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WEIGHTED BOUNDEDNESS OF MULTILINEAR OPERATORS FOR THE EXTREME CASES

Liu Lanzhe

Abstract. In this paper, the weighted boundedness of multilinear operators related to some non-convolution operators for the extreme cases are obtained. The operators include Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

1. Introduction

Let T be a Calderón-Zygmund operator, a classical result of Coifman, Rochberg and Weiss (see [8]) states that the commutator $[b, T] = bTf - T(bf)$(where $b \in BMO(R^n)$) is bounded on $L^p(R^n)$ for $1 < p < \infty$; Chanillo (see [2]) proved a similar result when T is replaced by the fractional integral operator. In [11], the endpoint boundedness of the commutators are obtained. The main purpose of this paper is to discuss the weighted endpoint boundedness of multilinear operators related to some non-convolution operators. In fact, we shall establish the weighted boundedness in the extreme cases of p for the multilinear operators related to some non-convolution operator only under certain size conditions of the operators. As application, the weighted endpoint boundedness of the multilinear operators related to the Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator are obtained.

2. Preliminaries

Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a cube Q and a locally integrable function f, let $f(Q) = \int_Q f(x)dx$,

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$f_Q = |Q|^{-1}\int_Q f(x)dx$ and $f^{\#}(x) = \sup_{x\in Q}|Q|^{-1}\int_Q |f(y) - f_Q|dy$. Moreover, for a weight function w, f is said to belong to $BMO(w)$ if $f^{\#} \in L^\infty(w)$ and define $||f||_{BMO(w)} = ||f^{\#}||_{L^\infty(w)}$, if $w = 1$, we denote that $BMO(w) = BMO(R^n)$. Also, we give the concepts of the atom and weighted H^1 space. A function a is called a H^1 atom if there exists a cube Q such that a is supported on Q, $||a||_{L^\infty(w)} \leq w(Q)^{-1}$ and $\int a(x)dx = 0$. It is well known that the weighted Hardy space $H^1(w)$ has the atomic decomposition characterization(see[1] [10]).

In this paper, we will consider a class of multilinear operators related to some integral operators as follows.

Let m be a positive integer and A be a function on R^n. We denote

$$R_{m+1}(A; x, y) = A(x) - \sum_{|\alpha|\leq m}\frac{1}{\alpha!}(x-y)^\alpha D^\alpha A(y)$$

and

$$Q_{m+1}(A; x, y) = R_m(A; x, y) - \sum_{|\alpha|=m}(x-y)^\alpha D^\alpha A(x).$$

Definition 1. Let $F(x, y, t)$ be a function defined on $R^n \times R^n \times [0, +\infty)$, and let

$$F_t(f)(x) = \int_{R^n} F(x, y, t)f(y)dy$$

and

$$F_t^A(f)(x) = \int_{R^n}\frac{R_{m+1}(A; x, y)}{|x-y|^m}F(x, y, t)f(y)dy.$$

Let H be a Banach space $H = \{h : ||h|| < \infty\}$ such that $F_t(f)(x)$ and $F_t^A(f)(x)$ may be viewed as a mapping from $[0, +\infty)$ to H for each fixed $x \in R^n$. Then, the multilinear operators related to F_t is defined by

$$T^A(f)(x) = ||F_t^A(f)(x)||.$$

We also define that $T(f)(x) = ||F_t(f)(x)||$.

In particular, we shall study the following sublinear operators.

Definition 2. Let $\varepsilon > 0$ and ψ be a fixed function which satisfies the following properties:

(1) $\int_{R^n}\psi(x)dx = 0$,

(2) $|\psi(x)| \leq C(1 + |x|)^{-(n+1)}$,

(3) $|\psi(x+y) - \psi(x)| \leq C|y|^\varepsilon(1 + |x|)^{-(n+1+\varepsilon)}$ for $2|y| < |x|$.

The multilinear Littlewood-Paley operator is defined by

$$g_\psi^A(f)(x) = \left(\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

where

$$F_t^A(f)(x) = \int_{R^n} \frac{R_{m+1}(A;x,y)}{|x-y|^m}\psi_t(x-y)f(y)dy,$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. Let $F_t(f) = \psi_t * f$. We also define

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

which is the Littlewood-Paley g function (see [17]).

We also consider the variant of g_ψ^A, which is defined by

$$\tilde{g}_\psi^A(f)(x) = \left(\int_0^\infty |\tilde{F}_t^A(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

where

$$\tilde{F}_t^A(f)(x) = \int_{R^n} \frac{Q_{m+1}(A;x,y)}{|x-y|^m}\psi_t(x-y)f(y)dy.$$

Definition 3. Let $0 < \gamma \le 1$ and Ω be homogeneous of degree zero on R^n such that $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$, that is, there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \le M|x-y|^\gamma$.

We denote $\Gamma(x) = \{(y,t) \in R_+^{n+1} : |x-y| < t\}$ and the characteristic of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The multilinear Marcinkiewicz operator is defined by

$$\mu_\Omega^A(f)(x) = \left[\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t^3}\right]^{1/2},$$

where

$$F_t^A(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \frac{R_{m+1}(A;x,y)}{|x-y|^m} f(y)dy.$$

Let

$$F_t(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

We also define that

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where the last is the Marcinkiewicz operator (see [18]). Also, the variant of μ_Ω^A is defined by

$$\tilde{\mu}_\Omega^A(f)(x) = \left(\int_0^\infty |\tilde{F}_t^A(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$\tilde{F}_t^A(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \frac{Q_{m+1}(A;x,y)}{|x-y|^m} f(y)dy.$$

Definition 4. Let $B_t^\delta(f)\hat{}(\xi) = (1-t^2|\xi|^2)_+^\delta \hat{f}(\xi)$. We denote

$$B_{\delta,t}^A(f)(x) = \int_{R^n} \frac{R_{m+1}(A;x,y)}{|x-y|^m} B_t^\delta(x-y)f(y)dy,$$

where $B_t^\delta(z) = t^{-n}B^\delta(z/t)$ for $t > 0$. The maximal multilinear Bochner-Riesz operator is defined by

$$B_{\delta,*}^A(f)(x) = \sup_{t>0} |B_{\delta,t}^A(f)(x)|.$$

We also define

$$B_*^\delta(f)(x) = \sup_{t>0} |B_t^\delta(f)(x)|,$$

which is the maximal Bochner-Riesz operator (see [12][14][15]). The variant of $B_{\delta,*}^A$ is defined by

$$\tilde{B}_{\delta,*}^A(f)(x) = \sup_{t>0} |\tilde{B}_{\delta,t}^A(f)(x)|,$$

where

$$\tilde{B}_{\delta,t}^A(f)(x) = \int_{R^n} \frac{Q_{m+1}(A;x,y)}{|x-y|^m} B_t^\delta(x-y)f(y)dy.$$

