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CONVERGENCE THEOREMS OF ITERATIVE ALGORITHMS FOR A FAMILY OF FINITE NONEXPANSIVE MAPPINGS

Jong Soo Jung

Abstract. Let E be a Banach space, C a nonempty closed convex subset of E, $f : C \to C$ a contraction, and $T_i : C \to C$ a nonexpansive mapping with nonempty $F := \bigcap_{i=1}^{N} Fix(T_i)$, where $N \geq 1$ is an integer and $Fix(T_i)$ is the set of fixed points of T_i. Let $\{x_t^n\}$ be the sequence defined by $x_t^n = tf(x_t^n) + (1-t)T_{n+N}T_{n+N-1}\cdots T_{n+1}x_t^n$ $(0 < t < 1)$. First, it is shown that as $t \to 0$, the sequence $\{x_t^n\}$ converges strongly to a solution in F of certain variational inequality provided E is reflexive and has a weakly sequentially continuous duality mapping. Then it is proved that the iterative algorithm $x_{n+1} = \lambda_{n+1}f(x_n) + (1-\lambda_{n+1})T_{n+1}x_n$ $(n \geq 0)$ converges strongly to a solution in F of certain variational inequality in the same Banach space provided the sequence $\{\lambda_n\}$ satisfies certain conditions and the sequence $\{x_n\}$ is weakly asymptotically regular. Applications to the convex feasibility problem are included.

1. INTRODUCTION

Let E be a real Banach space and C a nonempty closed convex subset of E. Recall that a mapping $f : C \to C$ is a *contraction* on C if there exists a constant $k \in (0, 1)$ such that $\|f(x) - f(y)\| \leq k\|x - y\|$ for all $x, y \in C$. We use $\Sigma_C = \{f : f : C \to C \text{ a contraction}\}$ to denote the collection of all contractions on C. Let $T : C \to C$ be a nonexpansive mapping (that is, $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$) and $Fix(T)$ denote the set of fixed points of T; that is, $Fix(T) = \{x \in C : x = Tx\}$.

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We consider the iterative algorithm: for $N \geq 1$, $T_1, T_2, \cdots, T_N$ nonexpansive mappings, $f \in \Sigma_C$ and $\lambda_n \in (0,1)$,

$$x_{n+1} = \lambda_{n+1} f(x_n) + (1 - \lambda_{n+1}) T_{n+1} x_n, \quad n \geq 0, \tag{1.1}$$

where $T_n := T_{n \mod N}$. As a special case of (1.1), the following algorithm

$$z_{n+1} = \lambda_{n+1} u + (1 - \lambda_{n+1}) T_{n+1} z_n, \quad n \geq 0, \tag{1.2}$$

where $u,\ z_0 \in C$ are arbitrary (but fixed), has been investigated by many author: see, for example, Browder [2], Halpern [7], Lions [14], Reich [19], Shioji and Takahashi [20], Wittmann [23], Xu [24] for $N = 1$ and Bauschke [1], Jung [8], Jung et al. [10], Jung and Kim [11], O'Hara et al. [17, 18], Takahashi et al. [22] and Zhou et al. [27] for $N > 1$, respectively. The authors above showed that the sequence $\{z_n\}$ generated by (1.2) converges strongly to a point in the fixed point set $Fix(T)$ for $N = 1$ and to a point in the common fixed point set $\bigcap_{i=1}^N Fix(T_i)$ for $N > 1$ under the following respective conditions in either Hilbert spaces or certain Banach spaces:

(C1) $\lim_{n\to\infty} \lambda_n = 0;$ (Halpern [7])

(C2) $\sum_{n=1}^{\infty} \lambda_n = \infty$ or, equivalently, $\prod_{n=1}^{\infty}(1 - \lambda_n) = 0;$ (Halpern [7])

(C3) $\lim_{n\to\infty} \frac{\lambda_n - \lambda_{n+1}}{\lambda_{n+1}^2} = 0;$ (Lions [14])

(C4) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty;$ (Wittmann [23])

(C5) $\sum_{n=1}^{\infty} |\lambda_{n+N} - \lambda_n| < \infty;$ (Bauschke [1])

(C6) $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+N}} = 1$ or, equivalently, $\lim_{n\to\infty} \frac{\lambda_n - \lambda_{n+N}}{\lambda_{n+N}} = 0.$

(O'Hara et al. []17,18)

In particular, in 2005, Jung et al. [10] considered the perturbed control condition with the necessary conditions (C1) and (C2) on the parameters

$$|\lambda_{n+N} - \lambda_n| \leq \circ(\lambda_{n+N}) + \sigma_n, \quad \sum_{n=1}^{\infty} \sigma_n < \infty \tag{C7}$$

to obtain the strong convergence of the sequence $\{z_n\}$ generated by (1.2) in a uniformly smooth Banach space having a weakly sequentially continuous duality mapping and gave an example which satisfies the conditions (C1), (C2) and (C7), but fails to satisfy the conditions (C5) and (C6). Using the Banach limit techniques and the weak asymptotic regularity on the sequence $\{x_n\}$ together with the conditions (C1) and (C2), Zhou et al. [27] also studied convergence of the sequence $\{z_n\}$ generated by (1.2) in a reflexive Banach space having a weakly sequentially continuous duality mapping and a uniformly Gâteaux differentiable norm together with the assumption that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings.

For $N = 1$, the viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping was proposed by Moudafi [16] in Hilbert space. In 2004, Xu [25] extended Theorem 2.2 of Moudafi [6] for the iterative algorithm (1.1) to a uniformly smooth Banach space using the condition (C1), (C2) and (C4) or (C6) for $N = 1$. Very recently, using the condition (C1), (C2) and (C7), Jung [9] improved the results of Xu [25] to the case of $N > 1$ in a reflexive Banach space E having a weakly sequentially continuous duality mapping and a uniformly Gâteaux differentiable norm together with the assumption that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings.

