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# ON $\mathcal{I}$-CAUCHY SEQUENCES

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**Abstract.** The concept of $\mathcal{I}$-convergence is a generalization of statistical convergence and it is dependent on the notion of the ideal $\mathcal{I}$ of subsets of the set $\mathbb{N}$ of positive integers. In this paper we prove a decomposition theorem for $\mathcal{I}$-convergent sequences and we introduce the notions of $\mathcal{I}$ Cauchy sequence and $\mathcal{I}^*$-Cauchy sequence, and then study their certain properties.

## 1. Introduction and Background

P. Kostyrko et al. [12] introduced the concept of $\mathcal{I}$-convergence of sequences in a metric space and studied some properties of this convergence. Note that $\mathcal{I}$-convergence is an interesting generalization of statistical convergence.

The concept of statistical convergence was introduced by Steinhaus [21] in $1951$ (see also Fast [5]) and has been discussed and developed by many authors including [2, 4, 7-9, 15-18, 20].

Let $\mathbb{N}$ denote the set of all positive integers and $(X, \rho)$ be a linear metric space. Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $X$ is said to be statistically convergent to $x \in X$ if the set $A(\varepsilon) = \{n \in \mathbb{N} : \rho(x_n, x) \geq \varepsilon\}$ has natural density zero for each $\varepsilon > 0$.

In general, statistically convergent sequences satisfy many of the properties of ordinary convergent sequences in metric spaces. For instance, a statistically convergent sequence is statistically Cauchy, ([7, 19]) in an arbitrary metric space. In this paper we investigate some properties of $\mathcal{I}$-convergent sequences in a linear metric space. In section 2 we prove the decomposition theorem of $\mathcal{I}$-convergent sequences in a linear metric space and give some results regarding this theorem. In section $3$ we introduce the notions of $\mathcal{I}$-Cauchy sequence and $\mathcal{I}^*$-Cauchy sequence, and study their certain properties.

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Now we give some definitions and notations.

**Definition 1.** [14] Let $Y \neq \emptyset$. A family $\mathcal{I} \subset 2^Y$ of subsets of $Y$ is said to be an ideal in $Y$ provided that the following conditions hold:
(a) $\emptyset \in \mathcal{I}$
(b) $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$
(c) $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$.

**Definition 2.** [11] Let $Y \neq \emptyset$. A non-empty family $\mathcal{F} \subset 2^Y$ is said to be a filter on $Y$ if the following are satisfied:
(a) $\emptyset \notin \mathcal{F}$
(b) $A, B \in \mathcal{F}$ imply $A \cap B \in \mathcal{F}$
(c) $A \in \mathcal{F}$, $A \subset B \subset Y$ imply $B \in \mathcal{F}$.

**Lemma 1.** [13] *Let $\mathcal{I}$ be a proper ideal in $Y$ (i.e. $Y \notin \mathcal{I}$), $Y \neq \emptyset$. Then the family of sets*

$$\mathcal{F}(\mathcal{I}) = \{M \subset Y : \exists A \in \mathcal{I} : M = Y \backslash A\}$$

*is a filter in $Y$. It is called the filter associated with the ideal $\mathcal{I}$.*

**Definition 3.** [13] A proper ideal $\mathcal{I}$ is said to be admissible if $\{x\} \in \mathcal{I}$ for each $x \in Y$.

**Definition 4.** [12, 13] Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a proper ideal in $\mathbb{N}$ and $(X, \rho)$ be a metric space. The sequence $x = (x_n)$ of elements of $X$ is said to be $\mathcal{I}$-convergent to $\xi \in X$ if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : \rho(x_n, \xi) \geq \varepsilon\}$ belongs to $\mathcal{I}$.

If $x = (x_n)$ is $\mathcal{I}$-convergent to $\xi$ then we write $\mathcal{I}\text{-}\lim\limits_{n\to\infty} x_n = \xi$. In this case the element $\xi \in X$ is called $\mathcal{I}$-limit of the sequence $x = (x_n) \in X$.

There are many examples of ideals $\mathcal{I} \subset 2^{\mathbb{N}}$ in [12, 13], and basic properties of $\mathcal{I}$-convergence have been studied in these works. Note that the $\mu$-statistical convergence of [1] is in a sense equivalent to $\mathcal{I}$-convergence (see [13]).

**Definition 5.** [12] An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to have the property (AP) if for any sequence $\{A_1, A_2, ...\}$ of mutually disjoint sets of $\mathcal{I}$, there is a sequence $\{B_1, B_2, ...\}$ of sets such that each symmetric difference $A_i \Delta B_i$ $(i = 1, 2, ...)$ is finite and $\bigcup\limits_{i=1}^{\infty} B_i \in \mathcal{I}$.