For g_ψ, let H be the space $H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t\right)^{1/2} < \infty\right\}$. Then for each fixed $x \in R^n$, $F_t^A(f)(x)$ and $F_t(f)(x)$ may be viewed as the mapping from $[0,+\infty)$ to H. It is clear that

$$g_\psi(f)(x) = ||F_t(f)(x)|| \text{ and } g_\psi^A(f)(x) = ||F_t^A(f)(x)||;$$

For μ_Ω, let H be the space $H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t^3\right)^{1/2} < \infty\right\}$. Then for each fixed $x \in R^n$, $F_t^A(f)(x)$ and $F_t(f)(x)$ may be viewed as the mapping from $(0,+\infty)$ to H, and it is clear that

$$\mu_\Omega^A(f)(x) = ||F_t^A(f)(x)|| \text{ and } \mu_\Omega(f)(x) = ||F_t(f)(x)||;$$

For $B_{\delta,*}$, let H be the space $H = \left\{h : ||h|| = \sup_{t>0} |h(t)| < \infty\right\}$. Then it is clear that

$$B_*^\delta(f)(x) = ||B_t^\delta(f)(x)|| \text{ and } B_{\delta,*}^A(f)(x) = ||B_{\delta,t}^A(f)(x)||.$$

It is obvious that Definition 2, 3 and 4 are the particular examples of Definition 1. Note that when $m = 0$, T^A, g_ψ^A, μ_Ω^A and $B_{\delta,*}^A$ is just the commutators generated by F_t and A(see[12-14][18]). It is well-known that multilinear operator, as an extension of commutator, is of great interest in harmonic analysis and has been widely studied by many authors (see [3-7, 9]).

We shall prove the following theorems in Section 3.

Theorem 1. *Let $w \in A_1$ and $D^\alpha A \in BMO(R^n)$ for all α with $|\alpha| = m$. Then*

(i) *g_ψ^A is bounded from $L^\infty(w)$ to $BMO(w)$;*

(i) *$\tilde{g}_\psi^A$ is bounded from $H^1(w)$ to $L^1(w)$;*

(iii) *g_ψ^A is bounded from $H^1(w)$ to weak $L^1(w)$;*

(iv) *If for any $H^1(w)$-atom a supported on certain cube Q and $u \in 3Q \setminus 2Q$, there is*

$$\int_{(4Q)^c} \left\|\sum_{|\alpha|=m} \frac{1}{\alpha!} \frac{(x-u)^\alpha}{|x-u|^m} \psi_t(x-u) \int_Q D^\alpha A(y) a(y) dy\right\| w(x)dx \le C,$$

then g_ψ^A is bounded from $H^1(w)$ to $L^1(w)$;

(v) *If for any cube Q and $u \in 3Q \setminus 2Q$, there is*

$$\frac{1}{w(Q)} \int_Q \left\|\sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha A(x) - (D^\alpha A)_Q) \int_{(4Q)^c} \frac{(u-y)^\alpha}{|u-y|^m} \psi_t(u-y) f(y) dy\right\|$$
$$w(x)dx \le C||f||_{L^\infty(w)},$$

then $\tilde{g}_\psi^A$ is bounded from $L^\infty(w)$ to $BMO(w)$.

Theorem 2. *Let $w \in A_1$ and $D^\alpha A \in BMO(R^n)$ for all α with $|\alpha| = m$. Then*

(i) *μ_Ω^A is bounded from $L^\infty(w)$ to $BMO(w)$;*

(ii) *$\tilde{\mu}_\Omega^A$ is bounded from $H^1(w)$ to $L^1(w)$;*

(ii) *μ_Ω^A is bounded from $H^1(w)$ to weak $L^1(w)$.*

(iv) *If for any $H^1(w)$-atom a supported on certain cube Q and $u \in 3Q \setminus 2Q$, there is*

$$\int_{(4Q)^c}\left\|\sum_{|\alpha|=m}\frac{1}{\alpha!}\frac{(x-u)^\alpha}{|x-u|^m}\frac{\Omega(x-u)}{|x-u|^{n-1}}\chi_{\Gamma(x)}(u,t)\int_Q D^\alpha A(y)a(y)dy\right\|w(x)dx\le C,$$

then μ_Ω^A is bounded from $H^1(w)$ to $L^1(w)$;

(v) *If for any cube Q and $u \in 3Q \setminus 2Q$, there is*

$$\frac{1}{w(Q)}\int_Q\left\|\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A(x)-(D^\alpha A)_Q)\int_{(4Q)^c}\frac{(u-y)^\alpha}{|u-y|^m}\frac{\Omega(u-y)\chi_{\Gamma(u)}(y,t)}{|u-y|^{n-1}}f(y)dy\right\|w(x)dx\le C||f||_{L^\infty(w)},$$

then $\tilde{\mu}_\Omega^A$ is bounded from $L^\infty(w)$ to $BMO(w)$.

Theorem 3. *Let $w \in A_1$ and $D^\alpha A \in BMO(R^n)$ for all α with $|\alpha| = m$. If $\delta > (n-1)/2$, then*

(i) *$B_{\delta,*}^A$ is bounded from $L^\infty(w)$ to $BMO(w)$;*

(ii) *$\tilde{B}_{\delta,*}^A$ is bounded from $H^1(w)$ to $L^1(w)$;*

(iii) *$B_{\delta,*}^A$ is bounded from $H^1(w)$ to weak $L^1(w)$.*

(iv) *If for any $H^1(w)$-atom a supported on certain cube Q and $u \in 3Q \setminus 2Q$, there is*

$$\int_{(4Q)^c}\left\|\sum_{|\alpha|=m}\frac{1}{\alpha!}\frac{(x-u)^\alpha}{|x-u|^m}B_t^\delta(x-u)\int_Q D^\alpha A(y)a(y)dy\right\|w(x)dx \le C,$$

then $B_{\delta,*}^A$ is bounded from $H^1(w)$ to $L^1(w)$;

(v) *If for any cube Q and $u \in 3Q \setminus 2Q$, there is*

$$\frac{1}{w(Q)}\int_Q\left\|\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A(x)-(D^\alpha A)_Q)\int_{(4Q)^c}\frac{(u-y)^\alpha}{|u-y|^m}B_t^\delta(u-y)f(y)dy\right\| w(x)dx \le C||f||_{L^\infty(w)},$$

then $\tilde{B}_{\delta,*}^A$ is bounded from $L^\infty(w)$ to $BMO(w)$.

Remark. In general, g_ψ^A, μ_Ω^A and $B_{\delta,*}^A$ are not (H^1, L^1) bounded.

3. Main Teorem and Proof

We first prove a general theorem.

Main Theorem. *Let $w \in A_1$ and $D^\alpha A \in BMO(R^n)$ for all α with $|\alpha| = m$. Suppose that F_t, T, T^A are the same as in Definition 1 and that T is bounded on $L^p(w)$ for any $1 < p \leq \infty$. If T satisfies the size condition:*

$$||F_t^A(f)(x) - F_t^A(f)(x_0)||| \leq C||D^\alpha A||_{BMO}||f||_{L^\infty(w)}$$

for any cube Q with supp$f \subset (2Q)^c$ and $x \in Q$. Then T^A is bounded from $BMO(w)$ to $L^\infty(w)$.

To prove the theorem, we need the following lemma.