The main purpose of this work is to remove the assumption of uniformly Gâteaux differentiable norm and the fixed point property (that is, the uniformly smoothness assumption) in the above mentioned results. More precisely, first we show the existence of a solution of certain variational inequality in a reflexive Banach space having a weakly sequentially continuous duality mapping. Then we establish the strong convergence of the sequence $\{x_n\}$ generated by the algorithm (1.1) for finitely many nonexpansive mappings to a solution of certain variational inequality in the same Banach space under the conditions (C1) and (C2) on the parameters $\{\lambda_n\}$ and the weak asymptotic regularity condition on the sequence $\{x_n\}$. Applications to the convex feasibility problem are also investigated. The main results improve and unify the corresponding results of Bauschke [1], Jung [8, 9], Jung et al. [10], Jung and Kim [11] and O'Hara et al. [17, 18], Xu [26] and others.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual. The value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f\rangle$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ (resp., $x_n \rightharpoonup x$, $x_n \overset{*}{\rightharpoonup} x$) will denote strong (resp., weak, weak*) convergence of the sequence $\{x_n\}$ to x.

A Banach space E is said to be *smooth* (and the norm of E is said to be *Gâteaux differentiable*) if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x,\ y$ in its unit sphere $U = \{x \in E : \|x\| = 1\}$. The (*normalized*) *duality mapping* J from E into the family of nonempty (by Hahn-Banach theorem) weak-star compact subsets of its dual E^* is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$$

for each $x \in E$. It is known (cf. [3]) that a Banach space E is smooth if and only if the duality mapping J is single-valued. The duality mapping J is said to be *weakly sequentially continuous* if J is single valued and weak-to-weak* continuous; that is, if $x_n \rightharpoonup x$ in E, $J(x_n) \overset{*}{\rightharpoonup} J(x)$ in E^*.

A Banach space E is said to satisfy *Opial's condition* if, for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$. It is well-known that, if E admits a weakly sequentially continuous duality mapping, then E satisfies Opial's condition.

Let C be a nonempty closed convex subset of E. C is said to have the *fixed point property* for nonexpansive mappings if every nonexpansive mapping of a bounded closed convex subset D of C has a fixed point. Let D be a subset of C. Then a mapping $Q : C \to D$ is said to be *retraction* from C onto D if $Qx = x$ for all $x \in D$. A retraction $Q : C \to D$ is said to be *sunny* if $Q(Qx + t(x - Qx)) = Qx$ for all $t \geq 0$ and $x + t(x - Qx) \in C$. A sunny nonexpansive retraction is a sunny retraction which is also nonexpansive. Sunny nonexpansive retractions are characterized as follows [5, p. 48]: If E is smooth, then $Q : C \to D$ is a sunny nonexpansive retraction if and only if the following inequality holds:

$$\langle x - Qx, J(z - Qx)\rangle \leq 0, \quad x \in C, \quad z \in D. \tag{2.1}$$

We need the following lemmas for the proof of our main results. For these lemmas, we refer to [3, 5, 6, 12, 15].

Lemma 2.1. *Let E be a real Banach space and J the duality mapping. Then, for any given $x,\ y \in E$, we have*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle$$

for all $j(x + y) \in J(x + y)$.

Lemma 2.2. (Demicloseness principle) *Let E be a reflexive Banach space with Opial's condition, C a nonempty closed convex subset of E and $T : C \to E$ a nonexpansive mapping. Then the mapping $I - T$ is demiclosed on C, where I is the identity mapping; that is, $x_n \rightharpoonup x$ in E and $(I - T)x_n \to y$ imply that $x \in C$ and $(I - T)x = y$.*

Lemma 2.3. *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1 - \lambda_n)s_n + \lambda_n\beta_n + \gamma_n, \quad n \geq 0,$$

where $\{\lambda_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *satisfying the condition*:

(i) $\{\lambda_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \lambda_n = \infty$ *or, equivalently,* $\prod_{n=0}^{\infty}(1-\lambda_n) := \lim_{n\to\infty} \prod_{k=0}^{n}(1-\lambda_k) = 0$,

(ii) $\limsup_{n\to\infty} \beta_n \le 0$ *or* $\sum_{n=1}^{\infty} \lambda_n \beta_n < \infty$,

(iii) $\gamma_n \ge 0 \ (n \ge 0),\ \sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4. *If E is a Banach space such that E^* is strictly convex, then E is smooth and any duality mapping is norm-to-weak*-continuous.*

Lemma 2.5. *Let E be a smooth Banach space, C a nonempty closed convex subset of E and $T : C \to C$ a nonexpansive mapping. If J is the duality mapping on E, then*

$$\langle (I-T)(x) - (I-T)(y), J(x-y) \rangle \ge 0, \quad \text{for all} \ \ x,\ y \in C.$$

Let μ be a continuous linear functional on l^∞ and $(a_0, a_1, \cdots) \in l^\infty$. We write $u_n(a_n)$ instead of $\mu((a_0, a_1, \cdots))$. μ is said to be *Banach limit* if μ satisfies $\|\mu\| = \mu_n(1) = 1$ and $u_n(a_{n+1}) = \mu_n(a_n)$ for all $(a_0, a_1, \cdots) \in l^\infty$. If μ is a Banach limit, the following are well-known:

(i) $for\ all\ n \ge 1, a_n \le c_n$ implies $\mu(a_n) \le \mu(c_n)$,

(ii) $\mu(a_{n+N}) = \mu(a_n)$ for any fixed positive integer N,

(iii) $\liminf\limits_{n\to\infty} a_n \le \mu_n(a_n) \le \limsup\limits_{n\to\infty} a_n$ for all $(a_0, a_1, \cdots) \in l^\infty$.

The following lemma was given in [27] as the revision of [20, Proposition 2].

Lemma 2.6. *Let $a \in \mathbb{R}$ be a real number and a sequence $\{a_n\} \in l^\infty$ satisfy the condition $\mu_n(a_n) \le a$ for all Banach limit μ. If $\limsup_{n\to\infty}(a_{n+N} - a_n) \le 0$ for $N \ge 1$, then $\limsup_{n\to\infty} a_n \le a$.*

Finally, the sequence $\{x_n\}$ generated by (1.1) is said to be *weakly asymptotically regular* [27] if for $N \ge 1$,

$$w - \lim_{n\to\infty}(x_{n+N} - x_n) = 0, \quad \text{that is,} \quad x_{n+N} - x_n \rightharpoonup 0$$

and *asymptotically regular* if for $N \ge 1$,

$$\lim_{n\to\infty}(x_{n+N} - x_n) = 0, \quad \text{that is,} \quad x_{n+N} - x_n \to 0,$$

respectively.

3. Main Results

First, we give conditions for the existence of solutions of certain variational inequality.