Definition 5 is similar to the condition (APO) used in [6].

In [12], the concept of $\mathcal{I}^*$-convergence which is closely related to the $\mathcal{I}$-convergence has been introduced.

**Definition 6.** [12] A sequence $x = \{x_n\}_{n\in\mathbb{N}}$ of elements of $X$ is said to be $\mathcal{I}^*$-convergent to $\xi \in X$ if and only if there exists a set $M \in \mathcal{F}(\mathcal{I})$, $M = \{m_1 < m_2 < ... < m_k < ...\} \subset \mathbb{N}$ such that $\lim_{k\to\infty} \rho(x_{m_k}, \xi) = 0$.

In paper [12] it is proved that $\mathcal{I}$ and $\mathcal{I}^*$-convergence are equivalent for admissible ideals with property (AP).

**Lemma 2.** ([12]) *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal with the property (AP) and $(X, \rho)$ be an arbitrary metric space. Then $\mathcal{I}$- $\lim_{n\to\infty} x_n = \xi$ if and only if there exists a set $P \in \mathcal{F}(\mathcal{I})$, $P = \{p_1 < p_2 < ... < p_k < ...\}$ such that $\lim_{k\to\infty} \rho(x_{p_k}, \xi) = 0$.*

**Remark 1.** Let $\mathcal{I} = \mathcal{I}_d$ and $X = \mathbb{R}$ with the usual metric, where $\mathcal{I}_d = \{A \subset \mathbb{N} : d(A) = 0\}$, and $d(A)$ is the natural density of the set $A \subset \mathbb{N}$. Then Lemma 2 is equivalent to the relation between statistical convergence and ”almost all” convergence of a real number sequence $(x_n)$ considered in [7].

## 2. The Decomposition Theorem

In this section we prove a decomposition theorem for $\mathcal{I}$ -convergent sequences.

**Theorem 1.** *Let $(X, \rho)$ be a linear metric space, $x = (x_n) \in X$ and $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal with property (AP). Then the following conditions are equivalent:*

$(a)$ $\mathcal{I}$- $\lim_{n\to\infty} x_n = \xi$

$(b)$ *There exist $y = (y_n) \in X$ and $z = (z_n) \in X$ such that $x = y + z$, $\lim_{n\to\infty} \rho(y_n, \xi) = 0$ and $supp\ z \in \mathcal{I}$, where $supp\ z = \{n \in \mathbb{N} : z_n \neq \theta\}$ and $\theta$ is the zero element of $X$.*

*Proof.* Let $\mathcal{I}$- $\lim_{n\to\infty} x_n = \xi$. Then by Lemma 2 we conclude that there exists a set $M \in \mathcal{F}(\mathcal{I})$, $M = \{m_1 < m_2 < ... < m_k < ...\}$ such that $\lim_{k\to\infty} \rho(x_{m_k}, \xi) = 0$. Now define the sequence $y = (y_n)$ in $X$ as

$$y = \begin{cases} x_n & , \ n \in M \\ \xi & , \ n \in \mathbb{N}\backslash M \end{cases} \tag{2.1}$$

It is clear that $\lim_{n\to\infty} \rho(y_n, \xi) = 0$. Further, put $z_n = x_n - y_n$, $n \in \mathbb{N}$. Since $\{k \in \mathbb{N} : x_k \neq y_k\} \subset \mathbb{N}\backslash M \in \mathcal{I}$ we have $\{k \in \mathbb{N} : z_k \neq 0\} \in \mathcal{I}$. It follows that $supp\ z \in \mathcal{I}$ and by $(2.1)$ we get $x = y + z$.

Now suppose that there exist two sequences $y = (y_n) \in X$ and $z = (z_n) \in X$ such that $x = y + z$, $\lim_{n\to\infty} \rho(y_n, \xi) = 0$ and $supp\ z \in \mathcal{I}$. We will prove that $\mathcal{I}\text{-}\lim_{n\to\infty} x_n = \xi$. Define $M = \{m_k\}$ to be a subset of $\mathbb{N}$ such that $M = \{m \in \mathbb{N} : z_m = 0\}$. Since $supp\ z = \{m \in \mathbb{N} : z_m \neq 0\} \in \mathcal{I}$, we have $M \in \mathcal{F}(\mathcal{I})$, hence $x_n = y_n$ if $n \in M$. Thus, we conclude that there exists a set $M = \{m_1 < m_2 < ...\}$, $M \in \mathcal{F}(\mathcal{I})$ such that $\lim_{k\to\infty} \rho(x_{m_k}, \xi) = 0$. Now, by Lemma 2 it follows that $\mathcal{I}\text{-}\lim_{n\to\infty} x_n = \xi$. Hence the proof is complete. ■