Lemma 1. (see [6]) *Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A; x, y)| \leq C|x - y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x, y)$ is the cube centered at x and having side length $5\sqrt{n}|x - y|$.

Proof of Main Theorem. It is only to prove that there exists a constant C_Q such that

$$\frac{1}{w(Q)} \int_Q |T^A(f)(x) - C_Q|w(x)dx \leq C||f||_{L^\infty(w)}$$

holds for any cube Q. Fix a cube $Q = Q(x_0, d)$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}} x^\alpha$, then $R_m(A; x, y) = R_m(\tilde{A}; x, y)$ and $D^\alpha \tilde{A} = D^\alpha A - (D^\alpha A)_{\tilde{Q}}$ for $|\alpha| = m$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n \setminus \tilde{Q}}$,

$$\begin{aligned} F_t^A(f)(x) = & \int_{R^n} \frac{R_m(\tilde{A}; x, y)}{|x - y|^m} F(x, y, t) f_1(y) dy \\ & - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \frac{F(x, y, t)(x - y)^\alpha}{|x - y|^m} D^\alpha \tilde{A}(y) f_1(y) dy \\ & + \int_{R^n} \frac{R_{m+1}(\tilde{A}; x, y)}{|x - y|^m} F(x, y, t) f_2(y) dy; \end{aligned}$$

then

$$\begin{aligned}
&\left|T^A(f)(x)-T^{\tilde{A}}(f_2)(x_0)\right| = \left|||F_t^A(f)(x)||-||F_t^{\tilde{A}}(f)(x_0)||\right| \\
\le\ & \left|\left|F_t^A(f)(x)-F_t^{\tilde{A}}(f)(x_0)\right|\right| \\
\le\ & \left|\left|F_t\left(\frac{R_m(\tilde{A};x,\cdot)}{|x-\cdot|^m}f_1\right)(x)\right|\right| \\
&+\sum_{|\alpha|=m}\frac{1}{\alpha!}\left|\left|F_t\left(\frac{(x-\cdot)^\alpha}{|x-\cdot|^m}D^\alpha\tilde{A}f_1\right)(x)\right|\right| + \left|\left|F_t^{\tilde{A}}(f_2)(x)-F_t^{\tilde{A}}(f_2)(x_0)\right|\right| \\
:=\ & I(x)+II(x)+III(x),
\end{aligned}$$

thus,

$$\begin{aligned}
&\frac{1}{w(Q)}\int_Q\left|T^A(f)(x)-T^{\tilde{A}}(f)(x_0)\right|w(x)dx \\
\le\ & \frac{1}{w(Q)}\int_Q I(x)w(x)dx+\frac{1}{w(Q)}\int_Q II(x)w(x)dx+\frac{1}{w(Q)}\int_Q III(x)w(x)dx \\
:=\ & I+II+III.
\end{aligned}$$

Now, let us estimate I, II and III, respectively. First, for $x\in Q$ and $y\in\tilde{Q}$, using Lemma 1, we get

$$R_m(\tilde{A};x,y)\le C|x-y|^m\sum_{|\alpha|=m}||D^\alpha A||_{BMO},$$

thus, by the $L^\infty(w)$-boundedness of T, we get

$$\begin{aligned}
I &\le \frac{C}{w(Q)}\int_Q|T(\sum_{|\alpha|=m}||D^\alpha A||_{BMO}f_1)(x)|w(x)dx \\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||T(f_1)||_{L^\infty(w)} \\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^\infty(w)};
\end{aligned}$$

Secondly, since $w\in A_1$, w satisfies the reverse of Hölder' inequality:

$$\left(\frac{1}{|Q|}\int_Q w(x)^q dx\right)^{1/q}\le\frac{C}{|Q|}\int_Q w(x)dx$$

for all cube Q and some $1<q<\infty$(see[10]), thus, taking $p>1$, by the $L^p(w)$-boundedness of T and the Hölder' inequality, we gain

$$\begin{aligned}
II &\leq \frac{C}{w(Q)}\int_Q |T(\sum_{|\alpha|=m}(D^\alpha A-(D^\alpha A)_{\tilde{Q}})f_1)(x)|w(x)dx \\
&\leq C\sum_{|\alpha|=m}\left(\frac{1}{w(Q)}\int_Q |T((D^\alpha A-(D^\alpha A)_{\tilde{Q}})f_1)(x)|^p w(x)dx\right)^{1/p} \\
&\leq C\sum_{|\alpha|=m}\left(\frac{1}{w(Q)}\int |(D^\alpha A(x)-(D^\alpha A)_{\tilde{Q}})f_1(x)|^p w(x)dx\right)^{1/p} \\
&\leq C\sum_{|\alpha|=m} w(Q)^{-1/p}\left(\int_{\tilde{Q}} |D^\alpha A(x)-(D^\alpha A)_{\tilde{Q}}|^{pq'}dx\right)^{1/pq'} \\
&\quad \left(\int_{\tilde{Q}} w(x)^q dx\right)^{1/pq}||f||_{L^\infty(w)} \\
&\leq C\sum_{|\alpha|=m}\left(\frac{1}{|Q|}\int_{\tilde{Q}} |D^\alpha A(x)-(D^\alpha A)_{\tilde{Q}}|^{pq'}dx\right)^{1/pq'}\left(\frac{1}{|Q|}\int_{\tilde{Q}} w(x)^q dx\right)^{1/pq} \\
&\quad \left(\frac{|Q|}{w(Q)}\right)^{1/p}||f||_{L^\infty(w)} \\
&\leq C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\left(\frac{1}{|Q|}\int_{\tilde{Q}} w(x)dx\right)^{1/p}\left(\frac{|Q|}{w(Q)}\right)^{1/p}||f||_{L^\infty(w)} \\
&\leq C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^\infty(w)};
\end{aligned}$$

For III, by the size condition of T, we have

$$III \leq C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^\infty(w)}.$$

This completes the proof of Main theorem.

To prove Theorem 1, 2 and 3, we need the following lemma.

Lemma 2. *Let $w \in A_1$, $1 < p < \infty$, $\delta > (n-1)/2$ and $D^\alpha A \in BMO(R^n)$ for all α with $|\alpha| = m$. Then g_ψ^A, μ_Ω^A and $B_{*,\delta}^A$ are all bounded on $L^p(w)$.*

Proof. By Minkowski' inequality , we get

$$\begin{aligned}
g_\psi^A(f)(x) &\leq \int_{R^n}\frac{|f(y)||R_{m+1}(A;x,y)|}{|x-y|^m}\left(\int_0^\infty |\psi_t(x-y)|^2\frac{dt}{t}\right)^{1/2}dy \\
&\leq C\int_{R^n}\frac{|f(y)||R_{m+1}(A;x,y)|}{|x-y|^m}\left(\int_0^\infty \frac{t^{-2n}}{(1+|x-y|/t)^{2(n+1)}}\frac{dt}{t}\right)^{1/2}dy
\end{aligned}$$

$$\le C\int_{R^n}\frac{|R_{m+1}(A;x,y)|}{|x-y|^{m+n}}|f(y)|dy,$$

$$\mu_\Omega^A(f)(x) \le \int_{R^n}\frac{|\Omega(x-y)||R_{m+1}(A;x,y)|}{|x-y|^{m+n-1}}|f(y)|\left(\int_{|x-y|}^\infty \frac{dt}{t^3}\right)^{1/2}dy$$
$$\le C\int_{R^n}\frac{|R_{m+1}(A;x,y)|}{|x-y|^{m+n}}|f(y)|dy,$$

and

$$|B_r^\delta(x-y)| \le Cr^{-n}(1+|x-y|/r)^{-(\delta+(n+1)/2)}$$

$$= C\left(\frac{r}{r+|x-y|}\right)^{\delta-(n-1)/2}\frac{1}{(r+|x-y|)^n} \le |x-y|^{-n},$$

thus,

$$B_{*,\delta}^A(f)(x) \le C\int_{R^n}\frac{|R_{m+1}(A;x,y)|}{|x-y|^{m+n}}|f(y)|dy,$$

so that, the lemma follows from [9].