For any $n \geq 1$, $T_{n+N}T_{n+N-1}\cdots T_{n+1} : C \to C$ is nonexpansive and so, for any $t \in (0,1)$ and $f \in \Sigma_C$, $tf + (1-t)T_{n+N}T_{n+N-1}\cdots T_{n+1} : C \to C$ defines a strict contraction mapping. Thus, by Banach contraction mapping principle, there exists a unique fixed point $x_t^{f,n}$ satisfying

$$x_t^{f,n} = tf(x_t^{f,n}) + (1-t)T_{n+N}T_{n+N-1}\cdots T_{n+1}x_t^{f,n}. \tag{A}$$

For simplicity we will write x_t^n for $x_t^{f,n}$ provided no confusion occurs.

The following result gives conditions under which we solves a variational inequality.

Theorem 3.1. *Let E be a Banach space such that E^* is strictly convex, C a nonempty closed convex subset of E and $T_1, \cdots, T_N$ nonexpansive mappings from C into itself with $F := \bigcap_{i=1}^N Fix(T_i) \neq \emptyset$ and*

$$F = Fix(T_N \cdots T_1) = Fix(T_1 T_N \cdots T_3 T_2) = \cdots = Fix(T_{N-1} \cdots T_1 T_N).$$

Suppose that $\{x_t^n\}$ defined by (A) *converges strongly to a point in F as $t \to 0^+$. If we define $Q : \Sigma_C \to F$ by*

$$Q(f) := \lim_{t \to 0^+} x_t^n, \quad f \in \Sigma_C,$$

then $Q(f)$ solves the variational inequality

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \leq 0, \quad f \in \Sigma_C, \quad p \in F.$$

Proof. For any $t \in (0,1)$ and $f \in \Sigma_C$, let $\{x_t^n\} \in C$ be the unique point that satisfies the equation

$$x_t^n = tf(x_t^n) + (1-t)T_{n+N}T_{n+N-1}\cdots T_{n+1}x_t^n.$$

Since $\lim_{t\to 0} x_t^n$ exists, if we define $Q^n : \Sigma_C \to Fix(T_{n+N}T_{n+N-1}\cdots T_{n+1})$ by

$$Q^n(f) = \lim_{t \to 0} x_t^n.$$

then $Q^n(f) = \lim_{t\to 0} x_t^n$ is well-defined. Since

$$(I-f)x_t^n = -\frac{1-t}{t}(I - T_{n+N}T_{n+N-1}\cdots T_{n+1})x_t^n,$$

by Lemma 2.5, we have for $p \in Fix(T_{n+N}T_{n+N-1}\cdots T_{n+1})$,

$$\begin{aligned}\langle (I-f)x_t^n, J(x_t^n - p)\rangle = -\frac{1-t}{t}\langle (I - T_{n+N}T_{n+N-1}\cdots T_{n+1})x_t^n \\ - (I - T_{n+N}T_{n+N-1}\cdots T_{n+1})p, J(x_t^n - p)\rangle \leq 0.\end{aligned}$$

Noting that J is norm-to-weak*-continuous by Lemma 2.4, and taking the limit as $t \to 0^+$, we obtain

$$\langle (I-f)Q^n(f), J(Q^n(f)-p)\rangle \leq 0, \quad \text{for } n \geq 1.$$

However, by our assumption, since

$$F = Fix(T_N\cdots T_1) = Fix(T_1T_N\cdots T_3T_2) = \cdots = Fix(T_{N-1}\cdots T_1T_N),$$

we know that $Q^n(f)$ solves the variational inequality

$$\langle (I-f)Q^n(f), J(Q^n(f)-p)\rangle \leq 0, \quad f \in \Sigma_C, \quad p \in F \quad \text{for } n \geq 1.$$

Since E is smooth, in F, there is at most one solution of the variational inequality

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \leq 0, \quad f \in \Sigma_C, \quad p \in F,$$

and so $Q^n(f) = Q(f)$ for all $n \geq 1$. Since $x_t^n \to Q(f) \in F$ as $t \to 0^+$ and $Q(f)$ is independent of n, we have

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \leq 0, \quad f \in \Sigma_C, \quad p \in F. \quad \square$$

The following lemma establishes conditions under which $\{x_t^n\}$ defined by (A) converges strongly to a point in F as $t \to 0^+$.

Lemma 3.1. *Let E be a reflexive smooth Banach space satisfying Opial's condition and having the duality mapping J weakly sequentially continuous at 0. Let C be a nonempty closed convex subset of E and $T_1, \cdots, T_N$ nonexpansive mappings from C into itself with $F := \bigcap_{i=1}^N Fix(T_i) \neq \emptyset$ and*

$$F = Fix(T_N\cdots T_1) = Fix(T_1T_N\cdots T_3T_2) = \cdots = Fix(T_{N-1}\cdots T_1T_N).$$

Then $\{x_t^n\}$ defined by (A) *converges strongly to a point in F as $t \to 0^+$.*

Proof. Let $t_m \in (0,1)$ be such that $t_m \to 0$ and let $\{x_m\} := \{x_{t_m}^n\}$ be a subsequence of $\{x_t^n\}$. Thus,

$$x_m = t_m f(x_m) + (1-t_m)T_{n+N}T_{n+N-1}\cdots T_{n+1}x_m.$$

Let $y \in Fix(T_{n+N}T_{n+N-1}\cdots T_{n+1})$. Then

$$x_m - y = t_m(f(x_m) - y) \\ + (1-t_m)(T_{n+N}T_{n+N-1}\cdots T_{n+1}x_m - T_{n+N}T_{n+N-1}\cdots T_{n+1}y).$$

Therefore

$$\|x_m - y\|^2 = \langle x_m - y, J(x_m - y)\rangle \\ \leq t_m\langle f(x_m) - y, J(x_m - y)\rangle + (1-t_m)\|x_m - y\|^2.$$

It follows that for all $y \in Fix(T_{n+N}T_{n+N-1}\cdots T_{n+1})$,

$$\|x_m - y\|^2 \leq \langle f(x_m) - y, J(x_m - y)\rangle. \tag{3.1}$$

Hence

$$\langle x_m - f(x_m), J(y - x_m)\rangle = \langle x_m - y, J(y - x_m)\rangle + \langle y - f(x_m), J(y - x_m)\rangle \\ \geq -\|x_m - y\|^2 + \|x_m - y\|^2 = 0.$$

