilon e_\alpha)^* = \lim_\alpha (e_\alpha^* \odot e_\alpha)^* \\ &= \lim_\alpha (e_\alpha^* \odot e_\alpha) = \lim_\alpha e_\alpha^* \epsilon e_\alpha = \epsilon.\end{aligned}$$ ■

The following proposition shows that when both A and A_ϵ are C^*-algebras, ϵ can not be an interior point of the unit ball of A.

Proposition 3.2. *Let A and A_ϵ be C^*-algebras with the same involution. Then $\|\epsilon\| = 1$.* *Proof.* It is known that any C^*-algebra admits an increasing bounded approximate unit. Let $\{e_\alpha\}$ be such an approximate unit with $\|e_\alpha\| = 1$ for all α's. Since A_ϵ is also a C^*-algebra, we have:

$$1 = \|e_\alpha\|^2 = \|e_\alpha \odot e_\alpha^*\| = \|e_\alpha \epsilon e_\alpha^*\| \quad \text{and} \quad \|e_\alpha \epsilon e_\alpha^*\| \longrightarrow \|\epsilon\|.$$

Consequently, $1 = \|\epsilon\|$. ■

Theorem 3.3. *Let A and A_ϵ be C^*-algebras where ϵ is invertible, then $Sp(\epsilon) \subseteq \{-1, 1\}$.*

Proof. First we show that when ϵ is invertible, there is a one-to-one correspondence between the irreducible representations of A and A_ϵ. Let $\{\pi, H\}$ be an irreducible representation of A. Then it is easy to see that $\{\pi_1, H\}$ is an irreducible representation on A_ϵ where $\pi_1(a) = \pi(\epsilon a)$ for all $a \in A$. Also if $\{\pi_1, H\}$ is an irreducible representation of A_ϵ, then $\{\pi, H\}$ is an irreducible representation of A in which $\pi(a) = \pi_1(\epsilon^{-1} a)$ for all $a \in A$. Now if moreover A and A_ϵ are C^*-algebras then by 2.7.1 and 2.7.3 of [2], for any $a \in A$, we have

$$\|a\| = SUP\{\|\pi(a)\| : \{\pi, H\} \text{ is an irreducible representation of } A\}$$

and

$$\|a\| = SUP\{\|\pi_1(a)\| : \{\pi_1, H\} \text{ is an irreducible representation of } A_\epsilon\}$$

so that by what we have shown above,

$$\|a\| = SUP\{\|\pi(a)\| : \{\pi, H\} \text{ is an irreducible representation of } A\}$$

$$= SUP\{\|\pi_1(\epsilon^{-1}a)\| : \{\pi_1, H\} \text{ is an irreducible representation of } A_\epsilon\} = \|\epsilon^{-1}a\|$$

similarly,

$$\|a\| = SUP\{\|\pi_1(a)\| : \{\pi_1, H\} \text{ is an irreducible representation of } A_\epsilon\}$$

$$= SUP\{\|\pi(a)\| : \{\pi, H\} \text{ is an irreducible representation of } A\} = \|\epsilon a\|$$

Hence, $\|a\| = \|\epsilon a\| = \|\epsilon^{-1}a\|$ for all $a \in A$. Therefore $1 = \|1\| = \|\epsilon\| = \|\epsilon^{-1}\|$.
This means that $0 \notin Sp\{\epsilon\}$, $Sp(\epsilon) \subseteq [-1,1]$ and $Sp(\epsilon^{-1}) \subseteq [-1,1]$. But $Sp(\epsilon^{-1}) = \{\frac{1}{\lambda} : \lambda \in Sp(\epsilon)\}$. Consequently $Sp(\epsilon) \subseteq \{-1,1\}$. ■

The next example shows that, $Sp(\epsilon) = \{-1,1\}$ is possible. So, one can not find some more restriction conditions of Theorem 3.3 on ϵ.

Example 3.4. Let $A = \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{C}.\right\}$. Then A is a C^*-algebra. Assume $\epsilon \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ Then $Sp(\epsilon) = \{-1,1\}$. For this ϵ ,A_ϵ is a C^*-algebra. Indeed,

$$\left\|\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right\| = r\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = Max\{|a|, |b|\}$$

and for $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$,

$$\begin{aligned} \|A \odot A^*\| &= \left\|\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right\| \\ &= \left\|\begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}\right\| = Max\{|a^2|, |b^2|\} = \|A\|^2. \end{aligned}$$

The following example shows that the condition $Sp(\epsilon) = \{-1,1\}$ by itself is not a sufficient condition for A_ϵ to be a C^*-algebra. In fact it shows that the converse of Theorem 3.3 does not hold if ϵ is not invertible.

