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ON CONVERGENT RATES OF ERGODIC HARRIS CHAINS INDUCED FROM DIFFUSIONS

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Abstract. We construct an irreducible ergodic Harris chain $\{X_n\}$ from a diffusion $\{S_t\}$ and barriers $\rho^{\pm}(x)$. We show that $\{X_n\}$ is exponentially uniformly ergodic in the sense of the operator norm under the Banach space C_β, where $\beta \in (0,1)$. Moreover, the sizes of the convergent rates $\alpha_X(\beta)$ and $\alpha_S(\beta)$ measured by the operator norm are studied. We give an upper bound of $\alpha_X(\beta)$ in terms of $\rho^{\pm}(x)$. The Ornstein-Uhlenbeck process and proper $\rho^{\pm}(x)$ are taken to show $\alpha_X(\beta) < \alpha_S(\beta)$ for $0 < \beta < 0.5$.

1. Introduction

Let S_t be a diffusion in natural scale with the generator $L = \frac{\partial^2}{m(x)\partial x^2}$, where $m(x)$ is positive and continuous. Throughout this article, we assume

(1) $$x^2 m(x) \to 0, \quad |x| \to \infty.$$

On the other hand, barriers $\rho^{\pm}(x)$ are both continuous functions of x and satisfy

(2) $$\rho^+(x) = c_+ x,\ \rho^-(x) = c_- x, \forall x \geq 1,$$

(3) $$\rho^+(x) = d_+ x,\ \rho^-(x) = d_- x, \forall x \leq -1,$$

where $c_+ > 1, 0 < c_- < 1$ and $d_- > 1, 0 < d_+ < 1$.

We consider a Harris chain $\{X_n\}$ defined by

(i) $X_0 \equiv S_0 \equiv x$ and $X_1 \equiv S_{1\wedge\tau}$, where

$$\tau^{\pm} \equiv \inf\{t \geq 0 :\ S_t = \rho^{\pm}(x)\}, \quad \tau \equiv \tau^+ \wedge \tau^-.$$

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(ii) $\{X_n\}$ has a stationary transition probability

$$p(x, dy) \equiv P_x(S_{1\wedge\tau} \in dy) = p_c(x, dy) + p_d(x, dy),$$

where $p_c(x, dy) \equiv P_x(S_1 \in dy, \tau > 1)$ and $p_d(x, dy) \equiv P_x(S_\tau \in dy, \tau \leq 1)$.

The consideration of $\{X_n\}$ has a background from Taiwan's stock market. In order to maintain a stable stock market, barriers of stocks are set at 7% of the closing price of the preceding business day in Taiwan's stock market. Concretely, if the final price of yesterday's stock was x, then the lower bound $\rho^-(x)$ of today's stock price is defined by $0.93x$, and the upper bound $\rho^+(x)$ of today's stock price is defined by $1.07x$. However, stock prices are determined by themselves in financial market. It seems unreasonable to settle barriers $\rho^\pm$ at daily stock price. The problem is what the influence of price limits is and what effect barriers bring. To investigate these problems, we use X_n to represent the final price at the nth day in Taiwan market, and S_n to represent the final price without barriers at the nth day. In [2], a fat tail's effect was found by comparing $\{X_n\}$ with $\{S_n\}$. Moreover, by [2], $\{X_n\}$ defined above is an irreducible ergodic Harris chain with the general state space $\mathbb{R}$. And there exists the unique invariant probability measure $\mu(\cdot)$ of $\{X_n\}$.

Before making our attempt obvious in this article, we give some settings and a definition at first. Fix $\beta \in (0, 1)$, $\eta > 0$ and introduce a smooth positive function ψ on $\mathbb{R}$ such that

$$\psi(x) = |x|^\beta + \eta, \text{if } |x| \geq 1,$$

and $\|f\|_\beta \equiv \sup_{x\in\mathbb{R}} |f(x)|(\psi(x))^{-1}$. Set

$$C_\beta \equiv \{f : f \text{ is continuous on } \mathbb{R} \text{ with } \|f\|_\beta < \infty\}, \text{ for } 0 < \beta < 1.$$

For $\beta = 0$, C_0 is the set of all bounded continuous functions on $\mathbb{R}$. Define T, H by

$$Tf(x) \equiv E_x f(X_1), \quad Hf(x) \equiv E_x f(S_1), \quad \forall f \in C_\beta.$$

Definition 1.1. An ergodic Harris chain $\{X_n\}$ is called "exponentially uniformly ergodic in the sense of the operator norm" iff, there exist two positive constants ε and C such that $\|T^n - \mu(\cdot)\| \leq Ce^{-n\varepsilon}$ for every positive integer n, where

$$\|T^n - \mu(\cdot)\| = \sup\{\|T^n f - \mu(f)\|_\beta : f \in C_\beta, \|f\|_\beta \leq 1\}.$$

Further, define

$$\alpha_X(\beta) \equiv \max\{\varepsilon : \ \|T^n - \mu(\cdot)\| \leq Ce^{-n\varepsilon}, \ \forall n \in \mathbb{N}\},$$

$\alpha_X(\beta)$ is called a "convergent rate" of $\{X_n\}$. Similarly, we define $\alpha_S(\beta)$ for $\{S_n\}$. Note that $\{S_n\}$ is obtained from $\{S_t\}$ by restricting values of t to non-negative integers.

Our purpose in this article is to study the convergent speed of $\{X_n\}$ and to compare the size of $\alpha_X(\beta)$ with $\alpha_S(\beta)$.

We find that $\{X_n\}$ is exponentially uniformly ergodic in the sense of the operator norm for $0 < \beta < 1$. Moreover, we obtain $\alpha_X(\beta) \leq (-\beta \ln c_-) \wedge (-\beta \ln d_+)$ under a mild condition. And if $0 < \beta < 0.5$, then $\alpha_S(\beta) \geq -\ln \lambda$, where λ is given in Section 2. In particular, if $0 < \beta < 0.5$, $\int_1^\infty y^2 m(y)dy = \infty$ and $\int_1^\infty y m(y)dy < \infty$, then $\alpha_S(\beta) = -\ln \lambda$. The Ornstein-Uhlenbeck process, $c_- = d_+ = 0.5$ and $c_+ = d_- = 1.5$ are taken to show $\alpha_X(\beta) < \alpha_S(\beta)$ for $0 < \beta < 0.5$.