**Corollary 1.** $\mathcal{I}\text{-}\lim_{n\to\infty} x_n = \xi$ *if and only if there exist* $(y_n) \in X$ *and* $(z_n) \in X$ *such that* $x_n = y_n + z_n$, $\lim_{n\to\infty} \rho(y_n, \xi) = 0$ *and* $\mathcal{I}\text{-}\lim_{n\to\infty} z_n = 0$.

*Proof.* Let $z_n = x_n - y_n$, where $(y_n)$ is the sequence defined by $(2.1)$. Then $\lim_{n\to\infty} \rho(y_n, \xi) = 0$, and by Theorem 1 in [13] we conclude that $\mathcal{I}\text{-}\lim_{n\to\infty} z_n = 0$.

Let $x_n = y_n + z_n$, where $\lim_{n\to\infty} \rho(y_n, \xi) = 0$ and $\mathcal{I}\text{-}\lim z_n = 0$. Since $\mathcal{I}\text{-}\lim_{n\to\infty} y_n = \xi$, then by Theorem 1 in [13] we get $\mathcal{I}\text{-}\lim_{n\to\infty} x_n = \xi$. ■

**Remark 2.** From the proof of Theorem 1, it is clear that if $(b)$ is satisfied then the ideal $\mathcal{I}$ need not have the property (AP). In fact, let $x_n = y_n + z_n$, $\lim_{n\to\infty} \rho(y_n, \xi) = 0$ and $supp\ z \in \mathcal{I}$ where $\mathcal{I}$ is an admissible ideal which has not the property (AP). Since $A(\varepsilon) = \{n \in \mathbb{N} : \rho(z_n, 0) \geq \varepsilon\} \subset \{n \in \mathbb{N} : z_n \neq 0\} \in \mathcal{I}$ for each $\varepsilon > 0$, we have $\mathcal{I}\text{-}\lim_{n\to\infty} z_n = 0$. Thus, we have $\mathcal{I}\text{-}\lim_{n\to\infty} x_n = \xi$.

**Remark 3.** By Theorem 1 we can obtain the decomposition theorem for a statistically convergent sequence considered in [1] and [20].

By Remark 2 and Theorem 1 we get the following theorem.

**Theorem 2.** *Let* $C_0(\mathcal{I}, X)$ *be the set of all sequences which are* $\mathcal{I}$*-convergent to the zero element of the linear metric space* $(X, \rho)$ *and let* $Supp\ (\mathcal{I}, X)$ *be the set of all sequences* $z \in C_0(\mathcal{I}, X)$ *with supp* $z \in \mathcal{I}$. *Then* $C_0(\mathcal{I}, X) \supset Supp\ (\mathcal{I}, X)$ *for each admissible ideal* $\mathcal{I}$.

## 3. $\mathcal{I}$-Cauchy Sequences

Now we introduce the notions of $\mathcal{I}$ Cauchy sequence and $\mathcal{I}^*$-Cauchy sequence.

**Definition 7.** Let $(X, \rho)$ be a linear metric space and $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. Then a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $X$ is called an $\mathcal{I}$-Cauchy sequence in $X$ if for every $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that

$$A(\varepsilon) = \{n \in \mathbb{N} : \rho(x_n, x_N) \geq \varepsilon\} \in \mathcal{I}.$$

**Definition 8.** Let $(X,\rho)$ be a linear metric space and $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. Then a sequence $x=(x_n)$ in $X$ is called an $\mathcal{I}^*$-Cauchy sequence if there exists a set $M=\{m_1<m_2<...<m_k<...\} \subset \mathbb{N}$, $M \in \mathcal{F}(\mathcal{I})$ such that the subsequence $x_M=(x_{m_k})$ is an ordinary Cauchy sequence in $X$, i.e.,

$$\lim_{k,p\to\infty} \rho\left(x_{m_k}, x_{m_p}\right)=0.$$

**Theorem 3.** *Let $\mathcal{I}$ be an admissible ideal. If $x=(x_n)$ is an $\mathcal{I}^*$-Cauchy sequence then it is $\mathcal{I}$-Cauchy.*

*Proof.* Let $x=(x_n)$ be an $\mathcal{I}^*$-Cauchy sequence. Then by definition, there exists a set $M=\{m_1<m_2<...<m_k<...\} \subset \mathbb{N}$, $M \in \mathcal{F}(\mathcal{I})$ such that $\rho\left(x_{m_k}, x_{m_p}\right)<\varepsilon$ for every $\varepsilon>0$ and for all $k,p>k_0=k_0(\varepsilon)$.