Example 3.5. Suppose A be the C^*-algebra of all complex 3×3 matrixes entries and let $\epsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Then it is clear that $Sp(\epsilon) = \{-1, 1\}$. But for $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ we have:

$$\begin{aligned} 9 = (r(A))^2 &= \|A\|^2 \neq \|A^* \odot A\| = \|A^* \epsilon A\| = r \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 4 \\ 0 & -2 & 11 \end{pmatrix} \\ &= \max\left\{1, |\frac{1}{-2}(-7 - \sqrt{193})|, |\frac{1}{-2}(-7 + \sqrt{193})|\right\} \end{aligned}$$

And this means that A_ϵ cannot be a C^*-algebra.

4. Derivations, Amenability, Arens Regularity of A_ϵ

In this section we investigate the derivations on A_ϵ and their relations with the derivations on A. Also we consider X-derivations where X is a A_ϵ-module, amenability of A_ϵ and it's relation with the amenability of A and finally we consider the Arens regularity of A_ϵ.

Definition 4.1. The linear operator $D : A \to A$ is called a derivation if

$$D(ab) = aD(b) + D(a)b.$$

The following proposition characterizes the derivations on A_ϵ with respect to the derivations on A.

Proposition 4.2.

(i) *Let D be a derivation on A such that $D(\epsilon) = 0$. Then D is a derivation on A_ϵ.*

(ii) *If A has a bounded approximate identity and D is a derivation on both A and A_ϵ, then $D(\epsilon) = 0$.*

Proof.

(i) Let D be a derivation on A such that $D(\epsilon) = 0$. Then for $a, b \in A$, we have

$$\begin{aligned} D(a \odot b) &= D(a\epsilon b) = D(a\epsilon)b + a\epsilon D(b) \\ &= D(a)\epsilon b + aD(\epsilon)b + a\epsilon D(b) = D(a) \odot b + a \odot D(b). \end{aligned}$$

Hence D is a derivation on A_ϵ.

(ii) Let $\{e_\alpha\}_{\alpha\in I}$ be a bounded approximate identity on A and D be a derivation on A and A_ϵ. Let $a, b \in A$, then since D is a derivation on A_ϵ, we have

$$D(a \odot b) = D(a) \odot b + a \odot D(b) = D(a)\epsilon b + a\epsilon D(b).$$

Also since D is a derivation on A,

$$D(a \odot b) = D(a\epsilon b) = D(a)\epsilon b + aD(\epsilon)b + a\epsilon D(b).$$

Therefore for all a and b in A, $aD(\epsilon)b = 0$. So that

$$0 = e_\alpha D(\epsilon)e_\alpha \to D(\epsilon).$$

Hence $D(\epsilon) = 0$. ■

The next proposition shows that in a special case any inner derivation on A_ϵ is an inner derivation on A.

Proposition 4.3. *If ϵ is in the algebraic center of A, then any inner derivation on A_ϵ is an inner derivation on A.*

Proof. Let δ_c^ϵ be the inner derivation corresponding to c on A_ϵ. Then:

$$\delta_c^\epsilon(a) = a \odot c - c \odot a = a\epsilon c - c\epsilon a = a(\epsilon c) - (\epsilon c)a = \delta_{\epsilon c}(a)$$

In which $\delta_{\epsilon c}$ is the inner derivation corresponding to ϵc on A. ■

Remark 4.4. If ϵ is an element in the algebraic center of A, then the identity $\delta_c(\epsilon) = c\epsilon - \epsilon c = 0$ and the proposition 4.2 implies that when ϵ is invertible, we have $\delta_c = \delta^\epsilon_{\epsilon^{-1}c}$. So that in this case the converse of the proposition 4.3 holds.

