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VECTOR F-COMPLEMENTARITY PROBLEMS WITH NONCONVEX PREFERENCES AND APPLICATIONS TO VECTOR OPTIMIZATION PROBLEMS

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Abstract. In this paper, several classes of vector F-complementarity problems (in short, VF-CP) and vector F-variational inequalities (in short, VF-VI) with relations determined by nonconvex preferences are introduced in Banach spaces. Some characterizations of solution sets for (VF-CP) and (VF-VI) are also presented. Furthermore, the results obtained are applied to vector optimization problems.

1. INTRODUCTION

Vector variational inequality (VVI) was first introduced and studied by Giannessi [5]. Recently, (VVIs) and vector complementarity problems (VCP) have been studied intensively because partly they can be efficient tools for investigating vector optimization problems (VOP) (see, for example, [1-3, 6, 7, 11-13, 15, 17, 18] and the references therein). As is well known, (VVIs), (VCP) and (VOP) are usually studied in ordered spaces with an ordering induced by a convex cone. Recently, Rubinov and Gasimov [16] considered (VOP) with preferences that are not necessarily a pre-order relation. Very recently, Huang, Rubinov and Yang [11] considered some (VVIs), (VCP) and (VOP) with relations determined by a nonconvex cone in Banach spaces. Since the relation, determined by a nonconvex cone, is not transitive, it is not an order relation. It is better to be called "nonconvex preferences" or

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"pseudo-ordering" or something else. Clearly, it is difficult to study (VVIs), (VCP) and (VOP) with nonconvex preferences by using the classical methods.

In 2001, Yin, Xu and Zhang [19] introduced a class of F-complementarity problem (F-CP) and proved the existence of solutions for (F-CP) under some assumptions with the F-pseudomonotonicity. Recently, the (F-CP) has been generalized to the vector F-complementarity problem (VF-CP) by Fang and Huang [4], Huang and Fang [8], the F-implicit complementarity problem (F-ICP) by Huang and Li [9], the vector F-implicit complementarity problem (VF-ICP) by Li and Huang [14], and the generalized vector F-complementarity problem (GVF-CP) with point-to-set mappings by Huang, Li and Thompson [10].

In this paper, we introduce several classes of vector F-complementarity problems (in short, VF-CP) and vector F-variational inequalities (in short, VF-VI) with relations determined by nonconvex preferences in Banach spaces. We also derive some characterizations of solution sets for (VF-CP) and (VF-VI). Furthermore, we apply these results obtained to vector optimization problems. The results of this paper generalizes and extends the corresponding results of Huang, Rubinov and Yang [11].

2. Preliminaries

Let Z be a Banach space. A nonempty subset P of Z is said to be a cone if $\lambda P \subseteq P$ for all $\lambda > 0$. P is called a convex cone if P is a cone and $P + P \subseteq P$. P is called a pointed cone if P is a cone and $P \cap \{-P\} = \{0\}$. Let P be a cone. Without other specifications, denote by $\mathcal{C}P$ the complement of P. Then $\mathcal{C}P$ is also a cone. An ordered Banach space (Z, P) is a Banach space Z with an partial ordering defined by a closed, convex and pointed cone $P \subseteq Z$ with apex at the origin, in the form of

$$x \geq_P y \Leftrightarrow x - y \geq_P 0 \Leftrightarrow x - y \in P \quad \forall x, y \in Z$$

and

$$x \geq_{\mathcal{C}P} y \Leftrightarrow x - y \geq_{\mathcal{C}P} 0 \Leftrightarrow x - y \in \mathcal{C}P \quad \forall x, y \in Z.$$

If the interior of P, say int$P$, is nonempty, then a weak ordering in Z is also defined by

$$x \geq_{\text{int}P} y \Leftrightarrow x - y \geq_{\text{int}P} 0 \Leftrightarrow x - y \in \text{int}P \quad \forall x, y \in Z$$

and

$$x \geq_{\mathcal{C}(\text{int}P)} y \Leftrightarrow x - y \geq_{\mathcal{C}(\text{int}P)} 0 \Leftrightarrow x - y \in \mathcal{C}(\text{int}P) \quad \forall x, y \in Z.$$

We also consider the following ordering

$$x \geq_{P\setminus\{0\}} y \Leftrightarrow x - y \geq_{P\setminus\{0\}} 0 \Leftrightarrow x - y \in P\setminus\{0\} \quad \forall x, y \in Z$$

and

$$x \geq_{\mathcal{C}(P\setminus\{0\})} y \Leftrightarrow x - y \geq_{\mathcal{C}(P\setminus\{0\})} 0 \Leftrightarrow x - y \in \mathcal{C}(P\setminus\{0\}) \quad \forall x, y \in Z.$$

Remark that for any $x, y \in Z$,

$$x \geq_P y \Leftrightarrow y \leq_P x; \quad x \geq_{CP} y \Leftrightarrow y \leq_{CP} x;$$

$$x \geq_{\text{int}P} y \Leftrightarrow y \leq_{\text{int}P} x; \quad x \geq_{C(\text{int}P)} y \Leftrightarrow y \leq_{C(\text{int}P)} x;$$

$$x \geq_{P\setminus\{0\}} y \Leftrightarrow y \leq_{P\setminus\{0\}} x; \quad x \geq_{C(P\setminus\{0\})} y \Leftrightarrow y \leq_{C(P\setminus\{0\})} x.$$

If cone P is not convex, then the relation given above is not transitive and so it is not an order relation. We call it "nonconvex preferences" or "pseudo-ordering" or something else.

Throughout this paper, without other specifications, let (X, K) be an ordered Banach space, where K is a closed, convex and pointed cone of X with apex at the origin. Let Y be a Banach space. Denote by $L(X, Y)$ the space of all continuous linear mappings from X to Y, and by (l, x) the value of $l \in L(X, Y)$ at $x \in X$. Let $C : K \to Y$ be a point-to-set mapping such that $C(x)$ is a closed cone in Y for each $x \in K$, with $C = \cap_{x \in K} C(x)$ and int$C \neq \emptyset$. It is easy to see that int$C(x)$, $\mathcal{C}(\text{int}C(x))$, $C(x)\setminus\{0\}$ and $\mathcal{C}(C(x)\setminus\{0\})$ are cones for all $x \in K$, and C is also a closed cone. Remark that $C(x)$ is not necessary convex for each $x \in K$, so is C.

Let $T : K \to L(X, Y)$ and $F : K \to Y$ be two mappings. In this paper, we consider the following vector F-complementarity problems with nonconvex preferences:

Strong Vector F-Complementarity Problem with Nonconvex Preferences (SVF-CP): find $x^* \in K$ such that

$$(Tx^*, x^*) + F(x^*) = 0, \quad (Tx^*, y) + F(y) \geq_{C(x^*)} 0 \qquad \forall y \in K;$$

Positive Vector F-Complementarity Problem with Nonconvex Preferences (PVF-CP): find $x^* \in K$ such that

$$(Tx^*, x^*) + F(x^*) \geq_{\mathcal{C}(\text{int}C(x^*))} 0, \quad (Tx^*, y) + F(y) \geq_{C(x^*)} 0 \qquad \forall y \in K;$$

Mild Strong Vector F-Complementarity Problem with Nonconvex Preferences (MSVF-CP): find $x^* \in K$ such that

$$(Tx^*, x^*) + F(x^*) = 0, \quad (Tx^*, y) + F(y) \leq_{\mathcal{C}(C(x^*)\setminus\{0\})} 0 \qquad \forall y \in K;$$

Weak Vector F-Complementarity Problem with Nonconvex Preferences (WVF-CP): find $x^* \in K$ such that

$$(Tx^*, x^*) + F(x^*) \geq_{\mathcal{C}(\text{int}C(x^*))} 0, \quad (Tx^*, y) + F(y) \leq_{\mathcal{C}(\text{int}C(x^*))} 0 \qquad \forall y \in K.$$