Let $N=N(\varepsilon)=m_{k_0+1}$. Then for every $\varepsilon>0$, we have

$$\rho\left(x_{m_k}, x_N\right)<\varepsilon, \qquad k>k_0.$$

Now let $H=\mathbb{N}\backslash M$. It is clear that $H \in \mathcal{I}$ and

$$A(\varepsilon)=\{n \in \mathbb{N} : \rho(x_n, x_N) \geq \varepsilon\} \subset H \cup \{m_1<m_2<...<m_{k_0}\} \tag{3.1}$$

Then the set on the right hand side of $(3.1)$ belongs to $\mathcal{I}$. Therefore, for every $\varepsilon>0$ we can find an $N=N(\varepsilon)$ such that $A(\varepsilon) \in \mathcal{I}$, i.e. $(x_n)$ is $\mathcal{I}$-Cauchy. Hence the proof is complete. ■

Now we will prove that $\mathcal{I}$-convergence implies the $\mathcal{I}$-Cauchy condition.

**Lemma 3.** *Let $\mathcal{I}$ be an arbitrary admissible ideal. Then $\mathcal{I}$-$\lim_{n\to\infty} x_n=\xi$ implies that $(x_n)$ is an $\mathcal{I}$-Cauchy sequence.*

*Proof.* Let $\mathcal{I}$-$\lim_{n\to\infty} x_n=\xi$. Then for each $\varepsilon>0$, we have $A(\varepsilon)=\{n \in \mathbb{N} : \rho(x_n,\xi) \geq \varepsilon\} \in \mathcal{I}$. Since $\mathcal{I}$ is an admissible ideal, there exists an $n_0 \in \mathbb{N}$ such that $n_0 \notin A(\varepsilon)$. Let $B(\varepsilon)=\{n \in \mathbb{N} : \rho(x_n, x_{n_0}) \geq 2\varepsilon\}$. Taking into account the inequality $\rho(x_n,\xi)+\rho(x_{n_0},\xi) \geq \rho(x_n, x_{n_0})$ we observe that if $n \in B(\varepsilon)$ then $\rho(x_n,\xi)+\rho(x_{n_0},\xi) \geq 2\varepsilon$.

On the other hand, since $n_0 \notin A(\varepsilon)$ we have $\rho(x_{n_0},\xi)<\varepsilon$. Here we conclude that $\rho(x_n,\xi)>\varepsilon$, hence $n \in A(\varepsilon)$. Observe that $B(\varepsilon) \subset A(\varepsilon) \in \mathcal{I}$ for each $\varepsilon>0$. This gives that $B(\varepsilon) \in \mathcal{I}$, i.e. $(x_n)$ is an $\mathcal{I}$-Cauchy sequence. ■

To prove that an $\mathcal{I}$-Cauchy sequence coincides with an $\mathcal{I}^*$-Cauchy sequence for admissible ideals with property (AP), we need the following lemma.

**Lemma 4.** *Let $\{P_i\}_{i=1}^{\infty}$ be a countable collection of subsets of $\mathbb{N}$ such that $P_i \in \mathcal{F}(\mathcal{I})$ for each $i$, where $\mathcal{F}(\mathcal{I})$ is a filter associate with an admissible ideal $\mathcal{I}$ with property (AP). Then there exists a set $P \subset \mathbb{N}$ such that $P \in \mathcal{F}(\mathcal{I})$ and the set $P \backslash P_i$ is finite for all $i$.*

*Proof.* Let $A_1 = \mathbb{N} \backslash P_1,\ A_2 = (\mathbb{N} \backslash P_2) \backslash A_1,\ A_3 = (\mathbb{N} \backslash P_3) \backslash (A_1 \cup A_2)$, and $A_m = (\mathbb{N} \backslash P_m) \backslash (A_1 \cup A_2 \cup ... \cup A_{m-1}),\ m = 2, 3, ....$ It is easy to see that $A_i \in \mathcal{I}$ for each $i$ and $A_i \cap A_j = \emptyset$, when $i \neq j$. Then by (AP) property of $\mathcal{I}$ we conclude that there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_j \Delta B_j$ is a finite set for $j \in \mathbb{N}$ and $B = \bigcup\limits_{j=1}^{\infty} B_j \in \mathcal{I}$. Put $P = \mathbb{N} \backslash B$. It is clear that $P \in \mathcal{F}(\mathcal{I})$.

