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ASYMPTOTIC BEHAVIOR FOR ALMOST-ORBITS OF ASYMPTOTICALLY NONEXPANSIVE TYPE MAPPINGS IN A METRIC SPACE

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Abstract. Let (M, ρ) be a metric space and τ a Hausdorff topology on M such that $\{M, \tau\}$ is sequentially compact. Let T be a ρ-asymptotically nonexpansive type self-mapping of M and $u = \{x_n\}$ a ρ-bounded almost-orbit of T. We study the τ-convergence of u in M when the triplet $\{M, \rho, \tau\}$ satisfies various types of τ-Opial conditions. Our results, which also hold for the continuous case of one-parameter semigroups, extend and unify many previously known results [1, 5, 9, 10, 12-19, 21, 22], and answers affirmatively an open question of S. Reich [20, p.550] in the very general context of a metric space.

1. Introduction

Let (M, ρ) be a metric space. A mapping $T : M \to M$ is called nonexpansive if $\rho(Tx, Ty) \leq \rho(x, y)$ for all $x, y \in M$. When M is a nonempty bounded closed and convex subset of a Hilbert space H, the first weak convergence theorem for the sequence of iterates $\{T^n x\}$ was proved by Z. Opial [19], namely that for each $x \in M$, $\{T^n x\}$ converges weakly to a fixed point of T, if and only if T is weakly asymptotically regular, i.e., $w - \lim_{n\to\infty}(T^{n+1}x - T^n x) = 0$ for each $x \in M$.

This result was extensively studied and extended in many directions, e.g. to one-parameter nonexpansive semigroups in H [21], and in a Banach space X [5, 12, 18], nonexpansive and almost nonexpansive sequences and curves in H [2-4, and the references therein], asymptotically nonexpansive mappings [1, 6, 7, 16, 22], more general semigroups of nonexpansive and asymptotically nonexpansive (resp.

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type) mappings [9, 10, 13, 14], as well as to Banach and metric spaces satisfying various types of Opial conditions via demiclosedness principles [1, 15, 17].

In this paper, we consider a Hausdorff topology τ on the metric space (M, ρ) such that $\{M, \tau\}$ is sequentially compact, and we study the τ-conve-rgence of a ρ-bounded almost-orbit $u = \{x_n\}$ of a ρ-asymptotically nonexpansive type self-mapping T of M when the triplet $\{M, \rho, \tau\}$ satisfies various types of τ-Opial conditions; see section 2 for appropriate definitions. In addition to the previous results, our results extend recent results of G. Li [14] and J. K. Kim and G. Li [9, 15] from Banach space to metric space, and from nonexpansive to asymptotically nonexpansive type mappings. We note that our results are new even in a Banach space X, since compared to [14], no other requirement than the appropriate Opial condition is assumed for the norm of X. Moreover, since in our case the τ-limit of u is not necessarily a fixed point of T, a new method of proof is required by introducing the notion of an asymptotic almost-orbit for T. Finally, our results, which hold for one-parameter semigroups too, answer affirmatively an open question of S. Reich [20, p.550] even in the very general context of a metric space and an asymptotically nonexpansive type semigroup.

2. Preliminaries

Throughout the paper (M, ρ) is a metric space and τ is a Hausdorff topology on M. Asymptotically nonexpansive type mappings were introduced by W. A. Kirk [11]. A self-mapping T of M is said to be of ρ-asymptotically nonexpansive type, if for each $x \in M$, we have:

$$\limsup_{n\to\infty} \sup_{y\in M} [\rho(T^n y, T^n x) - \rho(y, x)] \leq 0,$$

i.e., $\rho(T^n y, T^n x) \leq \rho(y, x) + \epsilon(n, x)$ for all $x, y \in M$ and $n \geq 0$, where $\epsilon \geq 0$ and for each $x \in M$, $\lim_{n\to\infty} \epsilon(n, x) = 0$.

Similarly, a family $\{S(t) : t \geq 0\}$ of self-mappings of M is called an asymptotically nonexpansive type semigroup on M if the following conditions are satisfied:

(1) $S(s + t)x = S(s)S(t)x$ for all $s, t \geq 0$ and $x \in M$.