We also consider the following problems with nonconvex preferences:

Strong Vector F-Variational Inequalities with Nonconvex Preferences (SVF-VI): find $x^* \in K$ such that

$$(Tx^*, y - x^*) + F(y) - F(x^*) \geq_{C(x^*)} 0 \qquad \forall y \in K;$$

Strong Minty Vector F-Variational Inequality with Nonconvex Preferences (SMVF-VI): find $x^* \in K$ such that

$$(Ty, y - x^*) + F(y) - F(x^*) \geq_{C(x^*)} 0 \qquad \forall y \in K;$$

Mild Strong Vector F-Variational Inequality with Nonconvex Preferences (MSVF-VI): find $x^* \in K$ such that

$$(Tx^*, y - x^*) + F(y) - F(x^*) \leq_{\mathcal{C}(C(x^*)\setminus\{0\})} 0 \qquad \forall y \in K;$$

Weak Vector F-Variational Inequality with Nonconvex Preferences (WVF-VI): find $x^* \in K$ such that

$$(Tx^*, y - x^*) + F(y) - F(x^*) \leq_{\mathcal{C}(\text{int}C(x^*))} 0 \qquad \forall y \in K.$$

We denote by S_{SVCP}, S_{PVCP}, S_{MSVCP}, S_{WVCP}, S_{SVVI}, S_{SMVVI}, S_{MSVVI} and S_{WVVI} the solutions set of (SVF-CP), (PVF-CP), (MSVF-CP), (WVF-CP), (SVF-VI), (SMVF-VI), (MSVF-VI) and (WVF-VI), respectively.

3. Characterization of Solution Sets for Vector F-Complementarity Problems with Nonconvex Preference

In this section, we establish the characterizations of solution sets for (SVF-CP), (PVF-CP), (MSVF-CP) and (WVF-CP), respectively. We first recall the following notions.

Definition 3.1. Let $\mathcal{K}$ be a cone of Y. A mapping $G : K \to L(X, Y)$ is said to be $\mathcal{K}$-monotone if for any $x, y \in K$,

$$(Gy - Gx, y - x) \geq_{\mathcal{K}} 0.$$

Example 3.1. Let $X = R, K = [0, +\infty), Y = R^2$ and $\mathcal{K} = \{(x, y) : x \geq 0, 0 \leq y \leq \frac{x}{4}\} \cup \{(x, y) : y \geq 0, 0 \leq x \leq \frac{y}{4}\}$. It is clear that $\mathcal{K}$ is a nonconvex cone. Define $G : K \to L(X, Y)$ by $\langle Gx, z\rangle = (xz, \frac{xz}{4})$ for any $x, z \in K$. Then G is $\mathcal{K}$-monotone.

Definition 3.2. A mapping $G : K \to L(X, Y)$ is said to be hemicontinuous if for any given $x, y \in K$, the mapping $t \mapsto (G(x + t(y - x)), y - x)$ is continuous at 0^+.

Definition 3.3. Let $\mathcal{K}$ be a cone of Y. A mapping $H : K \to Y$ is said to be

(i) $\mathcal{K}$-convex if for any $x, y \in K$ and $t \in [0, 1]$,

$$H(tx + (1 - t)y) \leq_{\mathcal{K}} tH(x) + (1 - t)H(y);$$

(ii) positively homogeneous if for any $x \in K$ and $\lambda > 0$,

$$H(\lambda x) = \lambda H(x).$$

Example 3.2. Let $X = R, K = [0, +\infty), Y = R^2$ and $\mathcal{K} = \{(x, y) : x \geq 0, 0 \leq y \leq \frac{x}{3}\} \cup \{(x, y) : y \geq 0, 0 \leq x \leq \frac{y}{4}\}$. Then $\mathcal{K}$ is a nonconvex cone. Let $H(x) = (x, \frac{x}{4})$ for all $x \in K$. It is easy to verify that H is both $\mathcal{K}$-convex and positively homogeneous.

In the main result of our paper, we also need the following lemma.

Example 3.1. Let Y be a Banach space, and $\mathcal{K}, \mathcal{P}$ be two cones in Y with $\mathcal{K} \subseteq \mathcal{P}$. Then

(i) $x - y \geq_{\mathcal{K}} 0$ implies that $x + z \geq_{\mathcal{K}} y + z$, for all $x, y, z \in Y$.
If $\mathcal{K} + \mathcal{P} \subseteq \mathcal{P}$, then

(ii) $x \geq_{\mathcal{K}} y$ and $y \geq_{\mathcal{P}} 0$ imply that $x \geq_{\mathcal{P}} 0$, for all $x, y \in Y$;

(iii) $x + y \geq_{\mathcal{P}} 0$ and $z \geq_{\mathcal{K}} y$ imply that $x + z \geq_{\mathcal{P}} 0$, for all $x, y, z \in Y$.
If $\mathcal{K} + \text{int}\mathcal{P} \subseteq \text{int}\mathcal{P}$, then

(iv) $x + y \leq_{\mathcal{C}(\text{int}\mathcal{P})} 0$ and $y \leq_{\mathcal{K}} z$ imply that $x + z \leq_{\mathcal{C}(\text{int}\mathcal{P})} 0$, for all $x, y, z \in Y$.

Proof.

(i) Since $x - y \geq_{\mathcal{K}} 0$, we obtain that $x - y \in \mathcal{K}$, and so $(x + z) - (y + z) = x - y \in \mathcal{K}$, i.e., $x + z \geq_{\mathcal{K}} y + z$;

(ii) Let $x \geq_{\mathcal{K}} y$ and $y \geq_{\mathcal{P}} 0$. Then $x - y \in \mathcal{K}$ and $y \in \mathcal{P}$. If $\mathcal{K} + \mathcal{P} \subseteq \mathcal{P}$, then it follows that $x = (x - y) + y \in \mathcal{K} + \mathcal{P} \subseteq \mathcal{P}$ and hence $x \geq_{\mathcal{P}} 0$;

(iii) Assume that $x + y \geq_{\mathcal{P}} 0$ and $z \geq_{\mathcal{K}} y$. Then $x + y \in \mathcal{P}$ and $z - y \in \mathcal{K}$. If $\mathcal{K} + \mathcal{P} \subseteq \mathcal{P}$, then $x + z = (x + y) + (z - y) \in \mathcal{P} + \mathcal{K} \subseteq \mathcal{P}$ and thus $x + z \geq_{\mathcal{P}} 0$;

(iv) Let $x + y \leq_{\mathcal{C}(\text{int}\mathcal{P})} 0$ and $y \leq_{\mathcal{K}} z$. Then one has $x + y \notin -\text{int}\mathcal{P}$ and $y - z \in -\mathcal{K}$. Suppose to the contrary that $x + z \leq_{\text{int}\mathcal{P}} 0$, i.e., $x + z \in -\text{int}\mathcal{P}$. Since $\mathcal{K}+\text{int}\mathcal{P} \subseteq \text{int}\mathcal{P}$, it follows that $x+y = (x+z)+(y-z) \in -\text{int}\mathcal{P}-\mathcal{K} \subseteq -\text{int}\mathcal{P}$, which is a contradiction. The proof is complete.

Remark 3.1. It is easy to see that if $\mathcal{K} \subseteq \mathcal{P}$, then the convexity of $\mathcal{P}$ implies that the condition $\mathcal{K} + \mathcal{P} \subseteq \mathcal{P}$ holds. Indeed, if $\mathcal{P}$ is a convex cone, then $\mathcal{K} + \mathcal{P} \subseteq \mathcal{P} + \mathcal{P} \subseteq \mathcal{P}$. But the converse is not true.