That is,

$$\langle x_m - f(x_m), J(y - x_m)\rangle \geq 0.$$

Now

$$\begin{aligned} &\|x_m - y\| \\ \leq\ & t_m\|f(x_m) - y\| \\ &+(1-t_m)\|T_{n+N}T_{n+N-1}\cdots T_{n+1}x_m - T_{n+N}T_{n+N-1}\cdots T_{n+1}y\| \\ \leq\ & t_m\|f(x_m) - y\| + (1-t_m)\|x_m - y\|. \end{aligned}$$

This gives that

$$\|x_m - y\| \leq \|f(x_m) - y\| \leq \|f(x_m) - f(y)\| + \|f(y) - y\| \\ \leq k\|x_m - y\| + \|f(y) - y\|,$$

and so $\|x_m - y\| \leq \frac{1}{1-k}\|f(y) - y\|$. In particular, $\{x_m\}$ is bounded, so are $\{f(x_m)\}$ and $\{T_{n+N}T_{n+N-1}\cdots T_{n+1}x_m\}$. Since E is reflexive, $\{x_m\}$ has a weakly convergent subsequence, say $x_{m_k} \rightharpoonup u \in E$. Since $t_m \to 0^+$,

$$x_m - T_{n+N}T_{n+N-1}\cdots T_{n+1}x_m = t_m(f(x_m) - T_{n+N}T_{n+N-1}\cdots T_{n+1}x_m) \to 0.$$

Hence by Lemma 2.2, $u \in Fix(T_{n+N}T_{n+N-1}\cdots T_{n+1})$. Therefore by (3.1) and the assumption that J is weakly sequentially continuous at 0, we obtain

$$\|x_{m_k} - u\|^2 \leq \langle f(x_{m_k}) - u, J(x_{m_k} - u)\rangle \to 0,$$

and so $x_{m_k} \to u$.

We will now show that every weakly convergent subsequence of $\{x_m\}$ has the same limit. suppose that $x_{m_k} \rightharpoonup u$ and $x_{m_j} \rightharpoonup v$. Then by the above proof, $u, v \in Fix(T_{n+N}T_{n+N-1} \cdots T_{n+1})$ and $x_{m_k} \to u$ and $x_{m_j} \to v$. It follows from (3.1) that

$$\|u - v\|^2 \leq \langle f(u) - v, J(u - v) \rangle, \tag{3.2}$$

and

$$\|v - u\|^2 \leq \langle f(v) - u, J(v - u) \rangle. \tag{3.3}$$

Adding (3.2) and (3.3) yields

$$2\|u - v\|^2 \leq \|u - v\|^2 + \langle f(u) - f(v), J(u - v) \rangle \leq (1 + k)\|u - v\|^2.$$

Since $k \in (0, 1)$, this implies that $u = v$. Hence x_m is strongly convergent to a point in $Fix(T_{n+N}T_{n+N-1} \cdots T_{n+1})$ as $t \to 0^+$. The same argument shows that if $t_l \to 0^+$, then the subsequence $\{x_l\} := \{x_{t_l}^n\}$ of $\{x_t^n\}$ is strongly convergent to the same limit. Thus, as $t \to 0^+$, $\{x_t^n\}$ converges strongly to a point in $Fix(T_{n+N}T_{n+N-1} \cdots T_{n+1})$. Therefore, by assumption, $\{x_t^n\}$ converges strongly to a point in F as $t \to 0^+$. ∎

Using Theorem 3.1 and Lemma 3.1, we show the existence of solutions of certain variational inequality in a reflexive Banach space having a weakly sequentially continuous duality mapping.

Theorem 3.2. *Let E be a reflexive Banach space having a weakly sequentially continuous duality mapping J. Let C be a nonempty closed convex subset of E and $T_1, \cdots, T_N$ nonexpansive mappings from C into itself with $F := \bigcap_{i=1}^N Fix(T_i) \neq \emptyset$ and*

$$F = Fix(T_N \cdots T_1) = Fix(T_1 T_N \cdots T_3 T_2) = \cdots = Fix(T_{N-1} \cdots T_1 T_N).$$

Then there exists the unique solution $Q(f) \in F$ of the variational inequality

$$\langle (I - f)Q(f), J(Q(f) - p) \rangle \leq 0, \quad f \in \Sigma_C, \quad p \in F, \tag{3.4}$$

where $Q : \Sigma_C \to F$ is defined by $Q(f) := \lim_{t \to 0^+} x_t^n$ and x_t^n is defined by (A).

Proof. We notice that the definition of the weak sequential continuity of the duality mapping J implies that E is smooth. Thus E^* is strictly convex for E reflexive. By Lemma 3.1, $\{x_t^n\}$ defined by (A) converges strongly to a point in F

as $t \to 0^+$. Hence by Theorem 3.1, $Q(f)$ is the unique solution of the variational inequality

$$\langle (I - f)Q(f), J(Q(f) - p)\rangle \le 0, \quad f \in \Sigma_C, \quad p \in F,$$

where $Q : \Sigma_C \to F$ is defined by $Q(f) = \lim_{t\to 0^+} x_t^n$ and x_t^n is defined by (A). In fact, suppose that $p, q \in F$ satisfy (3.4). Then it follows that

$$\langle (I - f)q, J(q - p)\rangle \le 0 \ \text{ and } \ \langle (I - f)p, J(p - q)\rangle \le 0.$$

Adding these two inequalities, we have

$$(1 - k)\|q - p\|^2 \le \langle (I - f)q - (I - f)p, J(q - p)\rangle \le 0,$$

and so $q = p$. ■

Remark 3.1. In Theorem 3.2, if $f(x) = u$, $x \in C$, is a constant, then it follows from (2.1) that (3.4) is reduced to the sunny nonexpansive retraction from C onto F; that is, Q satisfies the property:

$$\langle Q(u) - u, J(Q(u) - p)\rangle \le 0, \quad u \in C, \quad p \in F.$$

Remark 3.2. Theorem 3.1, Lemma 3.1 and Theorem 3.2 generalize Theorem 3.8, Lemma 3.9 and Theorem 3.10 in O'Hara et al. [18] to the viscosity approximation method for $N > 1$ finite mappings respectively. Theorem 3.2 also extends Theorems 3.1 in Xu [26] to the case of $N > 1$ finite mappings together with the contraction f.

Remark 3.3. In [9], Jung established Theorem 3.2 in a reflexive Banach space with a uniformly Gâteaux differentiable norm together with assumption that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings.