(2) For each $x \in M$, $\limsup_{t\to\infty} \sup_{y\in M} [\rho(S(t)y, S(t)x) - \rho(y, x)] \leq 0$.

Let F denote the fixed point set of T or the common fixed point set of the semigroup $\{S(t) : t \geq 0\}$.

A sequence $u = \{x_n\}$ is called an almost-orbit of T if

$$\lim_{n\to\infty} \left[\sup_{m\geq 0} \rho(x_{n+m}, T^m x_n) \right] = 0.$$

Definition 2.1. $u = \{x_n\}$ is called an asymptotic almost-orbit of T if

$$\lim_{n\to\infty}\left[\limsup_{m\to\infty}\rho(x_{n+m}, T^m x_n)\right] = 0.$$

Obviously every orbit of T is an almost-orbit for T, which itself is an asymptotic almost-orbit of T.

Definition 2.2. A sequence $\{x_n\}$ in M is said to be τ-asymptotically regular if for each $x \in M$ and each neighborhood V of x containing an infinite subsequence $\{x_{n_k}\}$ of $\{x_n\}$, there exists an integer $k_0\ (x, V)$ such that $x_{n_k+1} \in V$ for all $k \geq k_0$. $\{x_n\}$ is said to be ρ-asymptotically regular if $\lim_{n\to\infty}\rho(x_{n+1}, x_n) = 0$.

Similar definitions can be given in an obvious manner for a semigroup and for a function $u : R^+ \to M$ as well as for nets.

Now we define various types of τ-Opial conditions; see [8, 10, 17, 19].

Definition 2.3. The triplet $\{M, \rho, \tau\}$ is said to satisfy the τ-Opial condition if for each ρ-bounded net $\{x_\alpha : \alpha \in A\}$ in M that τ-converges to some $x \in M$, we have:

$$\limsup_{\alpha\in A}\rho(x_\alpha, x) < \limsup_{\alpha\in A}\rho(x_\alpha, y),$$

for all $y \neq x$, where A is a directed set.

It is said to satisfy the locally uniform τ-Opial condition if for each ρ-bounded net $\{x_\alpha\}$ in M that τ-converges to some $x \in M$, and every $\epsilon > 0$, there exists $\eta(\{x_\alpha\}, \epsilon) > 0$ such that for each $y \in M$ with $\rho(x, y) \geq \epsilon$, we have:

$$\limsup_{\alpha}\rho(x_\alpha, x) + \eta \leq \limsup_{\alpha}\rho(x_\alpha, y).$$

It is said to satisfy the uniform τ-Opial condition if for every $R > 0$ and every $\epsilon > 0$, there exists $\eta(R, \epsilon) > 0$ such that for every net $\{x_\alpha\}$ in M that τ-converges to some $x \in M$ with $\limsup_\alpha \rho(x_\alpha, x) \leq R$, and for every $y \in M$ with $\rho(x, y) \geq \epsilon$, we have:

$$\limsup_{\alpha}\rho(x_\alpha, x) + \eta \leq \limsup_{\alpha}\rho(x_\alpha, y).$$

It is clear that the uniform τ-Opial condition implies the locally uniform τ-Opial condition, which in turn implies the τ-Opial condition. It is also clear that in Definition 2.3 all the $\limsup_{\alpha\in A}$ can be replaced by $\liminf_{\alpha\in A}$.

The following lemma which gives an equivalent condition to the locally uniform τ-Opial condition is well-known; see [10, 17].