An outline of this article is as follows. In Section 2, we present the main theorems. Proofs of lemmas are given in the last section.

2. Main Theorems

Our main theorems are the followings.

Theorem 2.1. *If $0 < \beta < 1$, then $\{X_n\}$ is exponentially uniformly ergodic in the sense of the operator norm.*

Proof. It is clear that $\frac{T^n}{n}$ converges weakly to 0. Since $\rho^{\pm}(x)$ satisfies (2, 3), we obtain $\delta < 1$ in Lemma 3.3. This implies that T is a quasi-compact operator. Thus, by Theorem 2.8 of [5] page 91 (or Theorem 6-7 of [1] pages 713-714), we obtain that

(i) $T^n = \sum_{i=1}^k \lambda_i^n P_i + S^n$ for each positive integer n, where $\{\lambda_1, \lambda_2, \cdots, \lambda_k\}$ is the set of all eigenvalues of T with $\lambda_i = e^{2\pi i \theta_i}$, θ_i is rational and P_i is a projection with $\frac{TP_i}{\lambda_i} = \frac{P_i T}{\lambda_i} = P_i = P_i^2$ for $i = 1, 2, \cdots, k$,

(ii) $S = T - \sum_{i=1}^k \lambda_i P_i$ with $\|S^n\| \leq C\rho^n$ for each positive integer n, where $\rho \in (0,1)$ and C is a positive constant.

Let $n_0 = \min\{m \in \mathbb{N} : \lambda_i^m = 1 \text{ for all } i = 1, 2, \cdots, k\}$. It is clear that $\|T^{n_0 n} f - Pf\|_\beta \leq C\rho^n \|f\|_\beta$ for any $f \in C_\beta$ and any positive integer n, where $P = \sum_{i=1}^k P_i$. Notice that $Pf(x) = \int_{\mathbb{R}} f(x)\mu(dx)$ for any $f \in C_\beta$ and any $x \in \mathbb{R}$. To complete the proof, we must claim that $n_0 = 1$. Assume $n_0 \neq 1$. Then by the definition of n_0, we obtain that T has an eigenvalue $\lambda = e^{\frac{2\pi i}{n_0}}$. And $f \in C_\beta$ is an eigenfunction corresponding to λ. Thus $E_x f(X_n) = \lambda^n f(x)$ for every n. Let $Y_n = \lambda^{-n} f(X_n)$. Then $\{Y_n\}$ is a martingale under P_x. Moreover, since

$|f(x)| \le \|f\|_\beta \psi(x)$ and by Lemma 3.1, we get

$$E_x|Y_n| \le \|f\|_\beta E_x\psi(X_n) \le \|f\|_\beta(\theta_r \sum_{k=0}^{n-1} \gamma_r^k + \gamma_r^n \psi(x)).$$

This implies that there exists a finite random variable Y such that

$$P_x(\lim_{n\to\infty} Y_n = Y) = 1. \tag{4}$$

Under the assumption that T has an eigenvalue $\lambda = e^{i\theta}$ with $\theta \in (0, 2\pi)$, we will claim that $f(x)$ is a non-zero constant function firstly. We show this on the contrary. Suppose that $f(x)$ is not a constant function. Hence there exist a, b, $a \neq b$ and a positive constant ϵ such that $U_a \cap U_b = \emptyset$, where

$$U_a \equiv \{x \in \mathbb{R} : |f(x) - a| < \epsilon\},\ U_b \equiv \{x \in \mathbb{R} : |f(x) - b| < \epsilon\}.$$

On the other hand, it is not hard to obtain that $\{X_{n_0 n}\}_{n\ge 0}$ is positive recurrent. Further, since $p_c(x, y) > 0$ for any $y \in (\rho^-(x), \rho^+(x))$ and

$$\mu(U) = \int_{\mathbb{R}} p^n(y, U)\mu(dy),\ \forall n = 1, 2, \cdots,$$

we obtain $\mu(U) > 0$ for any open set U. Since $\{X_{n_0 n}\}$ is positive recurrent and $\mu(U_a) > 0$, $\mu(U_b) > 0$, we get

$$P_x\left(\sum_{n=0}^{\infty} 1_{U_a}(X_{n_0 n}) = \infty\right) = 1,\ P_x\left(\sum_{n=0}^{\infty} 1_{U_b}(X_{n_0 n}) = \infty\right) = 1. \tag{5}$$

Since $\lambda^{n_0} = 1$, we obtain that (5) contradicts (4). Consequently, we obtain that $f(x)$ is a non-zero constant. This implies $\lambda^n = 1$ for every positive integer n. But this contradicts that T has an eigenvalue $\lambda = e^{i\theta}$, $\theta \in (0, 2\pi)$. Hence $n_0 = 1$. This completes the proof. ■

Theorem 2.2. *If* $0 < \beta < 0.5$, *then* $\alpha_S(\beta) \ge -\ln\lambda$, *where*

$$\lambda = \max\left\{(f, \widehat{H}f)_m : \ f \in L^2(m),\ \int_{\mathbb{R}} f(x)m(x)dx = 0,\ \int_{\mathbb{R}} f^2(x)m(x)dx = 1\right\},$$

and $\widehat{H}f(x) = E_x f(S_1)$ *for all* $f \in L^2(\mathbb{R}, m(x)dx)$.

Remark 2.3. Assume $\int_1^\infty y^2 m(y)dy = \infty$ and $\int_1^\infty ym(y)dy < \infty$ in Theorem 2.2. Because the argument used in the proof of Lemma 3.5 can also work for $-\alpha(< 0)$ which is not the second largest eigenvalue, we obtain that other eigenfunctions of $\widehat{H}$ have the similar asymptotic behaviors like Lemma 3.5 (ii), This implies that all eigenfunctions of $\widehat{H}$ are bounded. It follows that all eigenfunctions of $\widehat{H}$ belong to C_β. This leads to the fact that the set of all eigenvalues of $\widehat{H}$ is a subset of all eigenvalues of H. Since $C_\beta \subset L^2(\mathbb{R}, m(x)dx)$ provided $0 < \beta < 0.5$, we get that the set of all eigenvalues of H is a subset of all eigenvalues of $\widehat{H}$. This establishes that the set of all eigenvalues of $\widehat{H}$ is the same as the set of all eigenvalues of H. Moreover, under the condition (1), it can be shown that $\widehat{H} : L^2(\mathbb{R}, m(x)dx) \to L^2(\mathbb{R}, m(x)dx)$ is a compact operator in terms of the Krein's spectral theory (cf. Theorem 2 of [4] page 252). By the argument in the proof of Theorem 2.2 below, we obtain that $e^{-\alpha_S(\beta)} = \lambda$ provided $0 < \beta < 0.5$, that is, $\alpha_S(\beta) = -\ln \lambda$.