Proof of Theorem 1. (i) First, by the proof of Lemma 2, we get

$$g_\psi(f)(x) \le C\int_{R^n}\frac{|f(y)|}{|x-y|^n}dy,$$

thus, g_ψ is $L^p(w)$-bounded for $p>1$ by [2]. Now, it suffices to verify that g_ψ satisfies the size condition in **Main Theorem**. For supp$f \subset (2Q)^c$, let $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A)_Q x^\alpha$.
Write

$$F_t^{\tilde{A}}(f)(x) - F_t^{\tilde{A}}(f)(x_0) = \int_{R^n}\left[\frac{\psi_t(x-y)}{|x-y|^m} - \frac{\psi_t(x_0-y)}{|x_0-y|^m}\right]R_m(\tilde{A};x,y)f(y)dy$$
$$+ \int_{R^n}\frac{\psi_t(x_0-y)f(y)}{|x_0-y|^m}[R_m(\tilde{A};x,y)-R_m(\tilde{A};x_0,y)]dy$$
$$- \sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\left(\frac{\psi_t(x-y)(x-y)^\alpha}{|x-y|^m} - \frac{\psi_t(x_0-y)(x_0-y)^\alpha}{|x_0-y|^m}\right)D^\alpha\tilde{A}(y)f(y)dy$$
$$:= I_1+I_2+I_3.$$

Note that $|x-y| \sim |x_0-y|$ for $x\in Q$ and $y \in R^n\setminus Q$. By Lemma 1 and the following inequality (see [16])

$$|b_{Q_1}-b_{Q_2}| \le C\log(|Q_2|/|Q_1|)||b||_{BMO} \text{ for } Q_1\subset Q_2,$$

we know that, for $x \in Q$ and $y \in 2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}$,

$$|R_m(\tilde{A}; x, y)| \le C|x-y|^m \sum_{|\alpha|=m} (||D^\alpha A||_{BMO} + |(D^\alpha A)_{\tilde{Q}(x,y)} - (D^\alpha A)_{\tilde{Q}}|)$$
$$\le Ck|x-y|^m \sum_{|\alpha|=m} ||D^\alpha A||_{BMO}.$$

Thus, similar to the proof of Lemma 2, we get

$$||I_1|| \le C \int_{R^n \setminus \tilde{Q}} \left(\frac{|x-x_0|}{|x_0-y|^{m+n+1}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{m+n+\varepsilon}} \right) |R_m(\tilde{A}; x, y)||f(y)|dy$$
$$\le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} \sum_{k=0}^\infty \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} k\left(\frac{|x-x_0|}{|x_0-y|^{n+1}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon}} \right)|f(y)|dy$$
$$\le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO}||f||_{L^\infty(w)} \sum_{k=1}^\infty k(2^{-k} + 2^{-\varepsilon k})$$
$$\le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO}||f||_{L^\infty(w)};$$

For I_2, by the formula (see [6]):

$$R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x_0, y) = \sum_{|\beta|<m} \frac{1}{\beta!} R_{m-|\beta|}(D^\beta \tilde{A}; x, x_0)(x-y)^\beta$$

and Lemma 1, we have

$$|R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x_0, y)| \le C \sum_{|\beta|<m} \sum_{|\alpha|=m} |x-x_0|^{m-|\beta|}|x-y|^{|\beta|}||D^\alpha A||_{BMO},$$

similar to the estimates of I_1, we get

$$||I_2|| \le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} \sum_{k=0}^\infty \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \frac{|x-x_0|}{|x_0-y|^{n+1}}|f(y)|dy$$
$$\le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO}||f||_{L^\infty(w)};$$

For I_3, similar to the estimates of I_1, we get

$$||I_3|| \le C \sum_{|\alpha|=m} \sum_{k=0}^\infty \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \left(\frac{|x-x_0|}{|x_0-y|^{n+1}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon}} \right) |D^\alpha \tilde{A}(y)||f(y)|dy$$
$$\le C \sum_{|\alpha|=m} \sum_{k=1}^\infty (2^{-k} + 2^{-\varepsilon k}) \left(|2^k\tilde{Q}|^{-1} \int_{2^k\tilde{Q}} |D^\alpha A(y) - (D^\alpha A)_{\tilde{Q}}|dy \right) ||f||_{L^\infty(w)}$$
$$\le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO}||f||_{L^\infty(w)}.$$

Thus

$$||F_t^{\tilde{A}}(f_2)(x) - F_t^{\tilde{A}}(f_2)(x_0)|| \leq C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO}||f||_{L^{n/\delta}}.$$

(ii). It suffices to show that there exists a constant $C > 0$ such that for every $H^1(w)$-atom a, we have

$$||\tilde{g}_\psi^A(a)||_{L^1(w)} \leq C.$$

We write

$$\int_{R^n} \tilde{g}_\psi^A(a)(x)w(x)dx = \left[\int_{2Q} + \int_{(2Q)^c}\right] \tilde{g}_\psi^A(a)(x)w(x)dx := J + JJ.$$

For J, by the following equality

$$Q_{m+1}(A; x, y) = R_{m+1}(A; x, y) + \sum_{|\alpha|=m} \frac{1}{\alpha!}(x-y)^\alpha(D^\alpha A(x) - D^\alpha A(y)),$$

we have, similar to the proof of Lemma 2,

$$\tilde{g}_\psi^A(a)(x) \leq g_\psi^A(a)(x) + C \sum_{|\alpha|=m} \int_{R^n} \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^n}|a(y)|dy,$$

thus, $\tilde{g}_\psi^A$ is $L^\infty(w)$-bounded by Lemma 2 and [2]. We get

$$J \leq C||\tilde{g}_\delta^A(a)||_{L^\infty(w)} w(2Q) \leq C||a||_{L^\infty(w)} w(Q) \leq C.$$