Lemma 2.4. *The triplet $\{M, \rho, \tau\}$ satisfies the locally uniform τ-Opial condition if and only if for every ρ-bounded net $\{x_\alpha\}$ in M that τ-converges to* *some* $x \in M$*, and for every net* $\{y_\beta\}$ *in* M *that satisfies*

$$\limsup_{\beta}\left[\limsup_{\alpha}\rho(x_\alpha, y_\beta)\right] \le \limsup_{\alpha}\rho(x_\alpha, x),$$

we have $\lim_\beta \rho(y_\beta, x) = 0.$

3. Asymptotic Behavior

In this section, unless otherwise stated, T is a ρ-asymptotically nonexpansive type self-mapping of M and $u = \{x_n\}$ is an almost-orbit of T. We study the τ-convergence of u in M. We denote the τ-convergence of a sequence $\{x_n\}$ to $x \in M$ by $\tau - \lim_{n\to\infty} x_n = x$ or by $x_n \xrightarrow{\tau} x$. $\omega_\tau(u)$ denotes the $\tau - \omega$-limit set of u, i.e., $\omega_\tau(u) = \{x \in M; \exists x_{n_k} \xrightarrow{\tau} x\}$.

$\omega_\tau(u) \neq \emptyset$ if $\{M, \tau\}$ is sequentially compact. Let $L(u) := \{p \in M;\ \lim_{n\to\infty} \rho(x_n, p)$ exists$\}$ and

$$AF = AF(T) := \{p \in M;\ \lim_{n\to\infty}\rho(T^n p, p) = 0.\}$$

Lemma 3.1. *If* $u = \{x_n\}$ *and* $v = \{y_n\}$ *are asymptotic almost-orbits of* T*, then* $\lim_{n\to\infty}\rho(x_n, y_n)$ *exists. In particular* $F \subset AF \subset L(u)$*.*

Proof. Let $a_n = \limsup_{m\to\infty}\rho(x_{n+m},\ T^m x_n)$ and $b_n = \limsup_{m\to\infty}\rho(y_{n+m},\ T^m y_n)$. Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$, and we have:

$$\begin{aligned}\rho(x_{n+k}, y_{n+k}) &\le \rho(x_{n+k}, T^k x_n) + \rho(T^k x_n, T^k y_n) + \rho(T^k y_n, y_{n+k})\\ &\le \rho(x_{n+k}, T^k x_n) + \rho(T^k y_n, y_{n+k}) + \rho(x_n, y_n) + \epsilon(k, x_n).\end{aligned}$$

Keeping n fixed and taking the limsup on k, we get:

$$\limsup_{k\to\infty}\rho(x_k, y_k) \le a_n + b_n + \rho(x_n, y_n).$$

Now taking the liminf on n we get:

$$\limsup_{k\to\infty}\rho(x_k, y_k) \le \liminf_{n\to\infty}\rho(x_n, y_n),$$

which implies that $\lim_{n\to\infty}\rho(x_n, y_n)$ exists.

Now to complete the proof of the lemma, the inclusion $F \subset AF$ is obvious, and we have $AF \subset L(u)$ since every element of AF is clearly an asymptotic almost-orbit of T. ∎

Lemma 3.2. *Assume $\{M,\tau\}$ is sequentially compact and $\{M,\rho,\tau\}$ satisfies the τ-Opial condition. Then an almost-orbit $u=\{x_n\}$ of T is τ-convergent in M if $\omega_\tau(u)\subset L(u)$.*

Proof. Since $\{M,\tau\}$ is sequentially compact, $\omega_\tau(u)\neq\emptyset$ and hence $L(u)\neq\emptyset$; therefore u is ρ-bounded. Assume $x_{n_k}\xrightarrow{\tau}p$ and $x_{m_l}\xrightarrow{\tau}q$. Then by assumption, $p,q\in L(u)$. If $p\neq q$, by using the τ-Opial condition we have:

$$\begin{aligned}\lim_{n\to\infty}\rho(x_n,p)&=\limsup_{k\to\infty}\rho(x_{n_k},p)<\limsup_{k\to\infty}\rho(x_{n_k},q)\\&=\lim_{n\to\infty}\rho(x_n,q)=\limsup_{l\to\infty}\rho(x_{m_l},q)\\&<\limsup_{l\to\infty}\rho(x_{m_l},p)=\lim_{n\to\infty}\rho(x_n,p)\end{aligned}$$

which is a contradiction. Therefore we must have $p=q$ which implies that $\omega_\tau(u)$ is a singleton. Since $\{M,\tau\}$ is sequentially compact, this implies the τ-convergence of $u=\{x_n\}$ in M. ∎

The following proposition plays a crucial role in the proof of our main result.