Example 3.3. Let $Y = R^2$,

$$\mathcal{K} = \{(x, y) : x \geq 0, 0 \leq y \leq \frac{x}{4}\} \cup \{(x, y) : y \geq 0, 0 \leq x \leq \frac{y}{2}\}$$

and

$$\mathcal{P} = \{(x, y) : xy \leq 0\} \cup \{(x, y) : x \geq 0,\ y \geq 0\}.$$

Then it is clear that $\mathcal{K} \subseteq \mathcal{P}$ and $\mathcal{K}+\mathcal{P} \subseteq \mathcal{P}$. However, $\mathcal{K}$ and $\mathcal{P}$ are two nonconvex cones. Furthermore, we can easily show that the condition $\mathcal{K} + \text{int}\mathcal{P} \subseteq \text{int}\mathcal{P}$ holds.

3.1. In The Case of (SVF-CP) and (PVF-CP)

In this subsection, we derive the characterizations of solution sets for (SVF-CP) and (PVF-CP), respectively.

Theorem 3.1. Let T be hemicontinuous and C-monotone, and F be C-convex. Assume $C + C(x) \subseteq C(x)$ holds for all $x \in K$. Then

$$S_{SVVI} = S_{SMVVI}.$$

Proof. Let $x^* \in S_{SVVI}$. Then $x^* \in K$ and

$$(Tx^*, y - x^*) + F(y) - F(x^*) \geq_{C(x^*)} 0 \quad \forall y \in K. \tag{3.1}$$

Since T is C-monotone, we have

$$(Ty - Tx^*, y - x^*) \geq_C 0 \quad \forall y \in K.$$

It follows that Lemma 3.1 (i) that

$$(Ty, y-x^*)+F(y)-F(x^*) ge_C (Tx^*, y-x^*)+F(y)-F(x^*) \quad \forall y \in K. \tag{3.2}$$

Since C and $C(x^*)$ are two cones with $C \subseteq C(x^*)$ and $C + C(x^*) \subseteq C(x^*)$, from Lemma 3.1 (ii), (3.1) and (3.2) imply that

$$(Ty, y - x^*) + F(y) - F(x^*) \geq_{C(x^*)} 0 \quad \forall y \in K,$$

and so $x^* \in S_{SMVVI}$. Conversely, suppose that $x^* \in S_{SMVVI}$. Then $x^* \in K$ and

$$(Ty, y - x^*) + F(y) - F(x^*) \geq_{C(x^*)} 0 \quad \forall y \in K.$$

For any $y \in K$, let $z = ty + (1-t)x^*$. Then $z \in K$ for $t \in (0,1)$. Substituting $z = ty + (1-t)x^*$ into the above inequality, we have

$$t(T(x^*+t(y-x^*)), y-x^*)+F(x^*+t(y-x^*))-F(x^*) \geq_{C(x^*)} 0 \quad \forall y \in K. \tag{3.3}$$

Since F is C-convex,

$$F(x^* + t(y - x^*)) \leq_C tF(y) + (1-t)F(x^*),$$

it follows that

$$F(x^* + t(y - x^*)) - F(x^*) \leq_C t(F(y) - F(x^*)). \tag{3.4}$$

Again since C and $C(x^*)$ are two cones with $C \subseteq C(x^*)$ and $C + C(x^*) \subseteq C(x^*)$, it follows from (3.3), (3.4) and Lemma 3.1 (iii) that

$$t\{(T(x^* + t(y - x^*)), y - x^*) + F(y) - F(x^*)\} \geq_{C(x^*)} 0 \quad \forall y \in K,$$

which implies that

$$(T(x^* + t(y - x^*)), y - x^*) + F(y) - F(x^*) \geq_{C(x^*)} 0 \quad \forall y \in K,$$

since $C(x^*)$ is a cone. Thus, the hemicontinuity of T and the closeness of $C(x^*)$ imply that

$$(T(x^*), y - x^*) + F(y) - F(x^*) \geq_{C(x^*)} 0 \quad \forall y \in K,$$

and so $x^* \in S_{SVVI}$. Thus, $S_{SVVI} = S_{SMVVI}$. The proof is complete.

Theorem 3.2. Assume that $C(x)$ is a pointed cone for each $x \in K$ and F is positively homogeneous. Then

$$S_{SVVI} = S_{SVCP}.$$

Proof. Let $x^* \in S_{SVCP}$. Then $x^* \in K$ and

$$(Tx^*, x^*) + F(x^*) = 0, \quad (Tx^*, y) + F(y) \geq_{C(x^*)} 0 \quad \forall y \in K.$$

Thus, for any $y \in K$,

$$\begin{aligned}(Tx^*, y - x^*) + F(y) - F(x^*) &= (Tx^*, y) + F(y) - \{(Tx^*, x^*) + F(x^*)\} \\ &= (Tx^*, y) + F(y) - 0 \\ &\geq_{C(x^*)} 0\end{aligned}$$

and so $x \in S_{SVVI}$. Conversely, suppose that $x^* \in S_{SVVI}$. Then $x^* \in K$ and

$$(Tx^*, y - x^*) + F(y) - F(x^*) \geq_{C(x^*)} 0 \quad \forall y \in K.$$

Since K is a cone and F is positively homogeneous, putting $y = 2x^*$ and $y = \frac{1}{2}x^*$ in the above inequality, we have

$$(Tx^*, x^*) + F(x^*) \geq_{C(x^*)} 0, \quad -\{(Tx^*, x^*) + F(x^*)\} \geq_{C(x^*)} 0$$

and so

$$(Tx^*, x^*) + F(x^*) \in C(x^*) \cap (-C(x^*)).$$

Since $C(x^*) \cap (-C(x^*)) = \{0\}$, we know that $(Tx^*, x^*) + F(x^*) = 0$. It follows that, for any $y \in K$,

$$\begin{aligned}(Tx^*, y) + F(y) &= (Tx^*, y - x^*) + F(y) - F(x^*) + \{(Tx^*, x^*) + F(x^*)\} \\ &= (Tx^*, y - x^*) + F(y) - F(x^*) + 0 \\ &\geq_{C(x^*)} 0,\end{aligned}$$

which implies that $x^* \in S_{SVCP}$ and so $S_{SVVI} = S_{SVCP}$. This completes the proof.

From Theorems 3.1 and 3.2, we obtain the following conclusion.

Corollary 3.1. Let T be hemicontinuous and C-monotone, F be C-convex and positively homogeneous. Assume that $C(x)$ is a pointed cone with $C + C(x) \subseteq C(x)$ for each $x \in K$. Then

$$S_{SVVI} = S_{SMVVI} = S_{SVCP}.$$

Let $C(x) = \bigcup_{i=1}^{n} C_i(x)$, where $C_i : K \to 2^Y$ is a point-to-set mapping such that $C_i(x)$ is closed cone in Y for each $x \in K$ $(i = 1, 2, \cdots, n)$. Now, we consider the following vector F-complementarity problems and vector F-variational inequalities:

the i-th Strong Vector F-Complementarity Problem with Nonconvex Preferences (i-SVF-CP): find $x^* \in K$, such that

$$(Tx^*, x^*) + F(x^*) = 0, \quad (Tx^*, y) + F(y) \geq_{C_i(x^*)} 0 \quad \forall y \in K;$$

the i-th Positive Vector F-Complementarity Problem with Nonconvex Preferences (i-PVF-CP): find $x^* \in K$, such that

$$(Tx^*, x^*) + F(x^*) \geq_{\mathcal{C}(\text{int}C_i(x^*))} 0, \quad (Tx^*, y) + F(y) \geq_{C(x^*)} 0 \quad \forall y \in K;$$

the i-th Strong Vector F-Variational Inequality with Nonconvex Preferences (i-SVF-VI): find $x^* \in K$ such that

$$(Tx^*, y - x^*) + F(y) - F(x^*) \geq_{C_i(x^*)} 0 \qquad y \in K;$$

the i-th Strong Minty Vector F-Variational Inequality with Nonconvex Preferences (i-SMVF-VI): find $x^* \in K$ such that

$$(Ty, y - x^*) + F(y) - F(x^*) \geq_{C_i(x^*)} 0 \qquad y \in K.$$

We denote by S^i_{SVCP}, S^i_{PVCP}, S^i_{SVVI} and S^i_{SMVVI} the solutions set of (i-SVF-CP), (i-PVF-CP), (i-SVF-VI) and (i-SMVF-VI), respectively.