To obtain the estimate of JJ, we denote that $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{2Q} x^\alpha$. Then $Q_m(A; x, y) = Q_m(\tilde{A}; x, y)$. We write, by the vanishing moment of a and for $x \in (2Q)^c$,

$$\begin{aligned}
\tilde{F}_t^A(a)(x) &= \int_{R^n} \frac{\psi_t(x-y)R_m(A; x, y)}{|x-y|^m} a(y)dy \\
&\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \frac{\psi_t(x-y)D^\alpha \tilde{A}(x)(x-y)^\alpha}{|x-y|^m} a(y)dy \\
&= \int_{R^n} \left[\frac{\psi_t(x-y)}{|x-y|^m} - \frac{\psi_t(x-x_0)}{|x-x_0|^m}\right] R_m(\tilde{A}; x, y)a(y)dy \\
&\quad + \int_{R^n} \frac{\psi_t(x-x_0)a(y)}{|x_0-x|^m}[R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x, x_0)]dy \\
&\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \left[\frac{\psi_t(x-y)(x-y)^\alpha}{|x-y|^m} - \frac{\psi_t(x-x_0)(x-x_0)^\alpha}{|x-x_0|^m}\right] \\
&\quad\quad D^\alpha \tilde{A}(x)a(y)dy, \\
&:= JJ_1 + JJ_2 + JJ_3.
\end{aligned}$$

Similar to the proof of Lemma 2 and (i), we obtain, for $x \in (2Q)^c$,

$$||JJ_1|| \le C \int_{R^n} \left[\frac{|y-x_0|}{|x-y|^{n+m+1}} + \frac{|y-x_0|^\varepsilon}{|x-y|^{n+m+\varepsilon}} \right] |R_m(\tilde{A};x,y)||a(y)|dy$$
$$\le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} w(Q)^{-1}$$
$$\left(|Q|^{1+1/n}|x-x_0|^{-n-1} + |Q|^{1+\varepsilon/n}|x-x_0|^{-n-\varepsilon} \right),$$

$$||JJ_2|| \le C \int_Q \frac{|R_m(\tilde{A};x,y) - R_m(\tilde{A};x,x_0)||a(y)|}{|x-y|^{m+n}} dy$$
$$\le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} \int_Q \frac{|x_0-y||a(y)|}{|x-x_0|^{n+1}} dy$$
$$\le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} w(Q)^{-1} |Q|^{1+1/n}|x-x_0|^{-n-1},$$

$$||JJ_3|| \le C \int_Q \frac{|x_0-y|}{|x-y|^{n+1}} \sum_{|\alpha|=m} |D^\alpha \tilde{A}(x)||a(y)|dy$$
$$\le C \sum_{|\alpha|=m} |D^\alpha \tilde{A}(x)| w(Q)^{-1} \Big| \left(|Q|^{1+1/n}|x-x_0|^{-n-1} + |Q|^{1+\varepsilon/n}|x-x_0|^{-n-\varepsilon} \right).$$

Note that if $w \in A_1$, then $\frac{w(Q_2)}{|Q_2|}\frac{|Q_1|}{w(Q_1)} \le C$ for all cubes Q_1, Q_2 with $Q_1 \subset Q_2$. Thus, by the Hölder' inequality and the reverse of Hölder' inequality for $w \in A_1$, we obtain

$$JJ \le \int_{(2Q)^c} ||JJ_1 + JJ_2 + JJ_3|| w(x)dx$$
$$\le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} \sum_{k=1}^\infty (2^{-k} + 2^{-\varepsilon k}) \left(\frac{|Q|}{w(Q)} \frac{w(2^{k+1}Q)}{|2^{k+1}Q|} \right)$$
$$+ C \sum_{|\alpha|=m} \sum_{k=1}^\infty (2^{-k} + 2^{-\varepsilon k}) \frac{|Q|}{w(Q)} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |D^\alpha \tilde{A}(x)|^{q'} dx \right)^{1/q'}$$
$$\left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} w(x)^q dx \right)^{1/q}$$
$$\le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} \sum_{k=1}^\infty k(2^{-k} + 2^{-\varepsilon k}) \left(\frac{w(2^{k+1}Q)}{|2^{k+1}Q|} \frac{|Q|}{w(Q)} \right)$$
$$\le C.$$

(iii). By the equality

$$R_{m+1}(A;x,y)=Q_{m+1}(A;x,y)+\sum_{|\alpha|=m}\frac{1}{\alpha!}(x-y)^{\alpha}(D^{\alpha}A(x)-D^{\alpha}A(y)),$$

similar to the proof of Lemma 2, we get

$$g_{\psi}^{A}(f)(x)\leq\tilde{g}_{\psi}^{A}(f)(x)+C\sum_{|\alpha|=m}\int_{R^n}\frac{|D^{\alpha}A(x)-D^{\alpha}A(y)|}{|x-y|^n}|f(y)|dy,$$

by (i)(ii) and [2], we obtain

$$\begin{aligned}&w(\{x\in R^n:g_{\psi}^{A}(f)(x)>\lambda\})\\ \leq\ &w(\{x\in R^n:\tilde{g}_{\psi}^{A}(f)(x)>\lambda/2\})\\ &+w\left(\left\{x\in R^n:\sum_{|\alpha|=m}\int_{R^n}\frac{|D^{\alpha}A(x)-D^{\alpha}A(y)|}{|x-y|^n}|f(y)|dy>C\lambda\right\}\right)\\ \leq\ &C||f||_{H^1(w)}/\lambda.\end{aligned}$$

(iv). Let a be $H^1(w)$-atom with supp$a\subset Q=Q(x_0,d)$. We write, by the vanishing moment of a and for $u\in 3Q\setminus 2Q$,

$$\begin{aligned}F_t^A(a)(x)\ =\ &\chi_{4Q}(x)F_t^A(a)(x)+\chi_{(4Q)^c}(x)\\ &\int_{R^n}\left[\frac{R_m(\tilde{A};x,y)\psi_t(x-y)}{|x-y|^m}-\frac{R_m(\tilde{A};x,u)\psi_t(x-u)}{|x-u|^m}\right]a(y)dy\\ &-\chi_{(4Q)^c}(x)\sum_{|\alpha|=m}\frac{1}{\alpha!}\\ &\int_{R^n}\left[\frac{\psi_t(x-y)(x-y)^{\alpha}}{|x-y|^m}-\frac{\psi_t(x-u)(x-u)^{\alpha}}{|x-u|^m}\right]D^{\alpha}\tilde{A}(y)a(y)dy\\ &-\chi_{(4Q)^c}(x)\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\frac{(x-u)^{\alpha}}{|x-u|^m}\psi_t(x-u)D^{\alpha}\tilde{A}(y)a(y)dy,\end{aligned}$$

then

$$\begin{aligned}&g_{\psi}^{A}(a)(x)=\left|\left|F_t^A(a)(x)\right|\right|\leq\chi_{4Q}(x)\left|\left|F_t^A(a)(x)\right|\right|\\ &+\chi_{(4Q)^c}(x)\left|\left|\int_{R^n}\left[\frac{R_m(\tilde{A};x,y)\psi_t(x-y)}{|x-y|^m}-\frac{R_m(\tilde{A};x,u)\psi_t(x-u)}{|x-u|^m}\right]a(y)dy\right|\right|\\ &+\chi_{(4Q)^c}(x)\left|\left|\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\left[\frac{\psi_t(x-y)(x-y)^{\alpha}}{|x-y|^m}-\frac{\psi_t(x-u)(x-u)^{\alpha}}{|x-u|^m}\right]D^{\alpha}\tilde{A}(y)a(y)dy\right|\right|\end{aligned}$$

$$+\chi_{(4Q)^c}(x)\left\|\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\frac{(x-u)^\alpha}{|x-u|^m}\psi_t(x-u)D^\alpha\tilde{A}(y)a(y)dy\right\|$$
$$= I_1(x)+I_2(x,u)+I_3(x,u)+I_4(x,u).$$