Now prove that the set $P \backslash P_i$ is finite for each $i$. Assume that there exists a $j_0 \in \mathbb{N}$ such that $P \backslash P_{j_0}$ has infinitely many elements. Since each $A_j \Delta B_j$ ( $j = 1, 2, ..., j_0$) is a finite set, there exists $n_0 \in \mathbb{N}$ such that

$$\bigcup_{j=1}^{j_0} B_j \cap \{n \in \mathbb{N} : n > n_0\} = \bigcup_{j=1}^{j_0} A_j \cap \{n \in \mathbb{N} : n > n_0\} \tag{3.2}$$

If $n > n_0$ and $n \notin B$, then $n \notin \bigcup\limits_{j=1}^{j_0} B_j$ and, by $(3.2)$ $n \notin \bigcup\limits_{j=1}^{j_0} A_j$. Since $A_{j_0} = (\mathbb{N} \backslash P_{j_0}) \backslash \bigcup\limits_{j=1}^{j_0-1} A_j$ and $n \notin A_{j_0},\ n \notin \bigcup\limits_{j=1}^{j_0-1} A_j$ we have $n \in P_{j_0}$ for $n > n_0$. Therefore, for all $n > n_0$ we get $n \in P$ and $n \in P_{j_0}$. This shows that the set $P \backslash P_{j_0}$ has a finite number of elements. This contradicts to our assumption that the set $P \backslash P_{j_0}$ is an infinite set. Hence the proof is complete. ∎

Theorem 4. *If $\mathcal{I}$ is an admissible ideal with property (AP) then the concepts $\mathcal{I}$-Cauchy sequence and $\mathcal{I}^*$-Cauchy sequence coincide.*

*Proof.* If a sequence is $\mathcal{I}^*$-Cauchy, then it is $\mathcal{I}$-Cauchy by Theorem $3$ where $\mathcal{I}$ need not have the (AP) property. Now it is sufficient to prove that $x = (x_n)$ in $X$ is a $\mathcal{I}^*$-Cauchy sequence under assumption that $(x_n)$ is an $\mathcal{I}$-Cauchy sequence. Let $x = (x_n)$ in $X$ be an $\mathcal{I}$-Cauchy sequence. Then by definition, there exists an $N = N(\varepsilon)$ such that

$$A(\varepsilon) = \{n \in \mathbb{N} : \rho(x_n, x_N) \geq \varepsilon\} \in \mathcal{I} \text{ for every } \varepsilon > 0.$$

Let $P_i = \left\{n \in \mathbb{N} : \rho(x_n, x_{m_i}) < \frac{1}{i}\right\},\ i = 1, 2, ...$ where $m_i = N\left(\frac{1}{i}\right)$. It is clear that $P_i \in \mathcal{F}(\mathcal{I})$ for $i = 1, 2, ...$ . Since $\mathcal{I}$ has the (AP) property, then by Lemma $4$

there exists a set $P \subset \mathbb{N}$ such that $P \in \mathcal{F}(\mathcal{I})$, and $P \backslash P_i$ is finite for all $i$. Now we show that

$$\lim_{\substack{n,m\to\infty \\ m,n\in P}} \rho(x_n, x_m) = 0.$$

To prove this, let $\varepsilon > 0$ and $j \in \mathbb{N}$ such that $j > \frac{2}{\varepsilon}$. If $m, n \in P$ then $P \backslash P_j$ is a finite set, so there exists $k = k(j)$ such that $m \in P_j$ and $n \in P_j$ for all $m, n > k(j)$. Therefore, $\rho\left(x_n, x_{m_j}\right) < \frac{1}{j}$ and $\rho\left(x_m, x_{m_j}\right) < \frac{1}{j}$ for all $m, n > k(j)$. Hence it follows that

$$\begin{aligned} \rho(x_n, x_m) &< \rho\left(x_n, x_{m_j}\right) + \rho\left(x_m, x_{m_j}\right) \\ &< \varepsilon \quad \text{for} \quad m, n > k(j). \end{aligned}$$

Thus, for any $\varepsilon > 0$ there exists $k = k(\varepsilon)$ such that for $n, m > k(\varepsilon)$ and $n, m \in P \in \mathcal{F}(\mathcal{I})$

$$\rho(x_n, x_m) < \varepsilon.$$

This shows that the sequence $(x_n)$ in $X$ is an $\mathcal{I}^*$-Cauchy sequence. ∎

Note that all these results imply the similar theorems for statistically Cauchy sequences which are investigated in [7] and [19].

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