Proof. Since $0 < \beta < 0.5$, we get $C_\beta \subset L^2(\mathbb{R}, m(x)dx)$. It is clear that eigenvalues of H are eigenvalues of $\widehat{H}$. Further, by Lemma 2.2, we obtain that H is a compact operator. Hence the spectrum of H consists of an at most countable set of points of the complex plane which has no point of accumulation except possibly zero (cf. Theorem 2 of [6] page 284). As mentioned in Remark 3.2, we know that $\widehat{H}$ is a compact operator on $L^2(\mathbb{R}, m(x)dx)$. Hence, every non-zero number in the spectrum of H(resp. $\widehat{H}$) is an eigenvalue of H(resp. $\widehat{H}$). Since $C_\beta \subset L^2(\mathbb{R}, m(x)dx)$, we get that the spectrum of H is contained in the spectrum of $\widehat{H}$. On the other hand, since $\widehat{H}$ is a non-negative self-adjoint compact operator on $L^2(\mathbb{R}, m(x)dx)$, we have that the largest eigenvalue of $\widehat{H}$ is 1 and the second largest eigenvalue λ is

$$\lambda = \max\{(f, \widehat{H}f)_m : f \in L^2(m),\ \int_{\mathbb{R}} f(x)m(x)dx = 0,\ \int_{\mathbb{R}} f^2(x)m(x)dx = 1\}.$$

Since a compact operator is also a quasi-compact operator, by Theorem 2.8 of [5] page 91, we have $H^n = P_1 + S^n$ such that $S = T - P_1$ and $\|S^n\| \le M\lambda^n$ for each positive integer n, where M is a positive constant and P_1 is the projection with $P_1 f(x) = \int_{\mathbb{R}} f(y)m(y)dy$ for any $f \in C_\beta$. By definition of $\alpha_S(\beta)$, we get $e^{-\alpha_S(\beta)} \le \lambda$. This leads $\alpha_S(\beta) \ge -\ln \lambda$. This completes the proof. ∎

Theorem 2.4. *If one of the following conditions holds*;

(i) $\int_{\mathbb{R}} x^2 m(x)dx < \infty$ *and* $0 < \beta < 1$,

(ii) $\int_{\mathbb{R}} x^2 m(x)dx = \infty$, $\int_{\mathbb{R}} |x|m(x)dx < \infty$ *and* $0 < \beta < 1$,

(iii) $\int_{\mathbb{R}} |x|m(x)dx = \infty$ *and* $\frac{1}{2} \le \beta < 1$,

then $\alpha_X(\beta) \le (-\beta \ln c_-) \wedge (-\beta \ln d_+)$.

Proof. By Lemma 3.5, we obtain $\varphi \in C_\beta$ under the condition (ii) or the condition (iii). Also notice that we can choose $\varphi(x) = c_1 + o(1)$ with $c_1 > 0$

in Lemma 3.5. It is clear that $\varphi(x) = c_1 + o(1)$ with $c_1 > 0$ is bounded. This gives that $\varphi \in C_\beta$ under the condition (i). Now we will claim that $\alpha_X(\beta) \leq (-\beta \ln c_-) \wedge (-\beta \ln d_+)$ under $\varphi \in C_\beta$. By Lemma 3.6, we obtain

$$E_x\varphi(X_n) = e^{-n\alpha}\varphi(x) + h_n(x),\ \forall x \in \mathbb{R},$$

where

$$h_n(x) = e^{-(n-1)\alpha}g(x) + \cdots + e^{-\alpha}E_x g(X_{n-2}) + E_x g(X_{n-1}).$$

Let

$$l_n = \frac{1}{c_-^n},\ k_n = \frac{1}{d_+^n},\ \kappa \equiv \inf\{g(x) : x \geq l_1\},\ \zeta \equiv \inf\{(-g)(x) : x \leq -k_1\}.$$

By Lemma 3.6, we get that $\kappa > 0$, $\zeta > 0$ and for $1 \leq i \leq n$,

$$E_x g(X_{i-1}) \geq \kappa, \forall x \geq l_n; \quad E_x(-g)(X_{i-1}) \geq \zeta, \forall x \leq -k_n.$$

Notice that under the condition $x \geq l_n$, we have $X_{i-1} \geq c_- X_{i-2} \geq \cdots \geq c_-^{i-1}x \geq l_1$ for each i with $1 \leq i \leq n-1$. This gives

$$\begin{aligned} h_n(x) &\geq \frac{\kappa(1-e^{-n\alpha})}{1-e^{-\alpha}},\ \forall x \geq l_n, \\ (-h_n)(x) &\geq \frac{\zeta(1-e^{-n\alpha})}{1-e^{-\alpha}},\ \forall x \leq -k_n. \end{aligned} \tag{6}$$

Since the value of $\mu(\varphi)$ has three possibilities, we consider the following cases;

Case 1. $\mu(\varphi) = 0$.

$$\begin{aligned} \|T^n\varphi - \mu(\varphi)\|_\beta &= \|T^n\varphi\|_\beta \\ &\geq \sup_{x \geq l_n} (\psi(x))^{-1}|T^n\varphi(x)| \\ &= \sup_{x \geq l_n} (\psi(x))^{-1}\left(e^{-\alpha n}\varphi(x) + h_n(x)\right). \end{aligned}$$

By (6), we obtain

$$\begin{aligned} \|T^n\varphi\|_\beta &\geq \sup_{x \geq l_n} (\psi(x))^{-1}\left(e^{-\alpha n}\varphi(x) + h_n(x)\right) \\ &\geq \sup_{x \geq l_n} (\psi(x))^{-1}\left(e^{-\alpha n}\varphi(x) + \frac{\kappa(1-e^{-n\alpha})}{1-e^{-\alpha}}\right) \\ &\geq \frac{\kappa c_-^{n\beta}(1-e^{-n\alpha})}{(1+\eta c_-^{n\beta})(1-e^{-\alpha})} + (\psi(l_n))^{-1}\varphi(l_n)e^{-\alpha n}. \end{aligned} \tag{7}$$