Similar to the proof of (i), we get

$$\begin{aligned}
\int_{R^n} I_1(x)w(x)dx &\le ||g_\psi^A(a)||_{L^p(w)}w(4Q)^{1-1/p}\le C||a||_{L^p(w)}w(Q)^{1-1/p}\le C;\\
\int_{R^n} I_2(x,u)w(x)dx &\le C\sum_{k=2}^\infty\int_{2^{k+1}Q\setminus 2^kQ}\int_Q\left(\frac{|y-u|}{|x-y|^{m+n+1}}+\frac{|y-u|^\varepsilon}{|x-y|^{m+n+\varepsilon}}\right)\\
&\quad |R_m(\tilde{A};x,y)||a(y)|dyw(x)dx\\
&\quad +\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=2}^\infty\int_{2^{k+1}Q\setminus 2^kQ}\int_Q\frac{|y-u|}{|x-y|^{n+1}}|a(y)|\\
&\quad dyw(x)dx\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=2}^\infty\int_{2^{k+1}Q\setminus 2^kQ}k\\
&\quad\left(\frac{d}{(2^kd)^{n+1}}+\frac{d^\varepsilon}{(2^kd)^{n+\varepsilon}}\right)||a||_{L^\infty(w)}|Q|w(x)dx\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=2}^\infty k(2^{-k}+2^{-\varepsilon k})\frac{w(2^{k+1}Q)}{|2^{k+1}Q|}\frac{|Q|}{w(Q)}\\
&\le C;\\
\int_{(4Q)^c} I_3(x,u)w(x)dx &\le C\sum_{|\alpha|=m}\sum_{k=2}^\infty\int_{2^{k+1}Q\setminus 2^kQ}\int_Q\left(\frac{|y-u|}{|x-y|^{n+1}}+\frac{|y-u|^\varepsilon}{|x-y|^{n+\varepsilon}}\right)\\
&\quad |D^\alpha\tilde{A}(y)||a(y)|dyw(x)dx\\
&\le C\sum_{|\alpha|=m}\sum_{k=2}^\infty\left(\frac{d}{(2^kd)^{n+1}}+\frac{d^\varepsilon}{(2^kd)^{n+\varepsilon}}\right)\left(\frac{1}{|Q|}\int_Q|D^\alpha\tilde{A}(y)|dy\right)\\
&\quad ||a||_{L^\infty(w)}|Q|w(2^{k+1}Q)\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=2}^\infty(2^{-k}+2^{-\varepsilon k})\frac{w(2^{k+1}Q)}{|2^{k+1}Q|}\frac{|Q|}{w(Q)}\\
&\le C.
\end{aligned}$$

Thus, using the condition of $I_4(x,u)$, we obtain

$$\int_{R^n} g_\psi^A(a)(x)w(x)dx\le C.$$

(v). For any cube $Q = Q(x_0, d)$, we write, for $f = f\chi_{4Q} + f\chi_{(4Q)^c} = f_1 + f_2$ and $u \in 3Q \setminus 2Q$,

$$\begin{aligned}
\tilde{F}_t^A(f)(x) \;=\; & \tilde{F}_t^A(f_1)(x) + \int_{R^n} \frac{R_m(\tilde{A}; x, y)}{|x-y|^m}\psi_t(x-y)f_2(y)dy \\
& - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A(x) - (D^\alpha A)_Q) \int_{R^n} \\
& \left[\frac{\psi_t(x-y)(x-y)^\alpha}{|x-y|^m} - \frac{\psi_t(u-y)(u-y)^\alpha}{|u-y|^m}\right] f_2(y)dy \\
& - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A(x) - (D^\alpha A)_Q) \int_{R^n} \frac{(u-y)^\alpha}{|u-y|^m}\psi_t(u-y)f_2(y)dy,
\end{aligned}$$

then

$$\begin{aligned}
& \left|\tilde{g}_\psi^A(f)(x) - g_\psi\left(\frac{R_m(\tilde{A}; x_0, \cdot)}{|x_0 - \cdot|^m} f_2\right)(x_0)\right| \\
= & \left|\left|\left|\tilde{F}_t^A(f)(x)\right|\right| - \left|\left|F_t\left(\frac{R_m(\tilde{A}; x_0, \cdot)}{|x_0 - \cdot|^m} f_2\right)(x_0)\right|\right|\right| \\
\le & \left|\left|\tilde{F}_t^A(f)(x) - F_t\left(\frac{R_m(\tilde{A}; x_0, \cdot)}{|x_0 - \cdot|^m} f_2\right)(x_0)\right|\right| \\
\le & \left|\left|\tilde{F}_t^A(f_1)(x)\right|\right| \\
& + \left|\left|\int_{R^n}\left[\frac{R_m(\tilde{A}; x, y)}{|x-y|^m}\psi_t(x-y) - \frac{R_m(\tilde{A}; x_0, y)}{|x_0-y|^m}\psi_t(x_0-y)\right] f_2(y)dy\right|\right| \\
& + \left|\left|\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A(x) - (D^\alpha A)_Q)\int_{R^n}\right.\right. \\
& \left.\left.\left[\frac{\psi_t(x-y)(x-y)^\alpha}{|x-y|^m} - \frac{\psi_t(u-y)(u-y)^\alpha}{|u-y|^m}\right] f_2(y)dy\right|\right| \\
& + \left|\left|\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A(x) - (D^\alpha A)_Q)\int_{R^n}\frac{(u-y)^\alpha}{|u-y|^m}\psi_t(u-y)f_2(y)dy\right|\right| \\
= & \; J_1(x) + J_2(x) + J_3(x,u) + J_4(x,u).
\end{aligned}$$