Theorem 3.3. The following statements hold:

(1) $\bigcup_{i=1}^n S^i_{SVCP} \subseteq S_{SVCP}$;

(2) $S_{PVCP} \subseteq \bigcap_{i=1}^n S^i_{PVCP}$;

(3) $\bigcup_{i=1}^n S^i_{SVVI} \subseteq S_{SVVI}$;

(4) $\bigcup_{i=1}^n S^i_{SMVVI} \subseteq S_{SMVVI}$.

Proof.

(1) Let $x^* \in \cup_{i=1}^n S^i_{SVCP}$. Then there exists $i \in \{1, 2, \cdots, n\}$ such that $x^* \in S^i_{SVCP}$, that is, $x^* \in K$ and

$$(Tx^*, x^*) + F(x^*) = 0, \quad (Tx^*, y) + F(y) \geq_{C_i(x^*)} 0 \quad \forall y \in K.$$

This implies that $(Tx^*, y) + F(y) \in C_i(x^*)$ for all $y \in K$. Since $C(x^*) = \cup_{i=1}^n C_i(x^*)$, we have $(Tx^*, y) + F(y) \in C(x^*)$ for all $y \in K$, which implies that $x^* \in S_{SVCP}$.

(2) Let $x^* \in S_{PVCP}$. Then $x^* \in K$ and

$$(Tx^*, x^*) + F(x^*) \geq_{\mathcal{C}(\text{int}C(x^*))} 0, \quad (Tx^*, y) + F(y) \geq_{C(x^*)} 0 \quad \forall y \in K.$$

This implies that $(Tx^*, x^*) + F(x^*) \notin \text{int}C(x^*)$. Since $C(x^*) = \cup_{i=1}^n C_i(x^*)$, we know that $\cup_{i=1}^n \text{int}C_i(x^*) \subseteq \text{int}C(x^*)$. It follows that $(Tx^*, x^*) + F(x^*) \notin \text{int}C_i(x^*)$ for all $i = 1, 2, \cdots, n$, and so $x^* \in S^i_{PVCP}$ for all $i = 1, 2, \cdots, n$. Thus, $x^* \in \cap_{i=1}^n S^i_{PVCP}$.

(3) Let $x^* \in \cup_{i=1}^n S^i_{SVVI}$. Then there exists $i \in \{1, 2, \cdots, n\}$ such that $x^* \in S^i_{SVVI}$, that is, $x^* \in K$ and

$$(Tx^*, y - x^*) + F(y) - F(x^*) \geq_{C_i(x^*)} 0 \quad \forall y \in K.$$

This implies that $(Tx^*, y - x^*) + F(y) - F(x^*) \in C_i(x^*)$ for all $y \in K$. Since $C(x^*) = \cup_{i=1}^n C_i(x^*)$, one has $(Tx^*, y-x^*)+F(y)-F(x^*) \in C(x^*)$ for all $y \in K$, which implies that $x^* \in S_{SVVI}$.

(4) Let $x^* \in \cup_{i=1}^n S^i_{SMVVI}$. Then there exists $i \in \{1, 2, \cdots, n\}$ such that $x^* \in S^i_{SMVVI}$, that is, $x^* \in K$ and

$$(Ty, y - x^*) + F(y) - F(x^*) \geq_{C_i(x^*)} 0 \quad \forall y \in K.$$

This implies that $(Ty, y-x^*)+F(y)-F(x^*) \in C_i(x^*)$ for all $y \in K$. Since $C(x^*) = \cup_{i=1}^n C_i(x^*)$, we obtain $(Ty, y - x^*) + F(y) - F(x^*) \in C(x^*)$ for all $y \in K$, which shows that $x^* \in S_{SMVVI}$. The proof is complete.

The following examples show that the converse inclusions of Theorem 3.3 may not hold.

Example 3.4. Let $X = R$, $K = [0, +\infty)$ and $Y = R^2$. Let $C_1(z) = \{(x, y) : x \geq 0, 0 \leq y \leq \frac{x}{2}\}$, $C_2(z) = \{(x, y) : y \geq 0, 0 \leq x \leq 2y\}$,

$$F(z) = \begin{cases} (z, z), & 0 \leq z \leq 1, \\ (3z, 0), & z > 1, \end{cases}$$

and $(Tz, u) = (u, u)$ for all $z, u \in K$. Then $C(z) = C_1(z) \bigcup C_2(z) = R^2_+$, $0 \in S_{SVCP}$, $0 \in S_{SVVI}$ and $0 \in S_{SMVVI}$. However, $0 \notin S^1_{SVCP} \bigcup S^2_{SVCP}$, $0 \notin S^1_{SVVI} \bigcup S^2_{SVVI}$ and $0 \notin S^1_{SMVVI} \bigcup S^2_{SMVVI}$.

Example 3.5. Let $X = R$, $K = [0, +\infty)$ and $Y = R^2$. Let $C_1(z) = \{(x, y) : x \geq 0, 0 \leq y \leq x\}$, $C_2(z) = \{(x, y) : y \geq 0, 0 \leq x \leq y\}$, $F(x) = (1, 1)$ and $(Tx, y) = (y, y)$ for all $x, y \in K$. Then it is easy to see that $C(z) = C_1(z) \bigcup C_2(z) = R^2_+$, and $0 \in S^1_{PVCP} \bigcap S^2_{PVCP}$. However, $0 \notin S_{PVCP}$.

From Corollary 3.1 and Theorem 3.3, we obtain the following.

Corollary 3.2. Let T be hemicontinuous and C-monotone, F be C-convex and positively homogeneous. Assume that $C(x)$ is a pointed cone with $C+C(x) \subseteq C(x)$ for each $x \in K$. Then

$$\bigcup_{i=1}^n \{S^i_{SVCP} \bigcup S^i_{SVVI} \bigcup S^i_{SMVVI}\} \subseteq S_{SVVI} \bigcap S_{SMVVI} \bigcap S_{SVCP}.$$

3.2. In the Case of (MSVF-CP)

In this subsection, we show the characterizations of solution sets for (MSVF-CP).

Theorem 3.4. The relation holds: $S_{MSVCP} \subseteq S_{MSVVI}$.

Proof. Let $x^* \in S_{MSVCP}$. Then $x^* \in K$ and

$$(Tx^*, x^*) + F(x^*) = 0, \quad (Tx^*, y) + F(y) \leq_{\mathcal{C}(C(x^*)\setminus\{0\})} 0 \quad \forall y \in K.$$

Thus, for any $y \in K$,

$$\begin{aligned}(Tx^*, y - x^*) + F(y) - F(x^*) &= (Tx^*, y) + F(y) - \{(Tx^*, x^*) + F(x^*)\} \\ &= (Tx^*, y) + F(y) - 0 \\ &\leq_{\mathcal{C}(C(x^*)\setminus\{0\})} 0\end{aligned}$$

and so $x^* \in S_{MSVVI}$. This completes the proof.

The following example shows that the converse inclusion of Theorem 3.4 may not hold.

