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GENERALIZED SKEW DERIVATIONS WITH ANNIHILATING ENGEL CONDITIONS

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Abstract. Let R be a noncommutative prime ring and $a \in R$. Suppose that f is a right generalized β-derivation of R such that $a[f(x), x]_k = 0$ for all $x \in R$, where k is a fixed positive integer. Then $a = 0$ or there exists $s \in C$ such that $f(x) = sx$ for all $x \in R$ except when $R = M_2(GF(2))$.

1. Introduction

Recently, C. L. Chuang, M. C. Chou and C. K. Liu [7] proved the following: Let R be a noncommutative prime ring and $a \in R$. Suppose that δ is a β-derivation of R such that $a[\delta(x), x]_k = 0$ for all $x \in R$, where k is a fixed positive integer. Then $a = 0$ or $\delta = 0$ except when $R = M_2(GF(2))$. This result generalizes several known results, see for instance, [12], [13] and [16]. In this paper we will extend [7] further to the so-called right generalized skew derivations.

Throughout this paper, R is always a prime ring with center Z. For $x, y \in R$, set $[x, y]_1 = [x, y] = xy - yx$ and $[x, y]_k = [[x, y]_{k-1}, y]$ for $k > 1$.

Let β be an automorphism of R. A β-derivation of R is an additive mapping $\delta : R \to R$ satisfying $\delta(xy) = \delta(x)y + \beta(x)\delta(y)$ for all $x, y \in R$. β-derivations are also called skew derivations. When $\beta = 1$, the identity map of R, β-derivations are merely ordinary derivations. If $\beta \neq 1$, then $1 - \beta$ is a β-derivation. An additive mapping $f : R \to R$ is a right generalized β-derivation if there exists a β-derivation $\delta : R \to R$ such that $f(xy) = f(x)y + \beta(x)\delta(y)$ for all $x, y \in R$. The right generalized β-derivations generalize both β-derivations and generalized derivations. If $a, b \in R$ and $\beta \neq 1$ is an automorphism of R, then $f(x) = ax - \beta(x)b$ is a right generalized β-derivation. Moreover, if δ is a β-derivation of R, then $f(x) = ax + \delta(x)$ is a right generalized β-derivation.

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We let $_{\mathcal{F}}R$ denote the right Martindale quotient ring of R and Q the two sided Martindale quotient ring of R. Let C be the center of Q and $_{\mathcal{F}}R$, which is called the extended centroid of R. Note that Q and $_{\mathcal{F}}R$ are also prime rings and C is a field (see [1]). It is known that automorphisms, derivations and β-derivations of R can be uniquely extended to Q and $_{\mathcal{F}}R$. In [2], we know that right generalized β-derivations of R can also be uniquely extended to $_{\mathcal{F}}R$. Indeed, if f is a right generalized β-derivation of R, then $f(x) = f(1)x + \delta(x)$ for all $x \in R$, where δ is a β-derivation of R (Lemma 2 in [2]).

A β-derivation δ of R is called X-inner if $\delta(x) = bx - \beta(x)b$ for some $b \in Q$. δ is called X-outer if it is not X-inner. An automorphism β is called X-inner if $\beta(x) = uxu^{-1}$ for some invertible $u \in Q$. β is called X-outer if it is not X-inner.

We are now ready to state the main result:

Main Theorem. *Let R be a noncommutative prime ring and $a \in R$. Suppose that f is a right generalized β-derivation of R such that $a[f(x), x]_k = 0$ for all $x \in R$, where k is a fixed positive integer. Then $a = 0$ or there exists $s \in C$ such that $f(x) = sx$ for all $x \in R$ except when $R = M_2(GF(2))$.*

We begin with two crucial lemmas.

Lemma 1. *Let R be a noncommutative prime ring and let $a, b, c \in R$, with $a \neq 0$. If $a[bx - xc, x]_k = 0$ for all $x \in R$, where k is a fixed positive integer. Then $b, c \in Z$.*

Proof. We claim first that $c \in Z$. If not, then

$$g(x) = a[bx - xc, x]_k = a\sum_{i=0}^{k}(-1)^i\binom{k}{i}x^i(bx - xc)x^{k-i} = 0$$

is a nontrivial GPI of R. By [3], $g(x) = 0$ is also a nontrivial GPI of Q. Let F be the algebraic closure of C if C is infinite, otherwise let F be C. By a standard argument [14, Proposition], $g(x) = 0$ is also a GPI of $Q \otimes_C F$. Since $Q \otimes_C F$ is a centrally closed prime F-algebra [8, Theorem 3.5], by replacing R, C with $Q \otimes_C F$ and F respectively, we may assume that R is centrally closed and the field C is either algebraically closed or finite. By [15, Theorem 3], R is a primitive ring having nonzero socle with field C as its associated division ring. By [9, p.75], R is isomorphic to a dense subring of the ring of linear transformations of a vector space V over C, containing nonzero linear transformations of finite rank. Since R is not commutative, we may assume that $\dim V_C \geq 2$.