Now we consider the relation between A-modules and A_ϵ-modules.
Let X be a Banach A-module. We define

$$\odot : A_\epsilon \ X \to X \ \text{ by } \ (a, x) \mapsto a \odot x = a\epsilon x.$$

Then X is a A_ϵ-module. Indeed,

$$(a_1 \odot a_2) \odot x = (a_1\epsilon a_2)\epsilon x = a_1\epsilon(a_2\epsilon x) = a_1 \odot (a_2 \odot x).$$

Also,

$$\|a \odot x\| = \|(a\epsilon)x\| \le k\|a\epsilon\| \ \|x\| \le k\|\epsilon\|\|a\|\|x\|.$$

Definition 4.5. The bounded linear operator $D : A \to X$ is called a X-derivation of A if $D(ab) = D(a)b + aD(b)$, for all $a, b \in A$.

The next proposition shows the relation between X-derivations of A and X-derivations of A_ϵ.

Proposition 4.6.

(i) *If D is a X-derivation of A such that $D(\epsilon) = 0$, then D is a X-derivation of A_ϵ.*

(ii) *If A has a bounded approximate identity for X, and D is a X-derivation of A and of A_ϵ, then $D(\epsilon) = 0$.*

Proof. Proof is similar to proposition 4.2. ■

Now we consider the amenability of A_ϵ. The following proposition shows that if A is commutative and ϵ is idempotent then the amenability of A implies the amenability of A_ϵ.

Proposition 4.7. *Let A be a Banach algebra and ϵ be an idempotent element of the algebraic center of A. If A is amenable, then A_ϵ is amenable.*

Proof. Let A be an amenable Banach algebra. Then $A \overset{\wedge}{\otimes} A$ (for its definition see [1]), is also amenable (see Theorem 4.3 of [6]). Now let:

$$f : A \overset{\wedge}{\otimes} A \to A_\epsilon \text{ be defined by } f(a \otimes b) = a\epsilon b$$

Then f is a continuous homomorphism of Banach algebras.
Indeed we have:

$$\begin{aligned} f((a_1 \otimes b_1)(a_2 \otimes b_2)) &= f(a_1 a_2 \otimes b_1 b_2) = a_1 a_2 \epsilon b_1 b_2 = a_1 a_2 \epsilon^3 b_1 b_2 \\ &= (a_1 \epsilon b_1)\epsilon(a_2 \epsilon b_2) = f(a_1 \otimes b_1) \odot f(a_2 \otimes b_2) \end{aligned}$$

Also, f is continuous, since for $u \in A \otimes A$, if

$$\sum_{i=1}^{n} a_i \otimes b_i$$

is one of the representations of u, then from the fact that $\|\epsilon\| \leq 1$ we have

$$\|f(u)\| = \|\sum_{i=1}^{n} a_i \epsilon b_i\| \leq \sum_{i=1}^{n} \|a_i\|\|\epsilon\|\|b_i\| \leq \sum_{i=1}^{n} \|a_i\|\|b_i\|$$

Consequently,

$$\|f(u)\| \leq inf\{\sum_{i=1}^{n} \|a_i\|\|b_i\| : \quad u = \sum_{i=1}^{n} a_i \otimes b_i\} = \|u\|.$$

Therefore $\|f\| \leq 1$. Also the range of f is A_ϵ, since for any $a \in A$

$$a = a1 = a\epsilon^2 = a\epsilon\epsilon,$$

and $a\epsilon\epsilon$ is an element in the range of f. Thus f is a continuous homomorphism of the amenable Banach algebra of $A \overset{\wedge}{\otimes} A$ onto A_ϵ. Now the amenability of Banach algebra A_ϵ is a consequence of Theorem 43.11 in [5]. ■

Remark 4.8. If in the above proposition, we also assume that ϵ is invertible, then the amenability of A_ϵ implies the amenability of A. This is because of the identity

$$A = (A_\epsilon)_{\epsilon^{-2}}.$$

We conclude this section with studding the Arens regularity of A_ϵ. In particular we show that if A is a left or right ideal of the Banach $(A^{**}, .)$, then A_ϵ is Arens regular for all ϵ in the unite ball of A.