Example 3.6. Let $X = R$, $K = [0, +\infty)$ and $Y = R^2$. Let

$$C(z) = \{(x, y) : x \geq 0, 0 \leq y \leq \frac{x}{2}\} \cup \{(x, y) : y \geq 0, 0 \leq x \leq \frac{y}{2}\}$$

and

$$F(z) = (z, 1), \quad (Tz, u) = (u, u)$$

for all $z, u \in K$. Then it is easy to check that $0 \in S_{MSVVI}$. However, $0 \notin S_{MSVCP}$.

Let $C(x) = \bigcup_{i=1}^{n} C_i(x)$, where $C_i : K \to 2^Y$ is a point-to-set mapping such that $C_i(x)$ is closed cone in Y for each $x \in K$ $(i = 1, 2, \cdots, n)$. Now, we consider the following vector F-complementarity problems and vector F-variational inequalities:

the i-th Mild Strong Vector F-Complementarity Problem with Nonconvex Preferences (i-MSVF-CP): find $x^* \in K$ such that

$$(Tx^*, x^*) + F(x^*) = 0, \quad (Tx^*, y) + F(y) \leq_{\mathcal{C}(C_i(x^*)\setminus\{0\})} 0 \qquad \forall y \in K;$$

the i-th Mild Strong Vector F-Variational Inequality with Nonconvex Preferences (i-MSVF-VI): find $x^* \in K$ such that

$$(Ty, y - x^*) + F(y) - F(x^*) \leq_{\mathcal{C}(C_i(x^*)\setminus\{0\})} 0 \qquad \forall y \in K.$$

We denote by S^i_{MSVCP} and S^i_{MSVVI} the solutions set of (i-MSVF-CP) and (i-MSVF-VI), respectively.

Theorem 3.5. *The following arguments are true:*

(1) $S_{MSVCP} = \bigcap_{i=1}^n S^i_{MSVCP}$;
(2) $S_{MSVVI} = \bigcap_{i=1}^n S^i_{MSVVI}$.

Proof.

(1) Let $x^* \in S_{MSVCP}$. Then $x^* \in K$ and

$$(Tx^*, x^*) + F(x^*) = 0, \quad (Tx^*, y) + F(y) \leq_{\mathcal{C}(C(x^*)\setminus\{0\})} 0 \quad \forall y \in K,$$

and therefore $-\{(Tx^*, y) + F(y)\} \notin C(x^*) \setminus \{0\}$ for all $y \in K$. Since $C(x^*) = \cup_{i=1}^n C_i(x^*)$, it follows that $-\{(Tx^*, y) + F(y)\} \notin C_i(x^*) \setminus \{0\}$ for all $y \in K$ and $i = 1, 2 \cdots, n$. Thus, $x^* \in S^i_{MSVCP}$ for all $i = 1, 2 \cdots, n$, and so $x^* \in \cap_{i=1}^n S^i_{MSVCP}$. Conversely, suppose that $x^* \in \cap_{i=1}^n S^i_{MSVCP}$. Then $x^* \in K$, $(Tx^*, x^*) + F(x^*) = 0$ and

$$(Tx^*, y) + F(y) \leq_{\mathcal{C}(C_i(x^*)\setminus\{0\})} 0 \quad \forall y \in K,\ i = 1, 2, \cdots, n.$$

This implies that $-\{(Tx^*, y) + F(y)\} \notin C_i(x^*) \setminus \{0\}$ for all $y \in K$ and $i = 1, 2, \cdots, n$. Again since $C(x^*) = \cup_{i=1}^n C_i(x^*)$, one has $-\{(Tx^*, y) + F(y)\} \notin C(x^*) \setminus \{0\}$ for all $y \in K$, which shows that $x^* \in S_{MSVCP}$.

(2) Let $x^* \in S_{MSVVI}$. Then $x^* \in K$ and

$$(Ty, y - x^*) + F(y) - F(x^*) \leq_{\mathcal{C}(C(x^*)\setminus\{0\})} 0 \quad \forall y \in K,$$

and hence $-\{(Ty, y - x^*) + F(y) - F(x^*)\} \notin C(x^*) \setminus \{0\}$ for all $y \in K$. Since $C(x^*) = \cup_{i=1}^n C_i(x^*)$, it follows that $-\{(Ty, y-x^*)+F(y)-F(x^*)\} \notin C_i(x^*) \setminus \{0\}$ for all $y \in K$ and $i = 1, 2 \cdots, n$. Thus, $x^* \in S^i_{MSVVI}$ for all $i = 1, 2 \cdots, n$, and so $x^* \in \cap_{i=1}^n S^i_{MSVVI}$. Conversely, suppose that $x^* \in \cap_{i=1}^n S^i_{MSVVI}$. Then $x^* \in K$ and

$$(Ty, y - x^*) + F(y) - F(x^*) \leq_{\mathcal{C}(C_i(x^*)\setminus\{0\})} 0 \quad \forall y \in K,\ i = 1, 2, \cdots, n.$$

This implies that $-\{(Ty, y - x^*) + F(y) - F(x^*)\} \notin C_i(x^*) \setminus \{0\}$ for all $y \in K$ and $i = 1, 2, \cdots, n$. Again since $C(x^*) = \cup_{i=1}^n C_i(x^*)$, one has $-\{(Ty, y - x^*) + F(y) - F(x^*)\} \notin C(x^*) \setminus \{0\}$ for all $y \in K$, which yields that $x^* \in S_{MSVVI}$. The proof is complete.

From Theorems 3.4 and 3.5, we obtain the following result.

Corollary 3.3. *The conclusion holds:* $\bigcap_{i=1}^n S^i_{MSVCP} \subseteq \bigcap_{i=1}^n S^i_{MSVVI}$.

3.3. In the Case of (WVF-CP)

In this subsection, we present the characterizations of solutions set for (WVF-CP).

Let $C(x) = \bigcup_{i=1}^{n} C_i(x)$, where $C_i : K \to 2^Y$ is a point-to-set mapping such that $C_i(x)$ is closed cone in Y for each $x \in K$ $(i = 1, 2, \cdots, n)$. Now, we consider the following vector F-complementarity problems and vector F-variational inequalities:

the i-th Weak Vector F-Complementarity Problem with Nonconvex Preferences (i-WVF-CP): find $x^* \in K$ such that

$$(Tx^*, x^*) + F(x^*) \geq_{\mathcal{C}(\text{int}C_i(x^*))} 0, \quad (Tx^*, y) + F(y) \leq_{\mathcal{C}(\text{int}C_i(x^*))} 0 \quad \forall y \in K;$$

the i-th Weak Vector F-Variational Inequality with Nonconvex Preferences (i-WVF-VI): find $x^* \in K$ such that

$$(Tx^*, y - x^*) + F(y) - F(x^*) \leq_{\mathcal{C}(\text{int}C_i(x^*))} 0 \quad \forall y \in K.$$

We denote by S^i_{WVCP} and S^i_{WVVI} the solutions set of (i-WVF-CP) and (i-WVF-VI), respectively.

Theorem 3.6. *The following statements are true:*

(1) $S_{WVCP} \subseteq \bigcap_{i=1}^{n} S^i_{WVCP}$;

(2) $S_{WVVI} \subseteq \bigcap_{i=1}^{n} S^i_{WVVI}$.

Proof.

(1) Let $x^* \in S_{WVCP}$. Then $x^* \in K$ and

$$(Tx^*, x^*) + F(x^*) \geq_{\mathcal{C}(\text{int}C(x^*))} 0, \quad (Tx^*, y) + F(y) \leq_{\mathcal{C}(\text{int}C(x^*))} 0 \quad \forall y \in K.$$

Since $\cup_{i=1}^{n} \text{int}C_i(x^*) \subseteq \text{int}C(x^*)$,

$$(Tx^*, x^*) + F(x^*) \notin \text{int}C_i(x^*), \quad (Tx^*, y) + F(y) \notin -\text{int}C_i(x^*)$$
$$\forall y \in K,\ i = 1, 2, \cdots, n.$$

It follows that $x^* \in S^i_{WVCP}$ for all $i = 1, 2, \cdots, n$, and so $x^* \in \cap_{i=1}^{n} S^i_{WVCP}$.