Now we study the strong convergence of the iterative algorithm (1.1) for a family of finite nonexpansive mappings.

For convenience, we list again the condition to be imposed on the sequence $\{\lambda_n\}$ of parameters in the iterative algorithm (1.1).

$$\text{(C1)} \ \lim_{n\to\infty} \lambda_n = 0; \quad \text{(C2)} \ \sum_{n=1}^{\infty} \lambda_n = \infty \ \text{ or, equivalently, } \prod_{n=1}^{\infty}(1 - \lambda_n) = 0.$$

Using Theorem 3.2, we give the following result in a reflexive Banach space having a weakly sequentially continuous duality mapping, which generalizes Theorem 5 in Zhou et al. [27] to the viscosity approximation method.

Proposition 3.1. *Let E be a reflexive Banach space having a weakly sequentially continuous duality mapping J. Let C be a nonempty closed convex subset of E and $T_1, \cdots, T_N$ nonexpansive mappings from C into itself with $F := \bigcap_{i=1}^N Fix(T_i) \neq \emptyset$ and*

$$F = Fix(T_N \cdots T_1) = Fix(T_1 T_N \cdots T_3 T_2) = \cdots = Fix(T_{N-1} \cdots T_1 T_N).$$

Let $\{\lambda_n\}$ be a sequence in $(0,1)$ which satisfies the conditions (C1), *$f \in \Sigma_C$ and $x_0 \in C$ chosen arbitrarily. Let $\{x_n\}$ be generated by*

$$x_{n+1} = \lambda_{n+1} f(x_n) + (1 - \lambda_{n+1}) T_{n+1} x_n, \quad n \geq 0$$

and μ a Banach limit. Then

$$\mu_n \langle (I - f)Q(f), J(Q(f) - x_n) \rangle \leq 0,$$

where $Q : \Sigma_C \to F$ is defined by $Q(f) = \lim_{t \to 0^+} x_t^n$ and x_t^n is defined by (A).

Proof. Note that the definition of the weak continuity of duality mapping J implies that E is smooth. Let x_t^n be defined by (A) and $n = r \mod N$ for some $r \in \{1, \cdots, N\}$. Then we can write $x_t^n := x_t^r$ and

$$x_t^r - x_{n+N} = (1-t)(T_{n+N} T_{n+N-1} \cdots T_{n+1} x_t^r - x_{n+N}) + t(f(x_t^r) - x_{n+N}).$$

Applying Lemma 2.1, we have

$$\begin{aligned} \|x_t^r - x_{n+N}\|^2 \leq & (1-t)^2 \|T_{n+N} T_{n+N-1} \cdots T_{n+1} x_t^r - x_{n+N}\|^2 \\ & + 2t \langle f(x_t^r) - x_{n+N}, J(x_t^r - x_{n+N}) \rangle. \end{aligned} \tag{3.5}$$

Let $p \in F$. As in the proof of Lemma 3.1, we have

$$\|x_t^r - p\| \leq \frac{1}{1-k} \|f(p) - p\|, \qquad t \in (0,1),$$

and hence $\{x_t^r\}$ is bounded. We also have

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{1}{1-k}\|f(p) - p\|\}$$

for all $n \geq 0$ and all $p \in F$ and so $\{x_n\}$ is bounded. Indeed, let $p \in F$ and $d = \max\{\|x_0 - p\|, \frac{1}{1-k}\|f(p) - p\|\}$. Then by the nonexpansivity of T_n and $f \in \Sigma_C$,

$$\begin{aligned} \|x_1 - p\| &\leq (1 - \lambda_1)\|T_1 x_0 - p\| + \lambda_1 \|f(x_0) - p\| \\ &\leq (1 - \lambda_1)\|x_0 - p\| + \lambda_1(\|f(x_0) - f(p)\| + \|f(p) - p\|) \\ &\leq (1 - (1-k)\lambda_1)\|x_0 - p\| + \lambda_1 \|f(p) - p\| \\ &\leq (1 - (1-k)\lambda_1)d + \lambda_1(1-k)d = d. \end{aligned}$$

Using an induction, we obtain $\|x_{n+1} - p\| \leq d$. Hence $\{x_n\}$ is bounded, and so are $\{T_{n+1}x_n\}$ and $\{f(x_n)\}$. As a consequence with the control condition (C1), we get

$$\|x_{n+1} - T_{n+1}x_n\| \leq \lambda_{n+1}\|T_{n+1}x_n - f(x_n)\| \to 0 \quad (n \to \infty).$$

By using the same method, we have

$$\|x_{n+N} - T_{n+N} \cdots T_{n+1}x_n\| \to 0 \quad (n \to \infty).$$

Indeed, noting that each T_i is nonexpansive and using just above fact, we obtain the finite table

$$\begin{aligned} x_{n+N} - &T_{n+N}x_{n+N-1} \to 0, \\ T_{n+N}x_{n+N-1} - &T_{n+N}T_{n+N-1}x_{n+N-2} \to 0, \\ &\vdots \\ T_{n+N} \cdots T_{n+2}x_{n+1} - &T_{n+N} \cdots T_{n+1}x_n \to 0. \end{aligned}$$

Adding up this table yields

$$x_{n+N} - T_{n+N} \cdots T_{n+1}x_n \to 0 \quad (n \to \infty).$$

Observe also that

$$\|T_{n+N}T_{n+N-1} \cdots T_{n+1}x_t^r - x_{n+N}\| \leq \|x_t^r - x_n\| + e_n,$$

where $e_n = \|x_{n+N} - T_{n+N}T_{n+N-1} \cdots T_{n+1}x_n\| \to 0$ as $n \to \infty$, and

$$\begin{aligned} &\langle f(x_t^r) - x_{n+N}, J(x_t^r - x_{n+N})\rangle \\ &= \langle f(x_t^r) - x_t^r, J(x_t^r - x_{n+N})\rangle + \|x_t^r - x_{n+N}\|^2. \end{aligned}$$

Thus it follows from (3.5) that

$$\begin{aligned} \|x_t^r - x_{n+N}\|^2 \leq &(1-t)^2(\|x_t^r - x_n\| + e_n)^2 \\ &+ 2t(\langle f(x_t^r) - x_t^r, J(x_t^r - x_{n+N})\rangle + \|x_t^r - x_{n+N}\|^2) \end{aligned} \tag{3.6}$$