We denote ”.” the first Arens product on A^{**}, which is defined as follows

$$< f.a, b >=< f, ab >$$

$$< n.f, a >=< n, f.a >$$

$$< m.n, f >=< m, n.f >$$

for $a, b \in A$, $f \in A^*$ and $m, n \in A^{**}$, and use ”Δ” for the second Arens product on A^{**} which is defined as follows

$$< b, a\Delta f >=< ba, f >$$

$$< a, f\Delta m >=< a\Delta f, m >$$

$$< f, m\Delta n >=< f\Delta m, n > .$$

Also the topological center Z_1 and Z_2 corresponding to the first and the second Arens product respectively, is defined by

$$Z_1 = \{m \in A^{**} : m.n = m\Delta n, \ \forall n \in A^*\}$$

$$Z_2 = \{n \in A^{**} : m.n = m\Delta n, \ \forall m \in A^{**}\}.$$

We refer to [3] and [4] for elementary definitions and more information about Arens products, topological center and Arens regularity of Banach algebras. The Banach algebra A is called Arens regular if and only if $Z_1 = A^{**}$ or $Z_2 = A^{**}$. We recall that if A is a Banach algebra $a \in A$ *and* $n \in A^{**}$, then $A \subseteq Z_1 \cap Z_2$, *so* $a.n = a\Delta n$ and $n.a = n\Delta a$.

Theorem 4.9. *Let A be a Banach algebra and A is a left or right ideal of A^{**}, with the products $a.n$ and $n.a$, $(a \in A,\ n \in A^{**})$. Then for each ϵ in the unit ball of A, A_ϵ is Arens regular.*

Proof. Let $\oplus$ denotes the first Arens product on A_ϵ^{**} *and* $\Delta_\oplus$ be the second Arens product on A_ϵ^{**}. Let $m, n \in A_\epsilon^{**}$, $f \in A_\epsilon^* = A^*$ and $a, b \in A_\epsilon$, we have

$$< f \oplus a, b >=< f, a \oplus b >=< f, a\epsilon b >=< f.a\epsilon, b >$$

so $f \oplus a = f.a\epsilon$, for all $a \in A$. Also

$$< n \oplus f, a >=< n, f \oplus a >=< n, f.a\epsilon >$$

$$=< (\epsilon\Delta n).f, a >=< \epsilon.n.f, a > .$$

The last equality holds, since $A \subseteq Z_1 \cap Z_2$ and so $\epsilon\Delta n = \epsilon.n$. Hence $n \oplus f = \epsilon.n.f$. Furthermore

$$< m \oplus n, f >=< m, n \oplus f >=< m, \epsilon.n.f >=< m.\epsilon.n, f > .$$

Thus $m \oplus n = m.\epsilon.n$. Similarly one can show that $m\Delta_\oplus n = m\Delta\epsilon\Delta n$. Now suppose A is a left ideal in A^{**}. This implies that for each $m, n \in A^{**}$, $\epsilon.n (= \epsilon\Delta n)$ belongs to Z_1 and

$$m \oplus n = m.(\epsilon.n) = m\Delta(\epsilon.n)$$

$$= m\Delta(\epsilon\Delta n) = m\Delta_\oplus n.$$

Hence A_ϵ^{**} is Arens regular. Similar arguments prove that A_ϵ is Arens regular when A is a right ideal of A^{**}. ∎

Remark 4.10. If A is Arens regular then the equalities $m \oplus n = m.\epsilon.n$ *and* $m \Delta_\oplus n = m\Delta n\Delta n$ implies that A_ϵ is Arens regular. But the converse is not true in general, for example let G be an infinite compact topological group. By Theorem [7] 4.1 we know that $A = L^1(G)$ is a right ideal in its second dual so by the previous Theorem for each ϵ in the unit ball of A, A_ϵ is Arens regular, but from [8] we know $L^1(G)$ is Arens regular if and only if G is finite, which shows that A is not Arens regular.

References

1. F. F. Bonsal, J. Duncan, *Complete normed algebras,* Springer-Verlag, 1973.
2. J. Dixmier, *C^*-Algebras,* N. H. Publishing Company, 1977.
3. J. Duncan and S. A. R. Hosseiniun, The second dual of a Banach algebra, *Proc. Roy. Edinburg*, **84A** (1979), 309-325.
4. A. T. Lau and A. Ulger, Topological center of certain dual algebras., *Trans. Amer. Math. Soc.*, **348(3)** (1996), 1191-1212.
5. G. J. Murphy, *C^*-Algebras and operator theory,* Academic Press Inc, 1990.
6. M. Takesaki, *Theory of operator algebras*, Vol. I, Springer-Verlag Heidelberg, 1979.
7. S. Watanabe, A Banach algebra which is an ideal in the second dual space, *Science Rep. of Niigata Univ. Series A.*, **11** (1974), 95-101.
8. N. J. Young, The irregularity of multiplication in group algebras, *Quart. J. Math.* (*Oxford*), **24(2)** (1973), 59-62.

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