Similar to the proof of (i) and (iv), we get

$$\frac{1}{w(Q)}\int_Q J_1(x)w(x)dx \le w(Q)^{-1/p}||\tilde{g}_\psi^A(f_1)||_{L^p(w)}$$

$$\begin{aligned}
&\le Cw(Q)^{-1/p}||f_1||_{L^p(w)} \le C||f||_{L^\infty(w)};\\
\frac{1}{w(Q)}\int_Q J_2(x)w(x)dx &\le C\frac{1}{w(Q)}\int_Q\sum_{k=2}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ} k\\
&\quad \left(\frac{|x-x_0|}{|x_0-y|^{n+1}}+\frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon}}\right)|f(y)|dyw(x)dx\\
&\le C||f||_{L^\infty(w)}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k}) \le C||f||_{L^\infty(w)};\\
\frac{1}{w(Q)}\int_Q J_3(x,u)w(x)dx &\le C\sum_{|\alpha|=m}\left(\frac{1}{|Q|}\int_Q |D^\alpha A(x)-(D^\alpha A)_Q|^{r'}dx\right)^{1/r'}\\
&\quad \left(\frac{1}{|Q|}\int_Q w(x)^r dx\right)^{1/r}|Q|w(Q)^{-1}\\
&\quad \times\sum_{k=2}^{\infty}\left(\frac{d}{(2^kd)^{n+1}}+\frac{d^\varepsilon}{(2^kd)^{n+\varepsilon}}\right)|2^kQ|||f||_{L^\infty(w)}\\
&\le C\sum_{k=2}^{\infty}(2^{-k}+2^{-\varepsilon k})||f||_{L^\infty(w)} \le C||f||_{L^\infty(w)}.
\end{aligned}$$

Thus, using the condition of $J_4(x,u)$, we obtain

$$\frac{1}{w(Q)}\int_Q\left|\tilde g_\psi^A(f)(x)-g_\psi\left(\frac{R_m(\tilde A;x_0,\cdot)}{|x_0-\cdot|^m}f_2\right)(x_0)\right|w(x)dx \le C||f||_{L^\infty(w)}.$$

This completes the proof of Theorem 1.

Proof of Theorem 2.

(i) First, by the proof of Lemma 2, we get

$$\mu_\Omega(f)(x) \le C\int_{R^n}\frac{|f(y)|}{|x-y|^n}dy,$$

thus, μ_Ω is $L^p(w)$-bounded for $p>1$. Now, it suffices to verify that μ_Ω satisfies the size condition in **Main Theorem**. For supp$f\subset(2Q)^c$, let $\tilde A(x)$ be the same as the proof of Theorem 1. We write

$$\begin{aligned}
&||F_t^{\tilde A}(f)(x)-F_t^{\tilde A}(f)(x_0)||\\
\le\ &\left(\int_0^\infty\left|\int_{|x-y|\le t}\frac{\Omega(x-y)R_m(\tilde A;x,y)}{|x-y|^{m+n-1}}f(y)dy\right.\right.
\end{aligned}$$

$$
\begin{aligned}
& - \int_{|x_0-y|\le t} \frac{\Omega(x_0-y)R_m(\tilde{A};x_0,y)}{|x_0-y|^{m+n-1}} f(y)dy\Bigg|^2 \frac{dt}{t^3}\Bigg)^{1/2} \\
& + \sum_{|\alpha|=m} \Bigg(\int_0^\infty \Bigg|\int_{|x-y|\le t} \Bigg(\frac{\Omega(x-y)(x-y)^\alpha}{|x-y|^{m+n-1}} \\
& - \int_{|x_0-y|\le t} \frac{\Omega(x_0-y)(x_0-y)^\alpha}{|x_0-y|^{m+n-1}}\Bigg) D^\alpha \tilde{A}(y) f(y)dy\Bigg|^2 \frac{dt}{t^3}\Bigg)^{1/2} \\
\le\ & \Bigg(\int_0^\infty \Bigg[\int_{|x-y|\le t,\ |x_0-y|>t} \frac{|\Omega(x-y)||R_m(\tilde{A};x,y)|}{|x-y|^{m+n-1}}|f(y)|dy\Bigg]^2 \frac{dt}{t^3}\Bigg)^{1/2} \\
& + \Bigg(\int_0^\infty \Bigg[\int_{|x-y|>t,\ |x_0-y|\le t} \frac{|\Omega(x_0-y)||R_m(\tilde{A};x_0,y)|}{|x_0-y|^{m+n-1}}|f(y)|dy\Bigg]^2 \frac{dt}{t^3}\Bigg)^{1/2} \\
& + \Bigg(\int_0^\infty \Bigg[\int_{|x-y|\le t, |x_0-y|\le t} \Bigg|\frac{\Omega(x-y)R_m(\tilde{A};x,y)}{|x-y|^{m+n-1}} \\
& - \frac{\Omega(x_0-y)R_m(\tilde{A};x_0,y)}{|x_0-y|^{m+n-1}}\Bigg| |f(y)|dy\Bigg]^2 \frac{dt}{t^3}\Bigg)^{1/2} \\
& + \sum_{|\alpha|=m} \Bigg(\int_0^\infty \Bigg|\int_{|x-y|\le t} \Bigg(\frac{\Omega(x-y)(x-y)^\alpha}{|x-y|^{m+n-1}} \\
& - \int_{|x_0-y|\le t} \frac{\Omega(x_0-y)(x_0-y)^\alpha}{|x_0-y|^{m+n-1}}\Bigg) D^\alpha \tilde{A}(y) f(y)dy\Bigg|^2 \frac{dt}{t^3}\Bigg)^{1/2} \\
:=\ & K_1 + K_2 + K_3 + K_4.
\end{aligned}
$$

Since $|x-y| \approx |x_0-y|$ when $y \in (2Q)^c$, we get

$$
\begin{aligned}
K_1 &\le C \int_{R^n\setminus\tilde{Q}} \frac{|f(y)||R_m(\tilde{A};x,y)|}{|x-y|^{m+n-1}} \left(\int_{|x-y|\le t<|x_0-y|} \frac{dt}{t^3}\right)^{1/2} dy \\
&\le C\int_{R^n\setminus\tilde{Q}} \frac{|f(y)||R_m(\tilde{A};x,y)|}{|x-y|^{m+n-1}} \frac{|x_0-x|^{1/2}}{|x-y|^{3/2}} dy \le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO}||f||_{L^\infty(w)};
\end{aligned}
$$

Similarly, we get $K_2 \le C\sum_{|\alpha|=m} ||D^\alpha A||_{BMO}||f||_{L^\infty(w)}$.

For K_3, by the following inequality (see [18]):

$$\left|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x_0-y)}{|x_0-y|^{n-1}}\right| \le \left(\frac{|x-x_0|}{|x_0-y|^n}+\frac{|x-x_0|^\gamma}{|x_0-y|^{n-1+\gamma}}\right),$$

we gain

$$\begin{aligned} K_3 \ &\le\ C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\int_{R^n\setminus\tilde{Q}} \\ &\left(\frac{|x-x_0|}{|x_0-y|^n}+\frac{|x-x_0|^\gamma}{|x_0-y|^{n-1+\gamma}}\right)\left(\int_{|x_0-y|\le t,|x-y|\le t}\frac{dt}{t^3}\right)^{1/2}|f(y)|dy \\ &\le\ C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=1}^\infty k(2^{-k}+2^{-\gamma k})||f||_{L^\infty(w)} \\ &\le\ C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^\infty(w)}; \end{aligned}$$

For K_4, similar to the proof of K_1, K_2 and K_3, we obtain

$$\begin{aligned} K_4 \ &\le\ C\sum_{|\alpha|=m}\sum_{k=1}^\infty\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\left(\frac{|x-x_0|}{|x_0-y|^{n+1}}+\frac{|x-x_0|^{1/2}}{|x_0-y|^{n+1/2}}+\frac{|x-x_0|^\gamma}{|x_0-y|^{n+\gamma}}\right) \\ &|D^\alpha\tilde{A}(y)||f(y)|dy \\ &\le\ C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=1}^\infty k(2^{-k}+2^{-k/2}+2^{-\gamma k})||f||_{L^\infty(w)} \\ &\le\ C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^\infty(w)}. \end{aligned}$$

A same argument as in the proof of Theorem 1 will get the proof of (ii),(iii),(iv) and (v), we omit the details. This completes the proof of Theorem 2.