(2) Let $x^* \in S_{WVVI}$. Then $x^* \in K$ and

$$(Tx^*, y - x^*) + F(y) - F(x^*) \leq_{\mathcal{C}(\text{int}C(x^*))} 0 \quad \forall y \in K.$$

Since $\cup_{i=1}^{n} \text{int}C_i(x^*) \subseteq \text{int}C(x^*)$,

$$-\{(Tx^*, y - x^*) + F(y) - F(x^*)\} \notin \text{int}C_i(x^*) \quad \forall y \in K,\ i = 1, 2, \cdots, n.$$

It follows that $x^* \in S^i_{WVVI}$ for all $i = 1, 2, \cdots, n$, and so $x^* \in \cap_{i=1}^{n} S^i_{WVVI}$. This completes the proof.

The following examples show that the converse inclusions of Theorem 3.6 may not hold.

Example 3.7. Let $X = R$, $K = [0, +\infty)$ and $Y = R^2$. Let $C_1(z) = \{(x, y) : x \geq 0, 0 \leq y \leq x\}$, $C_2(z) = \{(x, y) : y \geq 0, 0 \leq x \leq y\}$, $F(z) = (1, 1)$ and $(Tz, u) = (u, u)$ for all $z, u \in K$. Then it is easy to show that $C(z) = C_1(z) \bigcup C_2(z) = R_+^2$ and $0 \in S_{WVCP}^1 \bigcap S_{WVCP}^2$. However, $0 \notin S_{WVCP}$.

Example 3.8. Let $X = R$, $K = [0, +\infty)$ and $Y = R^2$. Let $C_1(z) = \{(x, y) : x \geq 0, 0 \leq y \leq x\}$, $C_2(z) = \{(x, y) : y \geq 0, 0 \leq x \leq y\}$, $F(z) = (1, 1)$ and $(Tz, u) = (-u, -u)$ for all $z, u \in K$. Then it is easy to see that $C(z) = C_1(z) \bigcup C_2(z) = R_+^2$ and $0 \in S_{WVVI}^1 \bigcap S_{WVVI}^2$. However, $0 \notin S_{WVVI}$.

Remark 3.1. If let $F(x) \equiv 0$ and $C(x) \equiv C$ for each $x \in K$, then Theorems 3.1-3.6 reduce the corresponding results obtained by Huang, Rubinov and Yang [11].

4. Applications to Vector Optimization Problems with Nonconvex Preferences

In this section, we apply the results derived in section 3 to vector optimization problems in cases of strong vector optimization problem (SVOP), mild strong vector optimization problem (MSVOP) and weak vector optimization problem (WVOP), respectively.

4.1. In the Case of (SVOP)

Let A be a nonempty subsets of X and $g : A \to Y$ a mapping. We say that $x^* \in A$ is a strongly (or an ideal) minimal point of the set A with respect to K if $x^* \leq_K y$ for all $y \in A$. We say that $g(x^*) \in g(A)$ is a strongly (or an ideal) minimal point of the set $g(A)$ with respect to C if $g(x^*) \leq_{C(x^*)} g(y)$ for all $y \in A$. The set of all strongly minimal points of A and $g(A)$ are denoted by $\text{Min}_s A$ and $\text{Min}_s g(A)$, respectively.

Define the feasible set associated with T as follows:

$$\mathcal{F}_s = \{x \in K : \ (Tx, y) + F(y) \geq_{C(x)} 0, \quad \forall y \in K\}.$$

Let $f(x) = (Tx, x)$ for all $x \in K$. We now consider the *Strong Vector Optimization Problem with Nonconvex Preferences* (SVOP):

$$\text{Min}_s\{f(x) + F(x)\} \qquad \text{subject to} \quad x \in \mathcal{F}_s.$$

A point x^* is called a strongly minimal solution of (SVOP) if $f(x^*) + F(x^*)$ is a strongly minimal point of (SVOP), i.e., $f(x^*) + F(x^*) \in \text{Min}_s\{f(\mathcal{F}_s) + F(\mathcal{F}_s)\}$.

We denote by E_s the set of all strongly minimal solutions of (SVOP) and by H_s the set of all strongly minimal points of (SVOP) . Then $f(E_s) + F(E_s) = H_s$.

Theorem 4.1. *Assume that $H_s \neq \emptyset$. Then the following conclusions hold:*

(1) *If there exists $x^* \in E_s$ such that $f(x^*) + F(x^*) = 0$, then $x^* \in S_{SVCP}$;*

(2) *If there exists $x^* \in E_s$ such that $f(x^*) + F(x^*) \geq_{\mathcal{C}(\text{int}C(x^*))} 0$, then $x^* \in S_{PVCP}$.*

Proof. Since $x^* \in E_s \subseteq \mathcal{F}_s$ and $f(x^*) + F(x^*) = 0$, it follows that $x^* \in K$ and

$$(Tx^*, x^*) + F(x^*) = f(x^*) + F(x^*) = 0, \quad (Tx^*, y) + F(y) \geq_{C(x^*)} 0 \quad \forall y \in K,$$

which implies that $x^* \in S_{SVCP}$ and thus conclusion (1) holds. Now we prove that (2) is true. Since $x^* \in E_s \subseteq \mathcal{F}_s$ and $f(x^*) + F(x^*) \geq_{\mathcal{C}(\text{int}C(x^*))} 0$, we have $x^* \in K$ and

$$\begin{aligned} &(Tx^*, x^*) + F(x^*) = f(x^*) + F(x^*) \geq_{\mathcal{C}(\text{int}C(x^*))} 0, \\ &(Tx^*, y) + F(y) \geq_{C(x^*)} 0 \quad \forall y \in K, \end{aligned}$$

which shows that $x^* \in S_{PVCP}$. The proof is complete.

We now consider the following problems.

Strong Vector Optimization Problem with Nonconvex Preferences (SVOP)$_l$: for a given $l \in L(X, Y)$, finding $x^* \in \mathcal{F}_s$ such that $l(x^*) \in \text{Min}_s l(\mathcal{F}_s)$;

Strongly Minimal Element Problem with Nonconvex Preferences (SMEP): finding $x^* \in \mathcal{F}_s$ such that $x^* \in \text{Min}_s \mathcal{F}_s$;

Strong Vector Unilateral Optimization Problem with Nonconvex Preferences (SVUOP): finding $x^* \in K$ such that $f(x^*) + F(x^*) \in \text{Min}_s\{f(K) + F(K)\}$.

Let X and Y be two Banach spaces. A map $f : X \to Y$ is Frechet differentiable at $x_0 \in X$ if there exists a linear bounded operator $Df(x_0)$ such that

$$\lim_{x \to 0} \frac{\|f(x_0 + x) - f(x_0) - (Df(x_0), x)\|}{\|x\|} = 0.$$

In this case, $Df(x_0)$ is said to be the Frechet derivative of f at x_0. The map f is said to be Frechet differentiable on X if f is Frechet differentiable at each point of X.