By the definition of $\alpha_X(\beta)$ and (7), we obtain for every n

$$e^{-n\alpha_X(\beta)} \geq \frac{1}{C\|\varphi\|_\beta}\left\{\frac{\kappa c_-^{n\beta}(1-e^{-n\alpha})}{(1+\eta c_-^{n\beta})(1-e^{-\alpha})}+(\psi(l_n))^{-1}\varphi(l_n)e^{-\alpha n}\right\}.$$

This implies

$$e^{-\alpha_X(\beta)} \geq \left(\frac{\kappa c_-^{n\beta}(1-e^{-n\alpha})}{C\|\varphi\|_\beta(1+\eta c_-^{n\beta})(1-e^{-\alpha})} \vee \frac{\varphi(l_n)e^{-n\alpha}}{C\|\varphi\|_\beta\psi(l_n)}\right)^{\frac{1}{n}}.$$

Let n approach to infinity, we obtain $e^{-\alpha_X(\beta)} \geq c_-^\beta \vee e^{-\alpha}$. Notice $\varphi \in C_\beta$, hence $\lim_{n\to\infty}(\psi(l_n))^{-\frac{1}{n}}(\varphi(l_n))^{\frac{1}{n}} \leq 1$. This gives $\alpha_X(\beta) \leq (-\beta \ln c_-)$.

Case 2. $\mu(\varphi) < 0$. It is trivial that

$$\begin{aligned}\|T^n\varphi-\mu(\varphi)\|_\beta &\geq \sup_{x\geq l_n}(\psi(x))^{-1}|T^n\varphi(x)-\mu(\varphi)|\\ &\geq \sup_{x\geq l_n}(\psi(x))^{-1}|T^n\varphi(x)|.\end{aligned}$$

By the same argument, we obtain

$$\|T^n\varphi\|_\beta \quad \geq \quad \frac{\kappa c_-^{n\beta}(1-e^{-n\alpha})}{(1+\eta c_-^{n\beta})(1-e^{-\alpha})}+(\psi(l_n))^{-1}\varphi(l_n)e^{-\alpha n}.$$

Therefore, we get $\alpha_X(\beta) \leq (-\beta \ln c_-)$.

Case 3. $\mu(\varphi) > 0$.
Since $\mu(-\varphi) < 0$ and (6), we obtain

$$\begin{aligned}\|T^n(-\varphi)-\mu(-\varphi)\|_\beta &\geq \sup_{x\leq -k_n}(\psi(x))^{-1}|T^n(-\varphi)(x)-\mu(-\varphi)|\\ &\geq \sup_{x\leq -k_n}(\psi(x))^{-1}|T^n(-\varphi)(x)|\\ &\geq \sup_{x\leq -k_n}(\psi(x))^{-1}\left(e^{-\alpha n}(-\varphi)(x)+(-h_n)(x)\right)\\ &\geq \sup_{x\leq -k_n}(\psi(x))^{-1}\left(e^{-\alpha n}(-\varphi)(x)+\frac{\zeta(1-e^{-n\alpha})}{1-e^{-\alpha}}\right)\\ &\geq \frac{\zeta d_+^{n\beta}(1-e^{-n\alpha})}{(1+\eta d_+^{n\beta})(1-e^{-\alpha})}+(\psi(-k_n))^{-1}(-\varphi)(-k_n)e^{-\alpha n}.\end{aligned}$$

In consequence, $\alpha_X(\beta) \leq (-\beta \ln d_+)$. Combining cases (I)(II)(III), we get $\alpha_X(\beta) \leq (-\beta \ln c_-) \wedge (-\beta \ln d_+)$. This completes the proof. ■

We conjecture that Theorem 2.4 also holds for $\int_{\mathbb{R}} |x|m(x)dx = \infty$ and $0 < \beta < \frac{1}{2}$. The following is an interesting example to show $\alpha_X(\beta) < \alpha_S(\beta)$ under $\int_{\mathbb{R}} |x|m(x)dx = \infty$ and $0 < \beta < \frac{1}{2}$.

Example 2.5. Assume that Z_t is an Ornstein-Uhlenbeck process with generator $\frac{\partial^2}{2\partial x^2} - x\frac{\partial}{\partial x}$. It is well-known that the spectrum of $\frac{\partial^2}{2\partial x^2} - x\frac{\partial}{\partial x}$ is $\{0, -1, -2, \cdots\}$ on $L^2(\mathbb{R}, e^{-x^2}dx)$. Let $S_t = s(Z_t)$, where $s(x) = \int_0^x e^{u^2}du$. It is clear that S_t has the generator $L = \frac{\partial^2}{m(x)\partial x^2}$, where $m(x) = 2e^{-2(s^{-1}(x))^2}$. It follows that

$$\lim_{|x|\to\infty} x^2 m(x) = 0, \qquad \int_{\mathbb{R}} |x|m(x)dx = \infty.$$

Also, the spectrum of L is $\{0, -1, -2, \cdots\}$ on $L^2(\mathbb{R}, m(x)dx)$. Now take $c_- = d_+ = 0.5, c_+ = d_- = 1.5$. Since $Ls^{-1}(x) = -s^{-1}(x)$ and $s^{-1} \in C_\beta$ for $0 < \beta < 1$, by the proof of Theorem 2.4, we obtain $\alpha_X(\beta) \leq \beta \ln 2$. If $\beta \in (0, 0.5)$, then we have $\lambda = e^{-1}$ in Theorem 2.2. By Theorem 2.2, we obtain $\alpha_S(\beta) \geq 1$. In consequence,

$$\alpha_X(\beta) \leq \beta \ln 2 < 1 \leq \alpha_S(\beta), \text{ for } 0 < \beta < 0.5.$$

3. Proofs of Lemmas

For $r > 1$, let δ_r^{-1} be

$$\begin{aligned}\delta_r^{-1} &= \sup_{|x|\geq r} \frac{-\psi(x)}{L\psi(x)} \\ &= \sup_{|x|\geq r} \frac{(1+\eta|x|^{-\beta})x^2 m(x)}{\beta(1-\beta)}.\end{aligned}$$

Then for any $r > 1$, there exists a constant c_r such that

$$L\psi(x) \leq c_r - \delta_r \psi(x) \text{ for any } x \in \mathbb{R}. \tag{8}$$

Lemma 3.1. *For any $r > 1$, there exists a constant ς_r such that*

$$H\psi(x) \leq \varsigma_r + e^{-\delta_r}\psi(x), \ \forall x \in \mathbb{R}.$$

Analogously, if $\rho^{\pm}(x)$ satisfy (2, 3)*, then*

$$T\psi(x) \leq \theta_r + \gamma_r \psi(x), \ \forall x \in \mathbb{R}.$$

where γ_r is a proper fixed constant with $\gamma_r \in (e^{-\delta_r}, 1)$ and θ_r is a proper constant depending on γ_r.