We claim that there exists $v \in V$ such that v and cv are C-independent. If not, v and cv are C-dependent for all $v \in V$. That is, for each $v \in V$ there exists $\lambda_v \in C$

such that $cv = v\lambda_v$. By [7, Lemma 1], there exists $\lambda \in C$ such that $cv = v\lambda$ for all $v \in V$. Then

$$(bx - xc)v = bxv - xcv = bxv - xv\lambda = bxv - cxv = (b - c)xv$$

for all $v \in V$. Since $a[bx - cx, x]_k = a\sum_{i=0}^{k}(-1)^i\binom{k}{i}x^i(bx - xc)x^{k-i} = 0$, we have

$$\begin{aligned} 0 &= \big(a[bx - xc, x]_k\big)v = \left(a\sum_{i=0}^{k}(-1)^i\binom{k}{i}x^i(bx - xc)x^{k-i}\right)v \\ &= \left(a\sum_{i=0}^{k}(-1)^i\binom{k}{i}x^i(b - c)x^{k-i}\right)xv = \big(a[b - c, x]_k x\big)v \end{aligned}$$

for all $v \in V$. Since V is faithful, we have

$$a[b - c, x]_k x = a[(b - c)x, x]_k = 0$$

for all $x \in R$. Since $bx - xc = (b - c)x + cx - xc$, we have

$$\begin{aligned} 0 &= a[bx - cx, x]_k = a[(b - c)x + cx - xc, x]_k \\ &= a[(b - c)x, x]_k + a[cx - xc, x]_k = a[c, x]_{k+1} \end{aligned}$$

and hence $a[c, x]_{k+1} = 0$ for all $x \in R$. By a result of Shiue [16], we can conclude that $a = 0$ or $c \in Z$, which is a contradiction. So there exists $v_0 \in V$ such that v_0 and cv_0 are C-independent.

Assume $\dim V_C \geq 3$. Choose $w \in V$ such that w, v_0 and cv_0 are C-independent. By the density of R there exists $x \in R$ such that

$$xv_0 = 0, xcv_0 = w, xw = w$$

and

$$\begin{aligned} a[bx - xc, x]_k v_0 &= a\sum_{i=0}^{k}(-1)^i\binom{k}{i}x^i(bx - xc)x^{k-i}v_0 \\ &= (-1)^{k+1}ax^{k+1}cv_0 = (-1)^{k+1}aw. \end{aligned}$$

Hence $aw = 0$. Since $w + v_0$ is also C-independent of v_0 and cv_0, we have $a(w + v_0) = 0$. Similarly $a(w + cv_0) = 0$. So $av_0 = 0$ and $acv_0 = 0$. Therefore $aV = 0$ and hence $a = 0$, a contradiction.

Now we may assume $\dim V_C = 2$. In this case, v_0 and cv_0 form a basis for V_C. If $w \notin v_0C$, then $w = v_0\lambda + cv_0\mu$, where $\mu \neq 0$. By the density of R, there exists $x \in R$ such that $xv_0 = 0$ and $xcv_0 = w$. This implies that $xw = x(v_0\lambda + cv_0\mu) = (xcv_0)\mu = w\mu$ and

$$\begin{aligned} 0 &= a[bx - xc, x]_k v_0 = a\left(\sum_{i=0}^{k}(-1)^i\binom{k}{i}x^i(bx - xc)x^{k-i}v_0\right) \\ &= a(-1)^{k+1}x^{k+1}cv_0 = (-1)^{k+1}ax^k w = (-1)^{k+1}aw\mu^k \end{aligned}$$

So $aw = 0$. Replacing w by $w + v_0$, we also have $a(w + v_0) = av_0 = 0$. Since w and v_0 are C-independent and $\dim V_C = 2$, we have $aV = 0$ and hence $a = 0$, a contradiction. This last contradiction shows $c \in Z$.