Applying the Banach limit μ to (3.6), we have

$$\begin{aligned} \mu_n(\|x_t^r - x_{n+N}\|^2) \leq &(1-t)^2\mu_n((\|x_t^r - x_n\| + e_n)^2) \\ &+ 2t\mu_n(\langle f(x_t^r) - x_t^r, J(x_t^r - x_{n+N})\rangle + \|x_t^r - x_{n+N}\|^2) \end{aligned} \tag{3.7}$$

and it follows from (3.7) that

$$\mu_n\langle x_t^r - f(x_t^r), J(x_t^r - x_n)\rangle \leq t\mu_n(\|x_t^r - x_n\|^2). \tag{3.8}$$

Since

$$t\|x_t^r - x_n\|^2 \leq t(\frac{2}{1-k}\|f(p) - p\| + \|x_0 - p\|)^2 \to 0 \quad (t \to 0),$$

we conclude from Theorem 3.2 and (3.8) that

$$\mu_n\langle (I-f)Q(f), J(Q(f) - x_n)\rangle \leq \limsup_{t\to 0} \mu_n\langle x_t^r - f(x_t^r), J(x_t^r - x_n)\rangle \leq 0,$$

where $Q : \Sigma_C \to F$ is defined by $Q(f) = \lim_{t\to 0} x_t^r$. ∎

Theorem 3.3. *Let E be a reflexive Banach space having a weakly sequentially continuous duality mapping J. Let C be a nonempty closed convex subset of E and $T_1, \cdots, T_N$ nonexpansive mappings from C into itself with $F := \bigcap_{i=1}^N Fix(T_i) \neq \emptyset$ and*

$$F = Fix(T_N \cdots T_1) = Fix(T_1 T_N \cdots T_3 T_2) = \cdots = Fix(T_{N-1} \cdots T_1 T_N).$$

Let $\{\lambda_n\}$ be a sequence in $(0,1)$ which satisfies the conditions (C1) *and* (C2)*, $f \in \Sigma_C$ and $x_0 \in C$ chosen arbitrarily. Let $\{x_n\}$ be generated by*

$$x_{n+1} = \lambda_{n+1} f(x_n) + (1-\lambda_{n+1})T_{n+1}x_n, \quad n \geq 0. \tag{3.9}$$

If the sequence $\{x_n\}$ is weakly asymptotically regular, then $\{x_n\}$ converges strongly to $Q(f)$, where $Q(f) \in F$ solves a variational inequality

$$\langle (I-f)Q(f), J(Q(f) - p)\rangle \leq 0 \quad f \in \Sigma_C, \quad p \in F.$$

Proof. Put $S = T_N \cdots T_1$. Then $Fix(S) = F = \bigcap_{i=1}^N Fix(T_i)$ by assumption. By Theorem 3.2, there exists a solution $Q(f)$ of a variational inequality

$$\langle (I-f)Q(f), J(Q(f) - p)\rangle \leq 0 \quad f \in \Sigma_C, \quad p \in F,$$

where $Q(f) = \lim_{t\to 0^+} x_t$ and $x_t = tf(x_t) + (1-t)Sx_t$ for $0 < t < 1$. We proceed with the following steps:

Step 1. $\|x_n - z\| \leq \max\{\|x_0 - z\|, \frac{1}{1-k}\|f(z) - z\|\}$ for all $n \geq 0$ and all $z \in F$ as in the proof of Proposition 3.1. Hence $\{x_n\}$ is bounded and so are $\{T_{n+1}x_n\}$ and $\{f(x_n)\}$.

Step 2. $\limsup_{n\to\infty}\langle (I-f)Q(f), J(Q(f) - x_n)\rangle \leq 0$. To this end, put

$$a_n := \langle (I-f)Q(f), J(Q(f) - x_n)\rangle, \quad n \geq 1.$$

Then Proposition 3.1 implies that $\mu_n(a_n) \le 0$ for any Banach limit μ. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}(a_{n+N} - a_n) = \lim_{j\to\infty}(a_{n_j+N} - a_{n_j})$$

and $x_{n_j} \rightharpoonup q \in E$. This implies that $x_{n_j+N} \rightharpoonup q$ since $\{x_n\}$ is weakly asymptotically regular. From the weak sequential continuity of duality mapping J, we have

$$w - \lim_{j\to\infty} J(Q(f) - x_{n_j+N}) = w - \lim_{j\to\infty} J(Q(f) - x_{n_j}) = J(Q(f) - q),$$

and so

$$\begin{aligned}
&\limsup_{n\to\infty}(a_{n+N} - a_n) \\
&\quad = \lim_{j\to\infty}\langle (I - f)Q(f), J(Q(f) - x_{n_j+N}) - J(Q(f) - x_{n_j})\rangle = 0.
\end{aligned}$$

Then Lemma 2.6 implies that $\limsup_{n\to\infty} a_n \le 0$, that is,

$$\limsup_{n\to\infty}\langle (I - f)Q(f), J(Q(f) - x_n)\rangle \le 0.$$

Step 3. $\lim_{n\to\infty}\|x_n - Q(f)\| = 0$. By using (3.9), we have

$$x_{n+1} - Q(f) = \lambda_{n+1}(f(x_n) - Q(f)) + (1 - \lambda_{n+1})(T_{n+1}x_n - Q(f)).$$

Applying Lemma 2.1, we obtain

$$\begin{aligned}
&\|x_{n+1} - Q(f)\|^2 \\
&\le (1 - \lambda_{n+1})^2\|T_{n+1}x_n - Q(f)\|^2 + 2\lambda_{n+1}\langle f(x_n) - Q(f), J(x_{n+1} - Q(f))\rangle \\
&\le (1 - \lambda_{n+1})^2\|x_n - Q(f)\|^2 + 2k\lambda_{n+1}\|x_n - Q(f)\|\|x_{n+1} - Q(f)\| \\
&\quad + 2\lambda_{n+1}\langle f(Q(f)) - Q(f), J(x_{n+1} - Q(f))\rangle \\
&\le (1 - \lambda_{n+1})^2\|x_n - Q(f)\|^2 + k\lambda_{n+1}(\|x_n - Q(f)\|^2 + \|x_{n+1} - Q(f)\|^2) \\
&\quad + 2\lambda_{n+1}\langle f(Q(f)) - Q(f), J(x_{n+1} - Q(f))\rangle.
\end{aligned}$$