Proposition 3.3. *Assume $\{M,\tau\}$ is sequentially compact and $u=\{x_n\}$ is a ρ-bounded and τ-asymptotically regular almost-orbit of T. Then $\omega_\tau(u)\subset AF$ if either one of the following (i) or (ii) holds :*

(*i*) *$\{M,\rho,\tau\}$ satisfies the uniform τ-Opial condition.*

(*ii*) *$\{M,\rho,\tau\}$ satisfies the locally uniform τ-Opial condition and u is moreover ρ-asymptotically regular.*

If T is ρ-nonexpansive, then we even have $\omega_\tau(u)\subset F$ if $\{M,\rho,\tau\}$ satisfies the τ-Opial condition.

Proof. We know $\omega_\tau(u)\neq\emptyset$. Let $p\in\omega_\tau(u)$ and $x_{n_k}\xrightarrow{\tau}p$; the τ-asymptotic regularity of u implies that $x_{n_k+m}\xrightarrow{\tau}p$ for each $m\geq1$.

Let $a_n=\sup_{m\geq0}\rho(x_{n+m},T^mx_n)$. Then $\lim_{n\to\infty}a_n=0$. Let $c_m=\limsup_{k\to\infty}\rho(x_{n_k+m},p)$ and $c=\inf_{m\geq0}c_m$. First assume that $\{M,\rho,\tau\}$ satisfies the τ-Opial condition. Then we have:

$$\begin{aligned}c_{m+l}&=\limsup_{k\to\infty}\rho(x_{n_k+m+l},p)\\&\leq\limsup_{k\to\infty}\rho(x_{n_k+m+l},T^mp)\\&\leq\limsup_{k\to\infty}\rho(x_{n_k+m+l},T^mx_{n_k+l})+\limsup_{k\to\infty}\rho(T^mx_{n_k+l},T^mp)\end{aligned}$$

$$\leq \limsup_{k\to\infty} a_{n_k+l} + \limsup_{k\to\infty} \rho(x_{n_k+l}, p) + \epsilon(m,p)$$
$$= c_l + \epsilon(m,p).$$

Keeping l fixed and letting $m \to \infty$, we get $\limsup_{n\to\infty} c_n \leq c_l$ for all $l \geq 0$. This implies that $\lim_{n\to\infty} c_n = \inf_{l\geq 0} c_l = c$.

Now let $\{\epsilon_n\}$ be an arbitrary sequence of positive numbers tending to zero (e.g., $\epsilon_n = \frac{1}{n}$), and let $\{O_\gamma : \gamma \in \Gamma\}$ be the family of all τ-open neighborhoods of p. Let $m \geq 1$ fixed. For each $l \geq 1$ we choose n_l so that $c_{n_l} \leq c + \epsilon_l$ and $a_n \leq \epsilon_l$ for all $n \geq n_l$.

Now we choose k_l so that $\rho(x_{n_i+n_l}, p) \leq c + 2\epsilon_l$ and $\rho(x_{n_i+n_l+m}, p) \geq c - \epsilon_l$ for all $i \geq k_l$.

Now for each τ-neighborhood O_γ of p we choose an integer $k_\gamma(m, l, O_\gamma) \geq k_l$ so that $x_{n_k+n_l+m} \in O_\gamma$ for all $k \geq k_\gamma$. This is possible, since for $m, l \geq 1$ fixed, we have $x_{n_k+n_l+m} \xrightarrow{\tau} p$. We now consider the set $I := N \times \Gamma$ directed by the relation: $(n_1, \gamma_1) \leq (n_2, \gamma_2)$ if and only if $n_1 \leq n_2$ and $O_{\gamma_2} \subset O_{\gamma_1}$. Then from our construction above, it is clear that for each $m \geq 1$ fixed, we have $\tau - \lim_{(l,\gamma)\in I} x_{n_{k_\gamma}+n_l+m} = p$ and for each $m, l \geq 1$ and $\gamma \in \Gamma$ we have the following inequalities:

$$\begin{aligned}
\rho(x_{n_{k_\gamma}+n_l+m}, T^m p) &\leq \rho(x_{n_{k_\gamma}+n_l+m}, T^m x_{n_{k_\gamma}+n_l}) + \rho(T^m x_{n_{k_\gamma}+n_l}, T^m p) \\
&\leq a_{n_{k_\gamma}+n_l} + \rho(x_{n_{k_\gamma}+n_l}, p) + \epsilon(m,p) \\
&\leq \epsilon_l + c + 2\epsilon_l + \epsilon(m,p) \\
&= c + 3\epsilon_l + \epsilon(m,p) \\
&\leq 4\epsilon_l + \rho(x_{n_{k_\gamma}+n_l+m}, p) + \epsilon(m,p).
\end{aligned} \tag{1}$$

First we note that if T is nonexpansive, then $\epsilon(m,p) = 0$ for all $m \geq 1$. Therefore taking $m = 1$ in (1) we deduce that

$$\limsup_{(l,\gamma)\in I} \rho(x_{n_{k_\gamma}+n_l+1}, Tp) \leq \limsup_{(l,\gamma)\in I} \rho(x_{n_{k_\gamma}+n_l+1}, p)$$

which implies by the τ-Opial condition that $Tp = p$, i.e., $p \in F$. Hence $\omega_\tau(u) \subset F$ and the proof to the last assertion of the proposition is now complete.

Assume now that (i) holds. Then for fixed $m \geq 1$, we get from (1) that

$$\limsup_{(l,\gamma)\in I} \rho(x_{n_{k_\gamma}+n_l+m}, T^m p) \leq \limsup_{(l,\gamma)\in I} \rho(x_{n_{k_\gamma}+n_l+m}, p) + \epsilon(m,p).$$

Since $\lim_{m\to\infty} \epsilon(m,p) = 0$, the uniform τ-Opial condition for $\{M, \rho, \tau\}$ implies that $\lim_{m\to\infty} \rho(T^m p, p) = 0$, i.e., $p \in AF$. Hence $\omega_\tau(u) \subset AF$ and the proof of the case (i) is now complete.

Now assume that (ii) holds. By the triangle inequality we have

$$\rho(x_{n_{k_\gamma}+n_l+m}, p) \leq \rho(x_{n_{k_\gamma}+n_l}, p) + \sum_{i=0}^{m-1} \rho(x_{n_{k_\gamma}+n_l+i}, x_{n_{k_\gamma}+n_l+i+1})$$

and

$$\rho(x_{n_{k_\gamma}+n_l+m}, T^m p) \geq \rho(x_{n_{k_\gamma}+n_l}, T^m p) - \sum_{i=0}^{m-1} \rho(x_{n_{k_\gamma}+n_l+i}, x_{n_{k_\gamma}+n_l+i+1}).$$

Hence for fixed $m \geq 1$, we get from (1) and the ρ-asymptotic regularity of u that

$$\limsup_{(l,\gamma)\in I} \rho(x_{n_{k_\gamma}+n_l}, T^m p) \leq \limsup_{(l,\gamma)\in I} \rho(x_{n_{k_\gamma}+n_l}, p) + \epsilon(m,p).$$

Now taking the limsup on m we get

$$\limsup_{m\to\infty} \left[\limsup_{(l,\gamma)\in I} \rho(x_{n_{k_\gamma}+n_l}, T^m p) \right] \leq \limsup_{(l,\gamma)\in I} \rho(x_{n_{k_\gamma}+n_l}, p).$$

Since $\{M, \rho, \tau\}$ satisfies the locally uniform τ-Opial condition, by Lemma 2.4 we conclude that $\lim_{m\to\infty} \rho(T^m p, p) = 0$, i.e., $p \in AF$. Hence $\omega_\tau(u) \subset AF$ and the proof of the proposition is now complete. ∎

Now we can state our main result.