Since $c \in Z$, we have $a[bx - xc, x]_k = a[bx, x]_k$ and hence

$$a[bx - xc, x]_k = a[b, x]_k x = 0 \tag{1}$$

for all $x \in R$. If $b \notin Z$, then

$$h(x) = a[b, x]_k x = a\sum_{i=0}^{k}(-1)^i\binom{k}{i}x^i b x^{k-i+1} = 0$$

is a nontrivial GPI of R. Again by the same argument as we did in first paragraph we can conclude that R is isomorphic to a dense subring of the ring of linear transformations of a vector space V over the field C, containing nonzero linear transformation of finite rank. Also, $\dim V_C \geq 2$.

Again, if bv and v are C-dependent for all $v \in V$, then as before, there exists $\lambda \in C$ such that $bv = v\lambda$ for all $v \in V$. This implies

$$\begin{aligned} [b, x]_k v &= \left(\sum_{i=0}^{k}(-1)^i\binom{k}{i}x^i b x^{k-i}\right)v \\ &= \sum_{i=0}^{k}(-1)^i\binom{k}{i}x^i x^{k-i}v\lambda \\ &= \left(\sum_{i=0}^{k}(-1)^i\binom{k}{i}\right)x^k v\lambda \\ &= 0 \end{aligned}$$

for all $v \in V$. Since V is faithful, we have $[b, x]_k = 0$ for all $x \in R$ and hence $b \in Z$ by [13], which is a contradiction. So we may assume that there exists $v_0 \in V$ such that bv_0 and v_0 are C-independent. By the density of R, there exists $x \in R$ such that $xv_0 = v_0$ and $xbv_0 = 0$. By (1) we have

$$\begin{aligned} 0 \;=\; a[b,x]_k x v_0 &= a\sum_{i=0}^{k}(-1)^i\binom{k}{i}x^i b x^{k-i+1}v_0 \\ &= a\sum_{i=0}^{k}(-1)^i\binom{k}{i}x^i b v_0 = abv_0 \end{aligned}$$

We also have $x \in R$ such that $xv_0 = v_0$ and $xbv_0 = v_0$. Again by (1) we get

$$\begin{aligned} 0 \;=\; a[b,x]_k x v_0 &= a\sum_{i=0}^{k}(-1)^i\binom{k}{i}x^i b x^{k-i+1}v_0 \\ &= a\sum_{i=0}^{k}(-1)^i\binom{k}{i}x^i b v_0 = abv_0 + a\sum_{i=1}^{k}(-1)^i\binom{k}{i}x^i b v_0 \\ &= a\sum_{i=1}^{k}(-1)^i\binom{k}{i}v_0 = -av_0 + a\left(\sum_{i=0}^{k}(-1)^i\binom{k}{i}\right)v_0 \\ &= -av_0 \end{aligned}$$

Now if $\dim V_C = 2$, then v_0 and bv_0 form a basis for V. Since $av_0 = 0$ and $abv_0 = 0$, we have $aV = 0$ and hence $a = 0$, a contradiction.

So we may assume that $\dim V_C \geq 3$. In this case, let $w \in V$ be C-independent of v_0 and bv_0. Again, by the density of R, there exists $x \in R$ such that $xv_0 = v_0$, $xbv_0 = w$ and $xw = w$. From (1) we get

$$\begin{aligned} 0 \;=\; a[b,x]_k x v_0 &= a\sum_{i=0}^{k}(-1)^i\binom{k}{i}x^i b x^{k-i+1}v_0 \\ &= a\sum_{i=0}^{k}(-1)^i\binom{k}{i}x^i b v_0 = abv_0 + a\sum_{i=1}^{k}(-1)^i\binom{k}{i}x^i b v_0 \\ &= a\sum_{i=1}^{k}(-1)^i\binom{k}{i}w = -aw \end{aligned}$$

Therefore $aV = 0$ and this implies $a = 0$, a contradiction. Hence $b \in Z$ and the proof is complete. ∎

Lemma 2. *Let R be a dense subring of the ring of linear transformations of a vector space V over a division ring D, where $\dim V_D \geq 2$ and let R contain nonzero linear transformations of finite rank. Let β be an automorphism of R. Suppose that $a, b, c \in R$ and $f(x) = bx - \beta(x)c$ satisfy $a[f(x), x]_k = 0$ for all $x \in R$, where k is a fixed positive integer. Then $a = 0$ or $b - c \in Z$ and $f(x) = (b-c)x$ for all $x \in R$ except $\dim V_D = 2$ and $D = GF(2)$, the Galois field of two elements.*

Proof. We will adopt the proof of Lemma 2 in [7] with some modification. We assume that $a \neq 0$ and proceed to show that $b-c \in Z$ and $f(x) = (b-c)x$ for all $x \in R$ except $\dim V_D = 2$ and $D = GF(2)$. Since R is a primitive ring with nonzero socle, by a result in [9, p. 79], there exists a semi-linear automorphism $T \in \mathrm{End}(V)$ such that $\beta(x) = TxT^{-1}$ for all $x \in R$. Hence $a[bx - \beta(x)c, x]_k = a[bx - TxT^{-1}c, x]_k = 0$ for all $x \in R$.