It then follows that

$$\begin{aligned}\|x_{n+1}-Q(f)\|^2 &\le \frac{1-(2-k)\lambda_{n+1}+\lambda_{n+1}^2}{1-k\lambda_{n+1}}\|x_n-Q(f)\|^2\\ &\quad+\frac{2\lambda_{n+1}}{1-k\lambda_{n+1}}\langle f(Q(f))-Q(f),J(x_{n+1}-Q(f))\rangle\\ &\le \frac{1-(2-k)\lambda_{n+1}}{1-k\lambda_{n+1}}\|x_n-Q(f)\|^2+\frac{\lambda_{n+1}^2}{1-k\lambda_{n+1}}M^2\\ &\quad+\frac{2\lambda_{n+1}}{1-k\lambda_{n+1}}\langle f(Q(f))-Q(f),J(x_{n+1}-Q(f))\rangle,\end{aligned} \tag{3.10}$$

where $M=\sup_{n\ge 0}\|x_n-Q(f)\|$. Put

$$\alpha_n=\frac{2(1-k)\lambda_{n+1}}{1-k\lambda_{n+1}},$$

$$\beta_n=\frac{M^2\lambda_{n+1}}{2(1-k)}+\frac{1}{1-k}\langle f(Q(f))-Q(f),J(x_{n+1}-Q(f))\rangle.$$

From (C1), (C2) and Step 2, it follows that

$$\alpha_n\to 0,\sum_{n=0}^{\infty}\alpha_n=\infty,\ \text{and}\ \limsup_{n\to\infty}\beta_n\le 0.$$

Since (3.10) reduces to

$$\|x_{n+1}-Q(f)\|^2\le(1-\alpha_n)\|x_n-Q(f)\|^2+\alpha_n\beta_n,$$

from Lemma 2.3 with $\gamma_n=0$, we conclude that $\lim_{n\to\infty}\|x_n-Q(f)\|=0$. This completes the proof. ∎

Corollary 3.1. *Let E, C, and $T_1,\cdots,T_N$ be as in Theorem 3.3. Let $\{\lambda_n\}$ be a sequence in $(0,1)$ which satisfies the conditions* (C1) *and* (C2)*, $f\in\Sigma_C$ and $x_0\in C$ chosen arbitrarily. Let $\{x_n\}$ be generated by*

$$x_{n+1}=\lambda_{n+1}f(x_n)+(1-\lambda_{n+1})T_{n+1}x_n,\quad n\ge 0. \tag{3.11}$$

If the sequence $\{x_n\}$ is asymptotically regular, then $\{x_n\}$ converges strongly to $Q(f)$, where $Q(f)\in F$ is the unique solution of the variational inequality

$$\langle (I-f)Q(f),J(Q(f)-p)\rangle\le 0\quad f\in\Sigma_C,\quad p\in F.$$

Remark 3.4. If $\{\lambda_n\}$ satisfies conditions (C1), (C2) and (C5), (or (C6),) or the perturbed control condition:

$$|\lambda_{n+N}-\lambda_n|\le\circ(\alpha_{n+N})+\sigma_n,\qquad \sum_{n=0}^{\infty}\sigma_n<\infty, \tag{C7}$$

then the sequence $\{x_n\}$ generated by (3.11) is asymptotically regular. Now we give only the proof for the condition (C7). Indeed, by Step 1 in the proof of Theorem 3.3, there exists a constant $L > 0$ such that for all $n \geq 0$, $\|f(x_n)\| + \|Tx_n\| \leq L$. Since for all $n \geq 1$, $T_{n+N} = T_n$, we have

$$\begin{aligned}
&\|x_{n+N} - x_n\| \\
&= \|(1-\lambda_{n+N})(T_{n+N}x_{n+N-1} - T_n x_{n-1}) \\
&\quad + (\lambda_{n+N} - \lambda_n)(f(x_{n-1}) - T_n x_{n-1}) + \lambda_{n+N}(f(x_{n+N-1}) - f(x_{n-1}))\| \\
&\leq (1-\lambda_{n+N})\|x_{n+N-1} - x_{n-1}\| + L|\lambda_{n+N} - \lambda_n| \\
&\quad + k\lambda_{n+N}\|x_{n+N-1} - x_{n-1}\| \\
&= (1-(1-k)\lambda_{n+N})\|x_{n+N-1} - x_{n-1}\| + (\circ(\lambda_{n+N}) + \sigma_n)L.
\end{aligned}$$

By taking $s_{n+1} = \|x_{n+N} - x_n\|$, $\alpha_n = (1-k)\lambda_{n+N}$, $\alpha_n\beta_n = \circ(\lambda_{n+N})L$ and $\gamma_n = \sigma_n L$, we have

$$s_{n+1} \leq (1-\lambda_n)s_n + \lambda_n\beta_n + \gamma_n,$$

and, by Lemma 2.3, $\lim_{n\to\infty} \|x_{n+N} - x_n\| = 0$.

In view of this observation, we have the following:

Corollary 3.2. *Let E, C, and $T_1, \cdots, T_N$ be as in Corollary 3.1. Let $\{\lambda_n\}$ be a sequence in $(0,1)$ which satisfies the conditions* (C1), (C2) *and* (C5) (or (C6)) *or* (C7), *$f \in \Sigma_C$ and $x_0 \in C$ chosen arbitrarily. Let $\{x_n\}$ be generated by*

$$x_{n+1} = \lambda_{n+1}f(x_n) + (1-\lambda_{n+1})T_{n+1}x_n, \quad n \geq 0.$$

Then $\{x_n\}$ converges strongly to $Q(f) \in F$, where $Q(f)$ is the unique solution of the variational inequality

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \leq 0, \quad f \in \Sigma_C, \quad p \in F.$$

Remark 3.5. (1) Theorem 3.3 and Corollary 3.1 extend Theorem 6 and Theorem 10 of Zhou et al. [27] to the viscosity approximation method without the assumption of uniformly Gâteaux differentiable norm and the fixed point property (that is, the uniform smoothness assumption), respectively.

(2) Theorem 3.3 (Corollary 3.1 and Corollary 3.2) also improves Theorem 2 (and Corollary 2) of Jung [9] because the assumption of uniformly Gâteaux differentiable norm and the fixed point property (that is, the uniform smoothness assumption) is removed.

(3) Corollary 3.2 extends Theorem 4.3 of O'Hara et al. [18] to the viscosity approximation method together with the condition (C7) in place of the condition (C6) on the parameters $\{\lambda_n\}$.