Proof. Ito's formula gives

$$\psi(S_t) = \psi(x) + M_t + \int_0^t (L\psi)(S_u)du$$

with a local martingale M_t. Let $\sigma = \inf\{t \geq 0 : S_t = a \text{ or } b\}$. For $a < x < b$, the optional stopping theorem shows

$$E_x\psi(S_{t\wedge\sigma}) = \psi(x) + E_x \int_0^{t\wedge\sigma} (L\psi)(S_u)du.$$

Taking the derivative of the above both sides, we see

$$\frac{\partial}{\partial t} E_x\psi(S_{t\wedge\sigma}) = E_x(L\psi)(S_{t\wedge\sigma})1_{\{\sigma\geq t\}}.$$

Hence (8) implies

$$\begin{aligned}\frac{\partial}{\partial t} E_x\psi(S_{t\wedge\sigma}) &\leq c_r - \delta_r E_x\psi(S_{t\wedge\sigma})1_{\{\sigma\geq t\}} \\ &= c_r - \delta_r E_x\psi(S_{t\wedge\sigma}) + \delta_r E_x\psi(S_\sigma)1_{\{\sigma<t\}}.\end{aligned}$$

Solving this differential inequality, we have

$$E_x\psi(S_{t\wedge\sigma}) \leq \frac{c_r(1-e^{-\delta_r t})}{\delta_r} + e^{-\delta_r t}\psi(x) + E_x\left(1 - e^{-\delta_r(t-\sigma)}\right)\psi(S_\sigma)1_{\{\sigma<t\}}. \tag{9}$$

Hence

$$E_x\psi(S_t)1_{\{\sigma\geq t\}} \leq \frac{c_r(1-e^{-\delta_r t})}{\delta_r} + e^{-\delta_r t}\psi(x). \tag{10}$$

Letting $a \to -\infty$, $b \to \infty$, we see

$$E_x\psi(S_t) \leq \frac{c_r(1-e^{-\delta_r t})}{\delta_r} + e^{-\delta_r t}\psi(x),$$

because $\sigma \uparrow \infty$, which concludes the first part of this lemma after setting $t = 1, \varsigma_r = \frac{c_r(1-e^{-\delta_r})}{\delta_r}$. On the other hand, let $b = \rho^+(x)$, $a = \rho^-(x)$. Hence $\sigma = \tau$. By $P_x(\tau^+ \leq \tau^-) = \frac{x-\rho^-(x)}{\rho^+(x)-\rho^-(x)}$, $P_x(\tau^- \leq \tau^+) = \frac{\rho^+(x)-x}{\rho^+(x)-\rho^-(x)}$, we obtain

$$E_x(1 - e^{\delta_r(\tau-t)})1_{\{\tau\leq t\}} \leq \left(\frac{x-\rho^-(x)}{\rho^+(x)-\rho^-(x)} + \frac{\rho^+(x)-x}{\rho^+(x)-\rho^-(x)}\right)(1-e^{-\delta_r}).$$

Further, since $\rho^{\pm}(x)$ satisfy (2, 3) and $-|x|^{\beta}$ is a convex function for $|x| > 1$, we obtain

$$\lim_{x\to\infty} \frac{E_x(1-e^{\delta_r(\tau-1)})\psi(S_\tau)1_{\{\tau\le 1\}}}{\psi(x)} \le \left\{\frac{c_+^{\beta}(1-c_-)}{c_+-c_-}+\frac{c_-^{\beta}(c_+-1)}{c_+-c_-}\right\}(1-e^{-\delta_r}) < 1-e^{-\delta_r}. \tag{11}$$

Similarly, we have

$$\lim_{x\to-\infty} (\psi(x))^{-1} E_x\left(1-e^{\delta_r(\tau-t)}\right)\psi(S_\tau)1_{\{\tau\le t\}} < 1-e^{-\delta_r}. \tag{12}$$

By (9, 11, 12), there exist positive constants J and γ_r with $\gamma_r \in (e^{-\delta_r}, 1)$ such that

$$E_x\psi(X_1) \le \varsigma_r + \gamma_r\psi(x), \text{ for } |x| \ge J, \tag{13}$$

Combine (9, 13), we get

$$E_x\psi(X_1) \le \theta_r + \gamma_r\psi(x), \text{ for } x \in \mathbb{R},$$

where

$$\theta_r \equiv \varsigma_r \vee \sup_{x\in[-J,J]} |E_x\psi(X_1) - \gamma_r\psi(x)|.$$

This completes the proof. ■

Lemma 3.2. *If* $0 < \beta < 1$, *then* $H : C_\beta \to C_\beta$ *is a compact operator.*

Proof. Set $\mathbb{B} = \left\{f \in C_\beta : \|f\|_\beta \le 1\right\}$, and choose any sequence $\{f_n\}_{n\ge1} \subset \mathbb{B}$. Since

$$Hf(x) = \int_R q(x,y)f(y)m(y)dy,$$

with a positive continuous kernel $q(x,y)$, then $\{Hf_n\}_{n\ge1}$ forms a relatively compact family on each compact interval because Hf_n is equi-bounded and equi-continuous (in n) on each fixed compact interval (cf. Ascoli-Arzelá Theorem in [6] page 85). Therefore we can pick up a subsequence $\{n(k)\}_{k\ge1}$ for which $Hf_{n(k)}$ converges to a $g \in C(\mathbb{R})$ uniformly on each compact interval. On the other hand, Lemma 3.1 shows

$$|Hf_n(x)| \le H\psi(x) \le \varsigma_r + e^{-\delta_r}\psi(x),$$

which implies $g \in C_\beta$, and

$$\begin{aligned}
&\|Hf_{n(k)} - g\|_\beta \\
\leq& \sup_{|x|\leq R} |Hf_{n(k)}(x) - g(x)|(\psi(x))^{-1} + \sup_{|x|>R} |Hf_{n(k)}(x) - g(x)|(\psi(x))^{-1} \\
\leq& \sup_{|x|\leq R} |Hf_{n(k)}(x) - g(x)|(\psi(x))^{-1} + 2 \sup_{|x|>R} \left[\varsigma_r(\psi(x))^{-1} + e^{-\delta_r}\right] \\
\leq& \sup_{|x|\leq R} |Hf_{n(k)}(x) - g(x)|(\psi(x))^{-1} + 2\varsigma_r R^{-\beta} + 2e^{-\delta_r}.
\end{aligned}$$