We claim that there exists $v_0 \in V$ such that v_0 and $T^{-1}cv_0$ are D-independent. If not, then v and $T^{-1}cv$ are D-dependent for all $v \in V$. As before there exists $\lambda \in D$ such that $T^{-1}cv = v\lambda$ for all $v \in V$. Then

$$\begin{aligned} f(x)v &= (bx - \beta(x)c)v = (bx - TxT^{-1}c)v \\ &= bxv - TxT^{-1}cv = bxv - T(xv\lambda) \\ &= bxv - T((xv)\lambda) = bxv - T(T^{-1}c)(xv) \\ &= bxv - cxv = (b-c)xv \end{aligned}$$

for all $x \in R$ and for all $v \in V$. Hence $(f(x) - (b-c)x)V = 0$ for all $x \in R$. Since V is faithful, we have $f(x) = (b-c)x$ for all $x \in R$ and therefore

$$a[(b-c)x, x]_k = 0 \tag{2}$$

for all $x \in R$. By (2) and Lemma 1, it follows that $b, c \in Z$. If $c = 0$, then we are done. So we may assume $c \neq 0$.

Since $f(x) = bx - \beta(x)c = (b-c)x + c(x - \beta(x))$, by the hypothesis and (2), we have

$$\begin{aligned} 0 &= a[f(x), x]_k = a[(b-c)x + c(x - \beta(x)), x]_k \\ &= a[(b-c)x, x]_k + a[c(x - \beta(x)), x]_k \\ &= ca[x - \beta(x), x]_k \end{aligned}$$

and hence $a[x - \beta(x), x]_k = 0$ for all $x \in R$. By the Main Theorem in [7] and assumption, we have $x - \beta(x) = 0$ for all $x \in R$ except $\dim V_D = 2$ and $D = GF(2)$ and hence $f(x) = (b-c)x$ for all $x \in R$ except $\dim V_D = 2$ and $D = GF(2)$.

So we may assume that v_0 and $T^{-1}cv_0$ are D-independent for some $v_0 \in V$. First assume $\dim V_D \geq 3$. Choose $w \in V$ such that w, v_0 and $T^{-1}cv_0$ are D-independent. By the density of R, there exists $x \in R$ such that

$$xv_0 = 0, xT^{-1}cv_0 = T^{-1}w, xw = w$$

This implies that

$$\begin{aligned} 0 &= a[bx - TxT^{-1}c, x]_k v_0 = a\sum_{i=0}^{k}(-1)^i \binom{k}{i} x^i(bx - TxT^{-1}c)x^{k-i}v_0 \\ &= (-1)^{k+1}ax^kTxT^{-1}cv_0 = (-1)^{k+1}ax^kw = (-1)^{k+1}aw \end{aligned}$$

and so $aw = 0$. Since v_0+w is also D-independent of v_0 and $T^{-1}cv_0$, we also have $a(v_0 + w) = 0$. Similarly, $a(T^{-1}cv_0 + w) = 0$. Therefore $av_0 = aT^{-1}cv_0 = 0$. But then $aV = 0$ and $a = 0$, a contradiction.

Second, assume $\dim V_D = 2$. Then v_0 and $T^{-1}cv_0$ form a basis for V_D. We claim that there exists $w \in V$ such that $w \notin v_0D$ and $Tw \notin v_0D$. Suppose on the contrary, for each $w \in V$ we have either $w \in v_0D$ or $w \in (T^{-1}v_0)D$. Then $V = v_0D \cup (T^{-1}v_0)D$. As a vector space cannot be the union of two proper subspaces, we must have $\dim V_D = 1$, a contradiction. For such w, $w \notin v_0D$ and $w \notin (T^{-1}v_0)D$, we write $w = v_0\lambda + (T^{-1}v_0)\mu$ and $Tw = v_0\sigma + (T^{-1}cv_0)\tau$, where $\lambda, \mu, \sigma, \tau \in D$ and $\mu, \tau \neq 0$. By the density of R, there exists $x \in R$ such that $xv_0 = 0$, $xT^{-1}cv_0 = w$. This implies that $xw = x(v_0\lambda + (T^{-1}cv_0)\mu) = x(T^{-1}cv_0)\mu = w\mu$ and $xTw = x(v_0\sigma + (T^{-1}cv_0)\tau) = w\tau$. Therefore,