Proof of Theorem 3.(i) First, by the proof of Lemma 2, we get

$$B_{*,\delta}(f)(x) \le C\int_{R^n}\frac{|f(y)|}{|x-y|^n}dy,$$

thus, $B_{*,\delta}$ is $L^p(w)$-bounded for $p>1$. Now, it suffices to verify that $B_{*,\delta}$ satisfies the size condition in **Main Theorem**. For supp$f \subset (2Q(x_0,d))^c$, let $\tilde{A}(x)$ be the same as the proof of Theorem 1. We write

$$\begin{aligned} B^{\tilde{A}}_{t,\delta}(f)(x)-B^{\tilde{A}}_{t,\delta}(f)(x_0)&=\int_{R^n}\left[\frac{B^\delta_t(x-y)}{|x-y|^m}-\frac{B^\delta_t(x_0-y)}{|x_0-y|^m}\right]R_m(\tilde{A};x,y)f(y)dy \\ &+\int_{R^n}\frac{B^\delta_t(x_0-y)}{|x_0-y|^m}[R_m(\tilde{A};x,y)-R_m(\tilde{A};x_0,y)]f(y)dy \end{aligned}$$

$$-\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\left(\frac{B_t^\delta(x-y)(x-y)^\alpha}{|x-y|^m}-\frac{B_t^\delta(x_0-y)(x_0-y)^\alpha}{|x_0-y|^m}\right)D^\alpha\tilde{A}(y)f(y)dy$$

$$= L_1+L_2+L_3.$$

We consider the following two cases:

Case 1. $0<t\le d$. In this case, notice that (see [14])

$$|B^\delta(z)|\le c(1+|z|)^{-(\delta+(n+1)/2)},$$

we obtain

$$|L_1|\le Ct^{-n}\int_{R^n\setminus\tilde{Q}}\frac{|f(y)||R_m(\tilde{A};x,y)|}{|x_0-y|^m}(1+|x-y|/t)^{-(\delta+(n+1)/2)}dy$$

$$\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}t^{-n}\sum_{k=0}^\infty k\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}|f(y)||(1+|x-y|/t)^{-(\delta+(n+1)/2)}dy$$

$$\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}(t/d)^{\delta-(n-1)/2}\sum_{k=1}^\infty k2^{k((n-1)/2-\delta)}||f||_{L^\infty(w)}$$

$$\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^\infty(w)},$$

$$|L_2|\le Ct^{-n}\int_{R^n\setminus\tilde{Q}}\frac{|f(y)||R_m(\tilde{A};x,y)-R_m(\tilde{A};x_0,y)|}{|x_0-y|^m}(1+|x-y|/t)^{-(delta+(n+1)/2)}dy$$

$$\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}t^{-n}\sum_{k=0}^\infty\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|x-x_0||f(y)|}{|x_0-y|}(1+|x-y|/t)^{-(\delta+(n+1)/2)}dy$$

$$\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^\infty(w)},$$

$$|L_3|\le C\sum_{|\alpha|=m}t^{-n}\sum_{k=0}^\infty\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}|f(y)||D^\alpha\tilde{A}(y)|(1+|x_0-y|/t)^{-(\delta+(n+1)/2)}dy$$

$$\le C\sum_{|\alpha|=m}(t/d)^{\delta-(n-1)/2}\sum_{k=0}^\infty 2^{k((n-1)/2-\delta)}\frac{1}{|2^{k+1}\tilde{Q}|}\int_{2^{k+1}\tilde{Q}}|f(y)||D^\alpha A(y)-(D^\alpha A)_{\tilde{Q}}|dy$$

$$\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^\infty(w)}.$$

Case 2. $t>d$. In this case, we choose δ_0 such that $(n-1)/2<\delta_0<\min(\delta,(n+1)/2)$, notice that (see [15])

$$|B^\delta(x-y)-B^\delta(x_0-y)|\le C|x-x_0|(1+|x-y|)^{-(\delta+(n+1)/2)},$$

similar to the proof of Case 1, we obtain

$$
\begin{aligned}
|L_1| &\le Ct^{-n}\int_{R^n\setminus\tilde{Q}}\frac{|f(y)||R_m(\tilde{A};x,y)|}{|x_0-y|^{m+1}}|x_0-x|(1+|x_0-y|/t)^{-(\delta_0+(n+1)/2)}dy\\
&\quad +Ct^{-n-1}\int_{R^n\setminus\tilde{Q}}\frac{|f(y)||R_m(\tilde{A};x,y)|}{|x_0-y|^{m}}|x_0-x|(1+|x_0-y|/t)^{-(\delta_0+(n+1)/2)}dy\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}(d/t)^{(n+1)/2-\delta_0}\sum_{k=1}^{\infty}k2^{k((n-1)/2-\delta_0)}||f||_{L^\infty(w)}\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^\infty(w)},\\
|L_2| &\le Ct^{-n}\int_{R^n\setminus\tilde{Q}}\frac{|f(y)||R_m(\tilde{A};x,y)-R_m(\tilde{A};x_0,y)|}{|x_0-y|^{m}}(1+|x_0-y|/t)^{-(\delta_0+(n+1)/2)}dy\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}(d/t)^{(n+1)/2-\delta_0}\sum_{k=1}^{\infty}2^{k((n-1)/2-\delta_0)}||f||_{L^\infty(w)}\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^\infty(w)},\\
|L_3| &\le C\sum_{|\alpha|=m}(d/t)^{(n+1)/2-\delta_0}\sum_{k=0}^{\infty}2^{k((n-1)/2-\delta_0)}\frac{1}{|2^{k+1}\tilde{Q}|}\int_{2^{k+1}\tilde{Q}}|f(y)||D^\alpha\tilde{A}(y)|dy\\
&\le C\sum_{|\alpha|=m}\sum_{k=1}^{\infty}k2^{k((n-1)/2-\delta_0)}\frac{1}{|2^{k+1}\tilde{Q}|}\int_{2^{k+1}\tilde{Q}}|f(y)||D^\alpha A(y)-(D^\alpha A)_{\tilde{Q}}|dy\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^\infty(w)}.
\end{aligned}
$$

These yield the desired results. A same argument as in the proof of Theorem 1 will give the proof of (ii), (iii), (iv) and (v), we omit the details. This completes the proof of Theorem 3.

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Liu Lanzhe
College of Mathematics and Computer,
Changsha University of Science and Technology,
Changsha 410077,
P. R. China
E-mail:lanzheliu@263.net