Choose $\varepsilon > 0$ and fix a sufficiently large r such that $2e^{-\delta_r} < \varepsilon$. Then choosing a sufficiently large R such that $2\varsigma_r R^{-\beta} < \varepsilon$, we have

$$\|Hf_{n(k)} - g\|_\beta \leq \sup_{|x|\leq R} |Hf_{n(k)}(x) - g(x)|(\psi(x))^{-1} + 2\varepsilon,$$

which completes the proof. ■

Let $Kf(x) = \int_{\rho^-(x)}^{\rho^+(x)} f(y)p_c(x,y)dy = E_x\left\{f(S_1) : \tau \geq 1\right\}, \forall f \in C_\beta$.

Lemma 3.3. *The following statements are valid.*

(i) *If* $0 < \beta < 1$, *then* $K : C_\beta \to C_\beta$ *is a compact operator.*

(ii) $\|T - K\| \leq \delta$, *where*

$$\delta \equiv \sup_{x\in\mathbb{R}} \left\{ \frac{\psi(\rho^+(x))O^+(x)}{\psi(x)} + \frac{\psi(\rho^-(x))O^-(x)}{\psi(x)} \right\},$$

$$O^+(x) = P_x\left(\tau^+ < \tau^-, \tau^+ < 1\right),\ O^-(x) = P_x\left(\tau^- < \tau^+, \tau^- < 1\right).$$

Proof. To show (i), we apply (10) for $\sigma = \tau$ and $t = 1$, then

$$K\psi(x) = E_x\psi(S_1)1_{\{\tau\geq 1\}} \leq \varsigma_r + e^{-\delta_r}\psi(x).$$

Since $p_c(x,y)$ is a a continuous kernel, the compactness of K can be proved exactly in the same manner as the proof of H. To show (ii), observe

$$\begin{aligned}
Tf(x) - Kf(x) =& E_x f(S_\tau)1_{\{\tau<1\}} \\
=& f(\rho^+(x))O^+(x) + f(\rho^-(x))O^-(x).
\end{aligned}$$

Therefore we have

$$\|Tf - Kf\|_\beta \leq \|f\|_\beta \sup_{x\in R} \left\{ \frac{\psi(\rho^+(x))O^+(x)}{\psi(x)} + \frac{\psi(\rho^-(x))O^-(x)}{\psi(x)} \right\},$$

which completes the proof. ∎

Remark 3.4. Set $E = \{x \in \mathbb{R} : \rho^-(x) \geq 1 \text{ or } \rho^+(x) \leq -1\}$.

$$c_1 = \sup_{x \in E} \left\{ \frac{\psi(\rho^+(x))O^+(x)}{\psi(x)} + \frac{\psi(\rho^-(x))O^-(x)}{\psi(x)} \right\}.$$

We remark that

$$\begin{aligned} O^+(x) < P_x\left(\tau^+ < \tau^-\right) &= \frac{x - \rho^-(x)}{\rho^+(x) - \rho^-(x)}, \\ O^-(x) < P_x\left(\tau^- < \tau^+\right) &= \frac{\rho^+(x) - x}{\rho^+(x) - \rho^-(x)}. \end{aligned}$$

Therefore the convexity of $\psi(x)$ for $|x| \geq 1$ implies

$$\begin{aligned} &\psi(\rho^+(x))O^+(x) + \psi(\rho^-(x))O^-(x) \\ &\quad \leq \psi\left(\frac{\rho^+(x)(x - \rho^-(x))}{\rho^+(x) - \rho^-(x)} + \frac{\rho^-(x)(\rho^+(x) - x)}{\rho^+(x) - \rho^-(x)} \right) \\ &\quad = \psi(x), \end{aligned}$$

hence $c_1 \leq 1$. On the other hand, generally we have

$$\frac{\psi(\rho^+(x))O^+(x)}{\psi(x)} + \frac{\psi(\rho^-(x))O^-(x)}{\psi(x)} \leq \frac{\psi(\rho^+(x)) \vee \psi(\rho^-(x))}{\psi(x)} P_x(\tau < 1).$$

Now set

$$\begin{aligned} c_2 &= \sup_{x \in A} \left\{ \frac{\psi(\rho^+(x)) \vee \psi(\rho^-(x))}{\psi(x)} P_x(\tau < 1) \right\}, \\ A &= \left\{ x \in \mathbb{R} : \rho^-(x) < 1, \rho^+(x) > -1 \right\}. \end{aligned}$$

Suppose A is bounded. Then $\sup_{x \in A} P_x(\tau < 1) < 1$, therefore if we choose an appropriate η, we can assume without loss of generality $c_2 < 1$. This is because

$$\frac{\psi(\rho^+(x)) \vee \psi(\rho^-(x))}{\psi(x)} \to 1, \text{ uniformly on } A \text{ as } \eta \uparrow \infty.$$

Since $\delta \leq c_1 \vee c_2$, a sufficient condition for $\delta < 1$ is

$$\sup_{x \notin A} \left\{ \frac{\psi(\rho^+(x))P_x(\tau^+ < \tau^-)}{\psi(x)} + \frac{\psi(\rho^-(x))P_x(\tau^+ < \tau^-)}{\psi(x)} \right\} < 1.$$

Under the condition (1), the operator L on $L^2(\mathbb{R}, m(x)dx)$ has a discrete spectrum (cf. Theorem 2 of [4] page 252 or Theorem 1-2 of [3] page 140-143). The largest eigenvalue is 0 and the eigenfunction is a constant. Let $-\alpha(<0)$ be the second largest eigenvalue and $\varphi(x)$ be its eigenfunction. It is also well known that $\varphi(x)$ has only one zero on $\mathbb{R}$ and for simplicity we let $\varphi(0) = 0$. Thus, without loss of generality, we assume $\varphi(x)$ is positive on $(0, \infty)$ and negative on $(-\infty, 0)$ in the sequel. The following lemma is consulted from [4].