Theorem 3.4. *Assume $\{M, \tau\}$ is sequentially compact and $u = \{x_n\}$ is a ρ-bounded and τ-asymptotically regular almost-orbit of T. Then u is τ-convergent in M if either one of the following (i), (ii) or (iii) holds.*

(i) *T is ρ-nonexpansive and $\{M, \rho, \tau\}$ satisfies the τ-Opial condition.*

(ii) *$\{M, \rho, \tau\}$ satisfies the uniform τ-Opial condition.*

(iii) *$\{M, \rho, \tau\}$ satisfies the locally uniform τ-Opial condition and u is moreover ρ-asymptotically regular.*

In (i) the τ-limit of u belongs to F. In (ii) and (iii) the τ-limit of u belongs to F if either T is ρ-continuous or T^N is ρ-nonexpansive for some $N \geq 1$.

Proof. By Lemma 3.1 we have $F \subset AF \subset L(u)$; by Proposition 3.3 we have $\omega_\tau(u) \subset F \subset L(u)$ in (i), and $\omega_\tau(u) \subset AF \subset L(u)$ in (ii) and (iii). Therefore an application of Lemma 3.2 gives the τ-convergence of u in all these cases. By Proposition 3.3, we know that in (i) the τ-limit of u belongs to F, and in (ii) and (iii) it belongs to AF. If T is ρ-continuous, then clearly $AF = F$, so the result follows in this case.

Now assume that T^N is ρ-nonexpansive for some $N \geq 1$. Then replacing m by N in the inequality (1) in Proposition 3.3 and noting that $\epsilon(N,p)=0$, we conclude by using the τ-Opial condition for $\{M,\rho,\tau\}$ that $T^N p = p$. Hence $T^{kN+1}p = Tp$, $\forall k \geq 0$; since $p \in AF$, i.e., $\lim_{n\to\infty} \rho(T^n p, p) = 0$, this implies by letting $k \to \infty$ that $Tp = p$, i.e., $p \in F$. This completes the proof of the theorem. ■

Remark 3.1. It is clear that every τ-convergent sequence is τ-asymptotically regular.

Now we state the corresponding result for ρ-asymptotically nonexpansive type semigroups on M whose proof can be done along the same lines.

Theorem 3.5. *Assume $\{M,\tau\}$ is sequentially compact, $\{S(t) : t \geq 0\}$ is a ρ-asymptotically nonexpansive type semigroup on M and $u = \{u(t) : t \geq 0\}$ is a ρ-bounded and τ-asymptotically regular almost-orbit of $\{S(t)\}$. Then u is τ-convergent in M as $t \to \infty$ if either one of the following (i), (ii) or (iii) holds.*

- (i) *$\{S(t)\}$ is a ρ-nonexpansive semigroup and $\{M,\rho,\tau\}$ satisfies the τ-Opial condition.*
- (ii) *$\{M,\rho,\tau\}$ satisfies the uniform τ-Opial condition.*
- (iii) *$\{M,\rho,\tau\}$ satisfies the locally uniform τ-Opial condition and u is moreover ρ-asymptotically regular.*

In (i) the τ-limit of u belongs to F. In (ii) and (iii) the τ-limit of u belongs to F if either $S(t)$ is ρ-continuous for each $t \geq 0$ or $S(t)$ is ρ-nonexpansive for some $t > 0$.

Remark 3.2. Theorem 3.5 gives an affirmative answer to an open question of S. Reich [20, p.550].

Remark 3.3. Theorems 3.4 and 3.5 extend recent results of G. Li [14] and J. K. Kim and G. Li [9, 15], and if M is a weakly (resp. weak star) compact subset of a Banach space and τ is the weak (resp. weak star) topology on M, then they extend many previously known results to asymptotically nonexpansive type mappings and semigroups, as mentioned in the introduction.

4. Some Open Problems

Our discussion leaves the following problems open.

(1) In Theorem 3.4 (ii) or (iii) does the conclusion hold if we assume only that $\{M,\rho,\tau\}$ satisfies the locally uniform τ-Opial condition?

(2) Is it possible to extend Theorems 3.4 and 3.5 to nonexpansive (resp. almost nonexpansive) sequences and curves? see [2-4, and the references therein] for appropriate definitions and an affirmative answer in the Hilbert space case. In this case, T is not defined anymore on $\omega_\tau(u)$.

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