$$\begin{aligned} 0 &= a[bx - TxT^{-1}c, x]_k v_0 = a\sum_{i=0}^{k}(-1)^i\binom{k}{i}x^i(bx - TxT^{-1}c)x^{k-i}v_0 \\ &= (-1)^{k+1}ax^kTxT^{-1}cv_0 = (-1)^{k+1}ax^kTw = (-1)^{k+1}ax^{k-1}w\tau \\ &= (-1)^{k+1}aw\mu^{k-1}\tau \end{aligned}$$

and so $aw = 0$. If there exists a nonzero $\lambda \in D$ such that $T(v_0\lambda + w) \notin v_0D$, then replacing w by $v_0\lambda + w$, we have $0 = a(v_0\lambda + w) = av_0\lambda$ and so $av_0 = 0$. Since w and v_0 are D-independent and $\dim V_D = 2$, we have $aV = 0$, again a contradiction. Thus $T(v_0\lambda + w) \in v_0D$ for all nonzero $\lambda \in D$. If $|D| > 2$, then we can choose two nonzero elements of D, say λ_1, λ_2 with $\lambda_1 \neq \lambda_2$. Then $T(v_0(\lambda_1 - \lambda_2)) = T(v_0\lambda_1 + w) - T(v_0\lambda_2 + w) \in v_0D$. Using semi-linearity of T, we have $T(v_0) \in v_0D$ and then $T(w) \in v_0D$, a contradiction. The proof is complete. ∎

Now we are ready to prove our Main Theorem.

Proof of Main Theorem. By [2, Lemma 2], we can write $f(x) = sx + \delta(x)$ for all $x \in R$, where $s = f(1) \in {}_{\mathcal{F}}R$ and δ is a β-derivation of R. By [3, Theorem 2],

$$a[sx + \delta(x), x]_k = 0 \tag{3}$$

for all $x \in {}_{\mathcal{F}}R$. Assume $a \neq 0$. If $\delta = 0$, then $f(x) = sx$ and $a[sx, x]_k = 0$ for all $x \in {}_{\mathcal{F}}R$. By Lemma 1, $s \in C$ and we are done. So we may assume $\delta \neq 0$. If δ is X-outer, then by [6, Theorem 1], we have $a[sx+y, x]_k = 0$ for all $x, y \in {}_{\mathcal{F}}R$. Pick $t \in {}_{\mathcal{F}}R\backslash C$ and replace y by $-xt$. Then we have $a[sx - xt, x]_k = 0$ for all $x \in {}_{\mathcal{F}}R$, which is contrary to Lemma 1. Hence we may assume that δ is X-inner and write $\delta(x) = cx - \beta(x)c$, where $c \in Q$. Suppose that β is X-inner. Thus there exists an invertible element $u \in Q$ such that $\beta(x) = uxu^{-1}$ for all $x \in R$. We rewrite (3) as

$$a[(s+c)x - uxu^{-1}c, x]_k = 0$$

for all $x \in R$ and also for all $x \in {}_{\mathcal{F}}R$. If $u^{-1}c \in C$, then $\delta(x) = cx - uxu^{-1}c = cx - u(u^{-1}c)x = cx - cx = 0$ for all $x \in R$, which is not the case. So we may assume that $u^{-1}c \notin C$. With this, we can see easily that

$$\begin{aligned} g(x) &= a[(s+c)x - uxu^{-1}c, x]_k \\ &= a\sum_{i=0}^{k-1}(-1)^i\binom{k}{i}x^i((s+c)x - uxu^{-1}c)x^{k-i} \\ &\quad +(-1)^k ax^k(s+c)x + (-1)^{k+1}ax^kuxu^{-1}c \\ &= 0 \end{aligned}$$

is a nontrivial GPI of R. By [3], $g(x) = 0$ is also a GPI of ${}_{\mathcal{F}}R$. By the same argument as we did in Lemma 1, we may assume that R is centrally closed and the field C is either finite or algebraically closed. By Martindale's theorem [15], R is a primitive ring having nonzero socle with the field C as its associated divising ring. By [9, p. 75] R is isomorphic to a dense subring of the ring of linear transformations of a vector space V over C, containing nonzero linear transformations of finite rank. Since R is not commutative, we may assume $\dim V_C \geq 2$. By Lemma 2, we are done in this case.