Lemma 3.5 *$\varphi(x)$ is increasing on $(0,\infty)$ and has the following asymptotic behavior depending on the condition of m.*

(i) *If $\int_1^\infty x^2 m(x)dx < \infty$, then $\varphi(x) = c_1 + c_2 x + o(1)$ as $x \to \infty$, where $c_1, c_2 \geq 0$ and $c_1 + c_2 > 0$. Conversely, for any $c_1, c_2 \geq 0$ with $c_1 + c_2 > 0$, there exists a unique eigenfunction $\varphi(x)$ satisfying $\varphi(x) = c_1 + c_2 x + o(1)$ as $x \to \infty$.*

(ii) *If $\int_1^\infty x^2 m(x)dx = \infty$ and $\int_1^\infty x m(x)dx < \infty$, then $\varphi(x) = c_1 + o(1)$ as $x \to \infty$ with $c_1 > 0$.*

(iii) *If $\int_1^\infty x m(x)dx = \infty$, then $\varphi(x) \uparrow \infty$ and $\varphi(x) = o(\sqrt{x})$ as $x \to \infty$.*

Proof. At first, we will claim that $\varphi'(x) > 0$ for every $x > 0$. Assume $\varphi'(b) = 0$ for some $b > 0$. Since $\varphi''(x) = -\alpha\varphi(x)m(x) < 0$, we have $\varphi'(b+\delta) < 0$ for some small $\delta > 0$. The above also implies that $\varphi'(x)$ is decreasing. Then $\varphi'(x) < \varphi'(b+\delta)$ if $x \geq b+\delta$. Therefore, for $x \geq b + \delta$,

$$\varphi(x) = \varphi(b+\delta) + \int_{b+\delta}^{x} \varphi'(u)du \leq \varphi(b+\delta) + \varphi'(b+\delta)(x - b - \delta).$$

From this, we see $\varphi(x) < 0$ if x is large enough. However, this is a contradiction. Hence $\varphi'(x)$ can not have zeroes on $[0, \infty)$. Apparently $\varphi'(0) > 0$, hence $\varphi'(x) > 0$ for every $x \geq 0$. This shows that $\varphi(x)$ is increasing on $(0, \infty)$. Now let

$$h(x) = \frac{\varphi'(x)}{\varphi(x)}, \text{ for } x > 0.$$

Then

$$\varphi(x) = \varphi(1)e^{\int_1^x h(y)dy} \text{ for } x \geq 1.$$

Since

$$h'(x) = -\alpha m(x) - h(x)^2 < 0, \tag{14}$$

we obtain that $h(x)$ is positive and decreasing to 0 as $x \to +\infty$. In fact, since $h(x) > 0$, we see there exists a constant $c_0 \geq 0$ such that $h(x) \to c_0$, as $x \to \infty$. Thus, (14) implies

$$h(x) - h(y) = \int_x^y (\alpha m(z) + h(z)^2) dz,$$

hence letting $y \to \infty$, we have

$$h(x) - c_0 = \int_x^\infty (\alpha m(z) + h(z)^2) dz. \tag{15}$$

Therefore

$$\int_1^{+\infty} h(x)^2 dx < \infty$$

holds, which in particular implies $c_0 = 0$. On the other hand, since

$$\varphi''(x) = -\alpha m(x)\varphi(x) < 0 \text{ and } \varphi'(x) > 0,$$

the limit $\varphi'(x) \to c \geq 0$ exists as $x \to \infty$. And it is clear that

$$\frac{\varphi(x)}{x} \to c, \text{ as } x \to +\infty.$$

Suppose the condition (i) holds. Let f, g be the solutions of integral equations

$$f(x) = 1 - \alpha \int_x^\infty (y - x) f(y) m(y) dy, \tag{16}$$

$$g(x) = x - \alpha \int_x^\infty (y - x) g(y) m(y) dy, \tag{17}$$

respectively. The existence of f, g can be shown as follows. Let $f_0(x) = 1$ and $f_n(x) = \int_x^\infty (y - x) f_{n-1}(y) m(y) dy$. Under the condition (i), $f_n(x)$ is well-defined for any $n \geq 0$. Moreover, it follows that $f_n(x) \leq \frac{1}{n!} B(x)^n$ for any fixed x because

$$\begin{aligned}
f_n(x) &= \int_x^\infty (y - x) f_{n-1}(y) m(y) dy \\
&\leq \frac{1}{(n-1)!} \int_x^\infty (y - x) B(y)^{n-1} m(y) dy \\
&\leq \frac{1}{(n-1)!} \int_x^\infty y B(y)^{n-1} m(y) dy \\
&= -\frac{1}{(n-1)!} \int_x^\infty B(y)^{n-1} dB(y) \\
&= \frac{1}{n!} B(x)^n \text{ for any } n \geq 0,
\end{aligned}$$

where $B(x) = \int_x^\infty ym(y)dy$. Thus, (16) can be solved by

$$f(x) = 1 + \sum_{n=1}^{\infty} (-\alpha)^n f_n(x).$$

Similarly, under the condition (i), (17) can be solved by letting

$$g_0(x) = x,\ g_n(x) = \int_x^\infty (y-x) g_{n-1}(y) m(y) dy,\ g(x) = x + \sum_{n=1}^{\infty} (-\alpha)^n g_n(x).$$