Theorem 4.2. *Let $T = Df$ be the Frechet derivative of an operator $f : X \to Y$ and F be C-convex. Assume that $C + C(x) \subseteq C(x)$ for each $x \in K$. Then x^* solves (SVUOP) implies that $x^* \in S_{SVVI}$.*

Proof. Let x^* be a solution of (SVUOP). Then $x^* \in K$ and $f(x^*) + F(x^*) \in \mathrm{Min}_s\{f(K) + F(K)\}$, i.e., $f(x^*) + F(x^*) \leq_{C(x^*)} f(y) + F(y)$ for all $y \in K$. Since K is a convex cone,

$$f(x^*)+F(x^*) \leq_{C(x^*)} f(x^*+t(w-x^*))+F(x^*+t(w-x^*)) \quad \forall w \in K,\ 0 < t < 1.$$

It follows that

$$\frac{1}{t}[f(x^*+t(w-x^*))-f(x^*)]+\frac{1}{t}[F(x^*+t(w-x^*))-F(x^*)] \geq_{C(x^*)} 0. \tag{4.1}$$

Since F is C-convex,

$$F(x^* + t(w - x^*)) \leq_C tF(w) + (1 - t)F(x^*) \quad \forall w \in K,$$

that is,

$$\frac{1}{t}[F(x^* + t(w - x^*)) - F(x^*)] \leq_C F(w) - F(x^*) \quad \forall w \in K. \tag{4.2}$$

Since C and $C(x^*)$ are two cones with $C \subseteq C(x^*)$ and $C + C(x^*) \subseteq C(x^*)$, it follows from (4.1), (4.2) and Lemma 3.1 (iii) that

$$\frac{1}{t}[f(x^* + t(w - x^*)) - f(x^*)] + F(w) - F(x^*) \geq_{C(x^*)} 0.$$

Since f is Frechet differentiable on X and $C(x^*)$ is closed, letting $t \to 0^+$, we get

$$(Df(x^*), w - x^*) + F(w) - F(x^*) \geq_{C(x^*)} 0 \quad \forall w \in K,$$

which implies that $x^* \in S_{SVVI}$. The proof is complete.

A linear operator $l : K \to Y$ is called strongly positive if, for any $x, y \in K$,

$$x \geq_K y \Longrightarrow l(x) \geq_{C(y)} l(y).$$

Example 4.1. Let $X = R$, $K = [0, +\infty)$, $Y = R^2$ and

$$C(z) = \{(x, y) : x \geq 0, 0 \leq y \leq \tfrac{x}{3}\} \cup \{(x, y) : y \geq 0, 0 \leq x \leq \tfrac{y}{3}\} \cup \{(x, y) \\ : x > 0, 0 \leq \tfrac{y}{x} \leq \min\{z, \tfrac{1}{2}\}\}$$

for all $z \in K$. It is clear that $C(z)$ is a closed nonconvex cone for each $z \in K$ and

$$\cap_{z\in K} C(z) = \{(x, y) : x \geq 0, 0 \leq y \leq \frac{x}{3}\} \cup \{(x, y) : y \geq 0, 0 \leq x \leq \frac{y}{3}\}.$$

Let $l : K \to Y$ such that $l(x) = (4x, x)$ for any $x \in K$. It is easy to verify l is strongly positive.

Theorem 4.3. *Let $l : K \to Y$ be a linear operator. If l is strongly positive, then x^* solves (SMEP) implies that x^* solves (SVOP)$_l$.*

Proof. Let x^* be a solution of (SMEP). Then $x^* \in \mathcal{F}_s$ and $x^* \leq_K x$ for all $x \in \mathcal{F}_s$, where

$$\mathcal{F}_s = \{x \in K : (Tx, y) + F(y) \geq_{C(x)} 0, \quad \forall y \in K\}.$$

For any $z \in \mathcal{F}_s$, we know that $x^* \leq_K z$. Since l is a strongly positive linear operator, it follows that $l(x^*) \leq_{C(x^*)} l(z)$ and so

$$l(x^*) \in \text{Min}_s l(\mathcal{F}_s),$$

which implies that x^* solves (SVOP)$_l$. The proof is complete.

4.2. In the Case of (MSVOP)

Let A be a nonempty subset of X and $g : A \to Y$ a mapping. We say that $g(x^*) \in g(A)$ is a mild strongly minimal point of the set $g(A)$ with respect to C if $g(y) \leq_{\mathcal{C}(C(x^*)\setminus\{0\})} g(x^*)$ for all $y \in A$. The set of all mild strongly minimal points of $g(A)$ is denoted by $\text{Min}_m g(A)$.

Define the feasible set associated with T as follows:

$$\mathcal{F}_m = \{x \in K : (Tx, y) + F(y) \leq_{\mathcal{C}(C(x)\setminus\{0\})} 0, \quad \forall y \in K\}.$$

Let $f(x) = (Tx, x)$ for all $x \in K$. We now consider the *Mild Strong Vector Optimization Problem with Nonconvex Preferences* (MSVOP):

$$\text{Min}_m\{f(x) + F(x)\} \qquad \text{subject to} \quad x \in \mathcal{F}_m.$$

A point x^* is called a mild strongly minimal solution of (MSVOP) if $f(x^*) + F(x^*)$ is a mild strongly minimal point of (MSVOP), i.e., $f(x^*) + F(x^*) \in \text{Min}_m\{f(\mathcal{F}_m) + F(\mathcal{F}_m)\}$. We denote by E_m the set of all mild strongly minimal solutions of (MSVOP), and by H_m the set of all mild strongly minimal points of (MSVOP). Then $f(E_m) + F(E_m) = H_m$.

Theorem 4.4. *Assume that $H_m \neq \emptyset$. If there exists $x^* \in E_m$ such that $f(x^*) + F(x^*) = 0$, then $x^* \in S_{MSVCP}$.*

Proof. Let $x^* \in E_m \subseteq \mathcal{F}_m$ and $f(x^*) + F(x^*) = 0$. Then $x^* \in K$ and

$$(Tx^*, x^*) + F(x^*) = f(x^*) + F(x^*) = 0, \;\; (Tx^*, y) + F(y) \leq_{\mathcal{C}(C(x^*)\setminus\{0\})} 0 \; \forall y \in K.$$

That is to say, $x^* \in S_{MSVCP}$. The proof is complete.

4.3. In the Case of (WVOP)

Let A be a nonempty subset of X and $g : A \to Y$ a mapping. We say that $x^* \in A$ is a weakly minimal point of the set A with respect to K if $y \leq_{\mathcal{C}(\mathrm{int}K)} x^*$ for all $y \in A$. We say that $g(x^*) \in g(A)$ is a weakly minimal point of the set $g(A)$ with respect to C if $g(y) \leq_{\mathcal{C}(\mathrm{int}C(x^*))} g(x^*)$ for all $y \in A$. The set of all weakly minimal points of A and $g(A)$ are denoted by $\mathrm{Min}_w A$ and $\mathrm{Min}_w g(A)$, respectively.

Define the feasible set associated with T as follows:

$$\mathcal{F}_w = \{x \in K : (Tx, y) + F(y) \leq_{\mathcal{C}(\mathrm{int}C(x))} 0, \quad \forall y \in K\}.$$

Let $f(x) = (Tx, x)$ for all $x \in K$. We now consider the *Weak Vector Optimization Problem with Nonconvex Preferences* (WVOP):

$$\mathrm{Min}_w\{f(x) + F(x)\} \qquad \text{subject to} \quad x \in \mathcal{F}_w.$$

A point x^* is called a weakly minimal solution of (WVOP) if $f(x^*) + F(x^*)$ is a weakly minimal point of (WVOP), i.e., $f(x^*)+F(x^*) \in \mathrm{Min}_w\{f(\mathcal{F}_w)+F(\mathcal{F}_w)\}$. We denote by E_w the set of all weakly minimal solutions of (WVOP), and by H_w the set of all weakly minimal points of (WVOP). Then $f(E_w) + F(E_w) = H_w$.

