

FIXED POINTS AND NEGATIVE CIRCUIT FREE IN FINITE LATTICES

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Abstract. Let X be a dimensional finite lattice (not necessary distributive) and let F be a mapping from X to X . Here we introduce a new notion of neighbours of an element of X and prove that if all the neighbours of each element of X are in X and there is no negative circuit in the interaction graph of F , then F has a fixed point.

1. INTRODUCTION

Our starting point is the following theorem, which was proven by Shih and Dong [4] and was conjectured by Shih and Ho [5].

Theorem 1.1. *If the mapping $F : \{0, 1\}^n \rightarrow \{0, 1\}^n$ has the property that all the Boolean eigenvalues of the discrete Jacobian matrix of each element of $\{0, 1\}^n$ are zero, then it has a unique fixed point.*

Theorem 1.1 is a discrete model of the fixed point conjecture [2] equivalent the long-standing Jacobian conjecture.

Equivalently, Theorem 1.1 can be formulated as the following:

Theorem 1.2. *If the mapping $F : \{0, 1\}^n \rightarrow \{0, 1\}^n$ has no fixed point or has multiple fixed points, then there exists $x \in \{0, 1\}^n$ such that the network $\Gamma(F'(x))$ has a circuit.*

Here $F'(x)$ is the discrete Jacobian matrix of F evaluated at x . Theorem 1.1 may be called the network perspective of the Jacobian conjecture. It should be noted that

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Theorem 1.2 fits perfectly with the conjecture by the biologist René Thomas [3, 6] concerning the rule of cell differentiation as pointed out by Professor Christophe Soulé at IHÉS. The network perspective of the Jacobian conjecture goes deeper, it may raise a universal mathematical principle that gives to the cell differentiation and stabilization of gene expression.

The purpose of this paper is to study one aspect of Theorem 1.2 in the framework of lattices.

2. ASYNCHRONOUS DYNAMICAL GRAPH AND INTERACTION GRAPH

Let (X, \leq) be a *lattice*. This means that X is partially ordered by \leq , and that any two elements, x and y of X , have the least upper bound or supremum (namely the span $x \vee y$, read as 'x join y') and the greatest lower bound or infimum (namely the intersection $x \wedge y$, read as 'x meet y'). Let $S \subseteq X$. Similarly we write $\vee S$ (the 'join of S ') and $\wedge S$ (the 'meet of S ') instead of $\sup S$ and $\inf S$ when these exist. If $\vee S$ and $\wedge S$ exist for all $S \subseteq X$, then X is called a *complete lattice*. Let (X, \leq) be a lattice. We say X has a "1", if there exists $1 \in X$ such that $a = a \wedge 1$ for all $a \in X$. Dually, X is said to have a "0", if there exists $0 \in X$ such that $a = a \vee 0$ for all $a \in X$. A lattice, (X, \leq) , having 0 and 1 is called *bounded*. A lattice is *finite* if it has finite cardinality. Recall that a finite lattice (X, \leq) is complete and bounded with $1 = \vee X$ and $0 = \wedge X$ [1].

For $a, b \in X$, by "a covers b", means that $b < a$ and $b < x < a$ is not satisfied by any x in X . Let $C \subseteq X$ such that for any two elements, x and y in C , either $x \leq y$ or $y \leq x$. Then C is said to be *simply ordered*, and called a *chain*. For a lattice (X, \leq) , a segment is then drawn from a to b whenever a covers b . Any figure obtained as so is called a *diagram* of X . Let (X, \leq) be a lattice with the least element 0. Then $a \in X$ is called an *atom* if $0 < a$ and there is no x in X satisfying $0 < x < a$. The set of atoms of X is denoted by $At(X)$. We say X is *atomic* if given $x \neq 0$ in X , there exists $a \in At(X)$ such that $a \leq x$. We write the cardinality of $At(X)$ by $\#At(X)$. Recall that every finite lattice is atomic.

Let (X, \leq) be a finite lattice. If the set of atoms of X has cardinality n , then we put $At(X) = \{a_1, a_2, \dots, a_n\}$. For $x \in X$, $1 \leq i \leq n$, we define the i -th dimension $d_i[x]$ is $r+1$, if $a_i \leq x$ where r is the maximum length of chains $a_i = x^0 < \dots < x^r = x$, in which X having x for the greatest element. Otherwise, $d_i[x] = 0$. Note that $d_i[a_i] = 1$ and $d_i[a_j] = 0$ if $i \neq j$. A *dimensional finite lattice* is a finite lattice X in which for all $x, y \in X$, if $d_i[x] = d_i[y]$ for all $i \in \{1, \dots, n\}$, then $x = y$.