Since f and g satisfy $\phi''(x) = -\alpha m(x)\phi(x)$ and they are linearly independent, we have

$$\varphi(x) = c_1 f(x) + c_2 g(x)$$

with some constants c_1, c_2. This completes the proof of the statement (i). Suppose the condition (ii) holds. Then $c = 0$, because, otherwise($c > 0$), $\varphi(x) \sim cx$, as $x \to \infty$, and $\varphi \in L^2(\mathbb{R}, m(x)dx)$ will imply

$$\int_1^\infty y^2 m(y) dy < \infty,$$

which contradicts the condition (ii). Now $\varphi'(x) = \alpha \int_x^\infty y\varphi(y)m(y)dy$, which implies, for $x \geq N$

$$\begin{aligned}
\varphi(x) &= \varphi(N) + \int_N^x \varphi'(y) dy \\
&= \varphi(N) + \alpha \int_N^x (y-N)\varphi(y)m(y)dy + \alpha(x-N) \int_x^\infty \varphi(y)m(y)dy \\
&\leq \varphi(N) + \alpha\varphi(x) \int_N^x (y-N)m(y)dy + \alpha(x-N) \int_x^\infty \varphi(y)m(y)dy,
\end{aligned}$$

since φ is increasing. Choosing sufficiently large N so that

$$\alpha \int_N^x (y-N)m(y)dy \leq \alpha \int_N^\infty ym(y)dy < 1,$$

we see that

$$\varphi(x) \leq A + B(x-N) \int_x^\infty \varphi(y)m(y)dy$$

with some $A, B > 0$. An iteration shows that $\varphi(x)$ is bounded under the condition (ii). This completes the proof of the statement (ii). $\varphi(x) \uparrow \infty$ can be accomplished by the identity (15), since

$$\begin{aligned}
\int_1^x h(y)dy &= \alpha \int_1^x dy \int_y^\infty m(z)dz + \int_1^x dy \int_y^\infty h(z)^2 dz \\
&\geq \alpha \int_1^x (y-1)m(y)dy + \alpha(x-1) \int_x^\infty m(y)dy.
\end{aligned}$$

Finally, the proof of $\varphi(x) = o(\sqrt{x})$ as $x \to \infty$ is given below. Under the condition $\int_1^\infty ym(y)dy = \infty$, we have $\varphi'(x) \to 0$, therefore

$$\begin{aligned}\varphi'(x) &= \alpha \int_x^\infty \varphi(y)m(y)dy \\ &\leq \alpha \left(\int_x^\infty \varphi(y)^2 m(y)dy \int_x^\infty m(y)dy \right)^{\frac{1}{2}}.\end{aligned}$$

However, the condition $m(x)x^2 \to 0$ implies

$$\int_x^\infty m(y)dy = o(x^{-1}).$$

This combined with $\varphi \in L^2(m)$ shows

$$\varphi'(x) = o(x^{-\frac{1}{2}}).$$

Since

$$\varphi(x) = \varphi(0) + \int_0^x \varphi'(y)dy,$$

we easily see that $\varphi(x) = o(\sqrt{x})$, as $x \to \infty$ holds. ∎

Lemma 3.6. *Assume*

(18) $$0 < \liminf_{x\to\infty} \frac{\rho^-(x)}{x} \leq \limsup_{x\to\infty} \frac{\rho^+(x)}{x} < \infty,$$

(19) $$0 < \liminf_{x\to-\infty} \frac{|\rho^+(x)|}{|x|} \leq \limsup_{x\to-\infty} |\frac{|\rho^-(x)|}{|x|} < \infty.$$

Then we have

$$T\varphi(x) = e^{-\alpha}\varphi(x) + g(x), \forall x \in \mathbb{R},$$

with

$$\inf_{x:x>0,\rho^-(x)>0}\{g(x)\} > 0, \qquad \inf_{x:x<0,\rho^+(x)<0}\{-g(x)\} > 0.$$

Proof. Firstly, we will claim $E_x\tau \to 0$ as $x \to \infty$. Observe

$$\begin{aligned}E_x\tau &= \frac{\rho^+(x) - x}{\rho^+(x) - \rho^-(x)} \int_{\rho^-(x)}^x (y - \rho^-(x))m(y)dy \\ &\quad + \frac{x - \rho^-(x)}{\rho^+(x) - \rho^-(x)} \int_x^{\rho^+(x)} (\rho^+(x) - y)m(y)dy.\end{aligned}$$

Therefore if $\rho^-(x) > 0$, we see

$$E_x\tau \le \left(\max_{z\in[\rho^-(x),\rho^+(x)]}(z^2 m(z))\right)(\rho^+(x)-\rho^-(x))\int_{\rho^-(x)}^{\rho^+(x)} y^{-2}dy$$
$$= \left(\max_{z\in[\rho^-(x),\rho^+(x)]}(z^2 m(z))\right)\frac{(\rho^+(x)-\rho^-(x))^2}{\rho^+(x)\rho^-(x)}.$$

Then (1) and (18) show that the right hand side converges to 0 as $x \to \infty$. This completes the claim. Secondly, consider

$$E_x\varphi(S_{1\wedge\tau}) = e^{-\alpha}\varphi(x) + E_x\left(1-e^{-\alpha(1-\tau)}\right)\varphi(S_\tau)1_{\{\tau\le 1\}}.$$

Therefore

$$g(x) = E_x\left(1-e^{-\alpha(1-\tau)}\right)\varphi(S_\tau)1_{\{\tau\le 1\}}.$$

Let $\varepsilon \in (0,1)$. Then $\rho^-(x) > 0$ implies

$$\begin{aligned}E_x\left(1-e^{-\alpha(1-\tau)}\right)\varphi(S_\tau)1_{\{\tau\le 1\}} &= E_x\left(1-e^{-\alpha(1-\tau)}\right)\varphi(S_\tau)1_{\{\tau\le \varepsilon\}}\\ &\quad + E_x\left(1-e^{-\alpha(1-\tau)}\right)\varphi(S_\tau)1_{\{\varepsilon<\tau\le 1\}}\\ &> E_x\left(1-e^{-\alpha(1-\tau)}\right)\varphi(S_\tau)1_{\{\tau\le \varepsilon\}}\\ &> \left(1-e^{-\alpha(1-\epsilon)}\right)E_x\varphi(S_\tau)1_{\{\tau\le \varepsilon\}}\\ &> \left(1-e^{-\alpha(1-\epsilon)}\right)\varphi(\rho^-(x))P_x(\tau\le\varepsilon).\end{aligned}$$

Since

$$\begin{aligned}P_x(\tau\le\varepsilon) &= 1-P_x(\tau>\varepsilon)\\ &\ge 1-\frac{E_x\tau}{\varepsilon}\to 1 \text{ as } x\to\infty,\end{aligned}$$

we obtain $\inf_{x:x>0,\rho^-(x)>0}\{g(x)\} > 0$. Similarly, $\inf_{x:x<0,\rho^+(x)<0}\{-g(x)\} > 0$ follows from (19). This completes the proof. ■

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