Theorem 4.5. *Assume that $H_w \neq \emptyset$. If there exists $x^* \in E_w$ such that $f(x^*) + F(x^*) \geq_{\mathcal{C}(\mathrm{int}C(x^*))} 0$, then $x^* \in S_{WVCP}$.*

Proof. Let $x^* \in E_w \subseteq \mathcal{F}_w$ and $f(x^*) + F(x^*) \geq_{\mathcal{C}\mathrm{int}C(x^*)} 0$. Then $x^* \in K$ and

$$(Tx^*, x^*) + F(x^*) = f(x^*) + F(x^*) \geq_{\mathcal{C}(\mathrm{int}C(x^*))} 0,$$
$$(Tx^*, y) + F(y) \leq_{\mathcal{C}(\mathrm{int}\,C(x^*))} 0 \quad \forall y \in K.$$

It follows that $x^* \in S_{WVCP}$. The proof is complete.

Example 4.2. Let $X = R$, $K = [0, +\infty)$, $Y = R^2$ and

$$C(z) \equiv \{(x, y) : x \geq 0, 0 \leq y \leq \frac{1}{2}\} \cup \{(x, y) : y \geq 0, 0 \leq x \leq \frac{1}{2}\}$$

for all $z \in K$. Let $(Tx, y) = (y, y)$ and $F(x) = (-x, -x)$ for all $x, y \in K$. Then it is easy to see that $\mathcal{F}_w = K$, $(0, 0) \in H_w$, $0 \in E_w$ and

$$f(0) + F(0) = (0, 0) \geq_{\mathcal{C}(\mathrm{int}C(0))} (0, 0).$$

This means that all conditions of Theorem 4.5 hold and so $0 \in S_{WVCP}$.

We now consider the following problems.

Weak Vector Optimization Problem with Nonconvex Preferences (WVOP)$_l$: for a given $l \in L(X, Y)$, finding $x^* \in \mathcal{F}_w$ such that $l(x^*) \in \text{Min}_w l(\mathcal{F}_w)$;

Weak Minimal Element Problem with Nonconvex Preferences (WMEP): finding $x^* \in \mathcal{F}_w$ such that $x^* \in \text{Min}_w \mathcal{F}_w$;

Weak Vector Unilateral Optimization Problem with Nonconvex Preferences (WVUOP): finding $x^* \in K$ such that $f(x^*) + F(x^*) \in \text{Min}_w\{f(K) + F(K)\}$.

Theorem 4.6. *Let $T = Df$ be the Frechet derivative of an operator $f : X \to Y$ and F be C-convex. Assume that $C + \text{int}C(x) \subseteq \text{int}C(x)$ for all $x \in K$. Then x^* solves (WVUOP) implies that $x^* \in S_{WVVI}$.*

Proof. Let x^* be a solution of (WVUOP). Then $x^* \in K$ and $f(x^*) + F(x^*) \in \text{Min}_w\{f(K) + F(K)\}$, i.e., $f(y) + F(y) \leq_{\mathcal{C}(\text{int}C(x^*))} \{f(x^*) + F(x^*)\}$ for all $y \in K$. Since K is a convex cone,

$$f(x^*) + F(x^*) \geq_{\mathcal{C}(\text{int}C(x^*))} \{f(x^* + t(w - x^*)) + F(x^* + t(w - x^*))\}$$
$$\forall w \in K, 0 < t < 1.$$

Since $C(x^*)$ is a cone, it follows that

$$\frac{1}{t}[f(x^* + t(w - x^*)) - f(x^*)] + \frac{1}{t}[F(x^* + t(w - x^*)) - F(x^*)] \leq_{C(\text{int}C(x^*))} 0. \tag{4.3}$$

Since F is C-convex, we obtain

$$F(x^* + t(w - x^*)) \leq_C tF(w) + (1 - t)F(x^*) \quad \forall w \in K,$$

that is,

$$\frac{1}{t}[F(x^* + t(w - x^*)) - F(x^*)] \leq_C F(w) - F(x^*) \quad \forall w \in K. \tag{4.4}$$

Note that C and $C(x^*)$ are two cones with $C \subseteq C(x^*)$ and $C + \text{int}C(x^*) \subseteq \text{int}C(x^*)$. It follows from (4.3), (4.4) and Lemma 3.1 (iv) that

$$\frac{1}{t}[f(x^* + t(w - x^*)) - f(x^*)] + F(w) - F(x^*) \leq_{\mathcal{C}(\text{int}C(x^*))} 0 \quad \forall w \in K.$$

Since f is Frechet differentiable on X and $\mathcal{C}(\text{int}C(x^*))$ is closed, letting $t \to 0^+$ in above relation, we get

$$(Df(x^*), w - x^*) + F(w) - F(x^*) \leq_{\mathcal{C}(\text{int}C(x^*))} 0 \quad \forall w \in K,$$

which implies that $x^* \in S_{WVVI}$. The proof is complete.

Example 4.3. Let $X = R, K = [0, +\infty), Y = R^2$ and

$$C(z) = \begin{cases} R^2_+, & 0 \le z \le \frac{\pi}{2}, \\ R^2_+ \bigcup \{(x, y) : -z + \frac{\pi}{2} \le \arctan \frac{y}{x} \le z\}, & \pi \ge z > \frac{\pi}{2}, \\ R^2 \backslash (-\mathrm{int} R^2_+), & z > \pi. \end{cases}$$

Let $f(x) = (-x, -x)$ and $F(x) = (x, x)$. Then $(Tx, y) = (Df(x), y) = (-y, -y)$ for all $x, y \in K$. It is easy to check that all assumptions in Theorem 4.6 hold and so 0 solves (WVUOP) implies that $0 \in S_{WVVI}$.

A linear operator $l : K \to Y$ is called weakly positive if, for any $x, y \in K$,

$$x \le_{\mathcal{C}(\mathrm{int}K)} y \Longrightarrow l(x) \le_{\mathcal{C}(\mathrm{int}C(y))} l(y).$$

Example 4.4. Let $X = R, K = [0, +\infty), Y = R^2$ and $C(z) = \{(x, y) : x \ge 0, 0 \le y \le \frac{x}{4}\} \cup \{(x, y) : y \ge 0, 0 \le x \le \frac{y}{3}\} \cup \{(x, y) : x > 0, 0 \le \frac{y}{x} \le \min\{z, \frac{1}{3}\}\}$ for all $z \in K$. It is clear that $C(z)$ is a closed nonconvex cone for each $z \in K$ and $\cap_{z \in K} C(z) = \{(x, y) : x \ge 0, 0 \le y \le \frac{x}{4}\} \cup \{(x, y) : y \ge 0, 0 \le x \le \frac{y}{3}\}$. Let $l : K \to Y$ such that $l(x) = (x, 4x)$ for any $x \in K$. Then l is weakly positive.

Theorem 4.7. *Let $l : K \to Y$ be a linear operator. If l is weakly positive, then x^* solves (WMEP) implies x^* solves (WVOP)$_l$.*

Proof. Let x^* be a solution of (WMEP). Then $x^* \in \mathcal{F}_w$ and $y \le_{\mathcal{C}(\mathrm{int}K)} x^*$ for all $y \in \mathcal{F}_w$, where

$$\mathcal{F}_w = \{x \in K,\ (Tx, y) + F(y) \le_{\mathcal{C}(\mathrm{int}C(x))} 0, \quad \forall y \in K\}.$$

For any $z \in \mathcal{F}_w$, we know that $z \le_{\mathcal{C}(\mathrm{int}K)} x^*$. Since l is a weakly positive linear operator, it follows that $l(z) \le_{\mathcal{C}(\mathrm{int}C(x^*))} l(x^*)$ for all $z \in \mathcal{F}_w$. Thus x^* solves (WVOP)$_l$. The proof is complete.

Remark 4.1. If let $F(x) \equiv 0$ and $C(x) \equiv C$ for each $x \in K$, then Theorems 4.1-4.7 reduce the corresponding results obtained by Huang, Rubinov and Yang [11].

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