So we may assume that β is X-outer. Since $a \neq 0$ and $c \neq 0$, R is a GPI-ring by [4] and ${}_{\mathcal{F}}R$ is also GPI-ring by [3]. By Martindale's theorem [15], ${}_{\mathcal{F}}R$ is a primitive ring having nonzero socle and its associated division ring D is finite dimensional over C. Hence ${}_{\mathcal{F}}R$ is isomorphic to a dense subring of the ring of linear transformations of a vector space V over D, containing nonzero linear transformations of finite rank. If $\dim V_D \geq 2$, then we are done by Lemma 2. Hence we may assume that $\dim V_D = 1$, that is ${}_{\mathcal{F}}R \cong D$. If C is finite, then $\dim D_C < \infty$ implies that D is also finite. Thus D is a field by Wedderburn's theorem [9, p. 183] and so ${}_{\mathcal{F}}R$ is commutative. In particular, R is commutative, a contradiction. Hence from now on we assume that C is infinite and ${}_{\mathcal{F}}R$ is a division ring. By the assumption $a \neq 0$, we have $[(s+c)x - \beta(x)c, x]_k = 0$ for all $x \in {}_{\mathcal{F}}R$.

Suppose that β is not Frobenius. Then by [5], $[(s+c)x - yc, x]_k = 0$ for all $x \in {}_{\mathcal{F}}R$. Putting $y = x$, we have $[(s+c)x - xc, x]_k = 0$ for all $x \in {}_{\mathcal{F}}R$. By Lemma 1, $c, s \in C$ and $[cx - \beta(x)c, x]_k = 0$ for all $x \in {}_{\mathcal{F}}R$. But then $cx - \beta(x)c = 0$ for all $x \in {}_{\mathcal{F}}R$ by the Main Theorem in [7], which is a contradiction.

Finally, we assume that β is Frobenius. Then $\operatorname{char} {}_{\mathcal{F}}R = p > 0$ and $\beta(\lambda) = \lambda^{p^n}$ for all $\lambda \in C$, where n is some fixed integer. Since β is X-outer, $n \neq 0$. Replacing

x by $x+\lambda$, where $0 \neq \lambda \in C$, we have from (3) that

$$\begin{aligned} 0 &= \big[(s+c)(x+\lambda) - \beta(x+\lambda)c, x+\lambda\big]_k \\ &= \big[(s+c)(x+\lambda) - \big(\beta(x) + \lambda^{p^n}\big)c, x\big]_k \\ &= \big[(s+c)x - \beta(x)c, x\big]_k + \big[(s+c)\lambda - c\lambda^{p^n}, x\big]_k \\ &= \big[(s+c)\lambda - c\lambda^{p^n}, x\big]_k \end{aligned}$$

for all $x \in {}_{\mathcal{F}}R$ and hence $(s+c)\lambda - c\lambda^{p^n} \in C$ by [13]. Since β is X-outer, there exists $t \in C$ such that $t \neq t^{p^n}$. Let $\lambda_1 = \lambda t$. Then we have $(s+c)\lambda - c\lambda^{p^n} = \tau \in C$ and $(s+c)\lambda_1 - c\lambda_1^{p^n} = \tau_1 \in C$. Solving these two equations, we have $s+c \in C$ and $c \in C$ and hence $s \in C$. Therefore $0 = [sx + cx - \beta(x)c, x]_k = c[\beta(x), x]_k$. Since $c \neq 0$ and $c[\beta(x) - x, x]_k = 0$ for all $x \in {}_{\mathcal{F}}R$, by the Main Theorem in [7], $\beta(x) - x = 0$ for all $x \in {}_{\mathcal{F}}R$, which is a contradiction. The proof is now complete. ■

The following example shows that the exceptional case does exist.

Example. Let $R = M_2(GF(2))$, $a = e_{11} + e_{12}$, $b = e_{21}$ and $c = e_{21} + e_{22}$. Let $\beta(x) = gxg^{-1}$, where $g = e_{12} + e_{21}$. Let $f(x) = bx - \beta(x)c$ for all $x \in R$. Then by a direct computation we have $a[[f(x), x], x] = 0$ for all $x \in R$.

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