Next, as an application of Theorem 3.3 or Corollary 3.1, we study the convex feasibility problem in a a strictly convex and reflexive Banach space with a weakly sequentially continuous duality mapping.

Using a nonlinear ergodic theorem, Crombez [4] considered the convex feasibility problem in a Hilbert space. Later, Kitahara and Takahashi [13], Takahashi and Tamura [21], Takhashi et al. [22] dealt with the convex feasibility problem by convex combinations of sunny nonexpansive retractions in uniformly convex Banch spaces. In particular, Zhou et al. [27] investigated the convex feasibility problem in a strictly convex and reflexive Banach space with a uniformly Gâteaux differentiable norm along with the assumption that every weakly compact convex subset of it has the fixed point property for nonexpansive mappings.

The following lemma was given by Takahashi et al. [22].

Lemma 3.2. [22] *Let E be a strictly convex Banach space and C a closed convex subset of E. Let $S_1, S_2, \cdots, S_N$ be nonexpansive mappings of C into itself such that the set of common fixed points of $S_1, S_2, \cdots, S_N$ is nonempty. Let $T_1, T_2, \cdots, T_N$ be mappings of C into itself given by $T_i = (1-\alpha_i)I + \alpha_i S_i$ for any $0 < \alpha_i < 1$, $(i = 1, 2, \cdots, N)$ where I denotes the identity mapping on C. Then $\{T_1, T_2, \cdots, T_N\}$ satisfies the following:*

$$\bigcap_{i=1}^{N} Fix(T_i) = \bigcap_{i=1}^{N} Fix(S_i)$$

and

$$\begin{aligned}\bigcap_{i=1}^{N} Fix(T_i) &= Fix(T_N \cdots T_1) = Fix(T_1 T_N \cdots T_2)\\ &= \cdots = Fix(T_{N-1} T_{N-2} \cdots T_1 T_N).\end{aligned}$$

Using Lemma 3.2 and Theorem 3.3 or Corollary 3.1 in the case of $f(x) = u$, $x \in C$, constant, we obtain the following:

Theorem 3.4. *Let E be a strictly convex and reflexive Banach space having a weakly sequentially continuous duality mapping J. Let C be a nonempty closed convex subset of E and $C_1, C_2 \cdots, C_N$ nonexpansive retracts of C into itself with $\bigcap_{i=1}^{N} C_i \neq \emptyset$. Define a family of finite $\{T_1, T_2, \cdots, T_N\}$ by $T_i = (1-\alpha_i)I + \alpha_i Q_{C_i}$ for any $0 < \alpha_i < 1$ $(i = 1, 2, \cdots, N)$, where Q_{C_i} is a nonexpansive retraction of C onto C_i. Let $\{\lambda_n\}$ and $\{x_n\}$ be as in Theorem 3.3 with $f(x) = u,\ x \in C$, constant. Then the sequence $\{x_n\}$ converges strongly to a point $z \in \bigcap_{i=1}^{N} C_i$. Moreover, if*

$Qu = \lim_{n\to\infty} x_n$ for any $u \in C$, then Q is a sunny nonexpansive retraction of C onto $\bigcap_{i=1}^N C_i$.

Proof. By Lemma 3.2, we have $\bigcap_{i=1}^N Fix(T_i) = \bigcap_{i=1}^N Fix(Q_{C_i}) = \bigcap_{i=1}^N C_i$ and

$$\begin{aligned}\bigcap_{i=1}^N Fix(T_i) &= Fix(T_N \cdots T_1) = Fix(T_1 T_N \cdots T_3 T_2)\\ &= \cdots = Fix(T_{N-1} T_{N-2} \cdots T_1 T_N).\end{aligned}$$

Thus, applying Theorem 3.3 with $f(x) = u$, $x \in C$, constant, we have the desired conclusion immediately. ■

Corollary 3.3. *Let E, C, C_i, T_i and Q_{C_i} $(i = 1,\ 2, \cdots, N)$ be as in Theorem 3.4. Let $\{\lambda_n\}$ and $\{x_n\}$ be as in Corollary 3.1 with $f(x) = u,\ x \in C$, constant. Then the sequence $\{x_n\}$ converges strongly to a point $z \in \bigcap_{i=1}^N C_i$. Moreover, if $Qu = \lim_{n\to\infty} x_n$ for any $u \in C$, then Q is a sunny nonexpansive retraction of C onto $\bigcap_{i=1}^N C_i$.*

Theorem 3.5. *Let E be a strictly convex and reflexive Banach space having a weakly sequentially continuous duality mapping J. Let C be a nonempty closed convex subset of E and $S_1, S_2 \cdots, S_N$ nonexpansive mappings of C into itself with $F := \bigcap_{i=1}^N Fix(S_i) \neq \emptyset$. Define a family of finite $\{T_1, T_2, \cdots, T_N\}$ by $T_i = (1-\alpha_i)I + \alpha_i S_i$ for any $0 < \alpha_i < 1$, $(i = 1, 2, \cdots, N)$. Let $\{\lambda_n\}$ and $\{x_n\}$ be as in Corollary 3.1 with $f(x) = u,\ x \in C$, constant. Then the sequence $\{x_n\}$ converges strongly to a common fixed point of $S_1, S_2, \cdots, S_N$. Moreover, if $Qu = \lim_{n\to\infty} x_n$ for any $u \in C$, then Q is a sunny nonexpansive retraction of C onto $\bigcap_{i=1}^N Fix(S_i)$.*

Proof. By Lemma 3.2, we have $\bigcap_{i=1}^N Fix(T_i) = \bigcap_{i=1}^N Fix(S_i) = F$ and

$$\begin{aligned}\bigcap_{i=1}^N Fix(T_i) &= Fix(T_N \cdots T_1) = Fix(T_1 T_N \cdots T_3 T_2)\\ &= \cdots = Fix(T_{N-1} T_{N-2} \cdots T_1 T_N).\end{aligned}$$

Thus, applying Corollary 3.1 with $f(x) = u$, $x \in C$, constant, the sequence $\{x_n\}$ converges strongly to a common fixed point of $S_1, S_2, \cdots, S_N$. This completes the proof. ■

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Jong Soo Jung
Department of Mathematics,
Dong-A University,
Busan 604-714
Korea
E-mail: jungjs@mail.donga.ac.kr