Recall that, a *distributive lattice* is a lattice in which for all x, y and $z \in X$,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

We see the two typical examples of nondistributive lattices M_3 and N_5 , they are dimensional finite lattice. Hence a dimensional finite lattice is not necessary distributive.

Consider a map F from a dimensional finite lattice X to itself. For $x \in X$, set $x = (x_1, \dots, x_n)$, $x_i = d_i[x]$, $1 \leq i \leq n$. Then we can set $F(x) = (f_1(x), \dots, f_n(x))$, $f_i(x) = d_i[F(x)]$, $1 \leq i \leq n$. Here n can be the number of interacting automata. For $i \in \{1, \dots, n\}$, the state x_i for an automaton i , is a finite integer. We consider the dimensional finite lattice X as the set of states of a dynamical system. The dynamics of the network is then described by a map $F : X \rightarrow X$.

For $x \in X$, $1 \leq i \leq n$, the state x_F^i is defined by $x_F^i = (x_1, \dots, f_i(x), \dots, x_n)$. If $x_F^i \in X$, $1 \leq i \leq n$, then we set:

$$U_F(x) = \{x_F^1, \dots, x_F^n\},$$

$$N_F(x) = \{i \in \{1, \dots, n\} \mid x_F^i \neq x\}.$$

Remark 2.1. For any $x \in X$, the number of different states of $U_F(x) \setminus \{x\}$ equals to the number of indices of $N_F(x)$, denoted as $|U_F(x) \setminus \{x\}| = |N_F(x)|$.

A *trajectory* in the dynamics is a sequence of states $\{x^1, \dots, x^r\}$ such that for each $t = 1, \dots, r - 1$, $x^{t+1} \in U_F(x^t) \setminus \{x^t\}$. In terms of *strategy* $\varphi : \{1, \dots, r - 1\} \rightarrow \{1, \dots, n\}$, there exists $\varphi(t) \in \{1, \dots, n\}$ such that

$$x^{t+1} = (x^t)_F^{\varphi(t)} \neq x^t.$$

A *cycle* is a trajectory in the form $\{x^1, \dots, x^r, x^1\}$ with $r > 1$. In this paper, *the asynchronous dynamical graph*, denoted $\Gamma(F)$, is the directed graph whose set of vertices is X and whose set of arcs is

$$\{(x, x_F^i) : x \in X \text{ and } i \in N_F(x)\}.$$

Remark 2.2. The following conditions are mutually equivalent:

- (a) x is a fixed point of F .
- (b) $U_F(x) = \{x\}$.
- (c) $N_F(x) = \phi$.
- (d) $|U_F(x)| = 1$.
- (e) $|N_F(x)| = 0$.
- (f) $\Gamma(F)$ has no arc from x to any vertex.

It is clear that there is an arc from x to y in $\Gamma(F)$ only if there exists a $\varphi(t) \in \{1, \dots, n\}$ such that $y = x_F^{\varphi(t)} \neq x$. A *trap domain* of the asynchronous dynamical graph is a non-empty subset $D \subseteq X$, such that for every arc (x, x_F^i) of the asynchronous dynamical graph that $x \in D$, $x_F^i \in D$. An *attractor* of the asynchronous dynamical graph is the smallest trap domain with respect to the inclusion. An attractor of cardinality at least two is called a *cycle*, and a fixed point of F is an attractor of

$\Gamma(F)$ with cardinality one. If x and y belong to the same attractor, then there exists a trajectory from x to y . A cycle C is called a *attractive cycle* if it is an attractor.

For $x \in X$, $1 \leq i \leq n$, the i -neighbour of x is denoted as x^{+i} and defined by

$$x^{+i} = (x_1, \dots, x_i + 1, \dots, x_n).$$

The *discrete derivative* of F at $x \in X$ is the Boolean $n \times n$ matrix, defined by

$$F'(x) = (f_{ij}(x)),$$

where $f_{ij}(x) = 1$ if $x^{+j} \in X$ and $f_i(x) \neq f_i(x^{+j})$, otherwise $f_{ij}(x) = 0$.

An *interaction graph*, denoted as $G(F)$, is the directed graph whose set of vertices is $\{1, \dots, n\}$ and whose set of arcs is

$$\{(j, s, i) : \exists x \in X, \text{ such that } f_{ij}(x) = 1\}.$$

For any arc (j, s, i) in $G(F)$, sign s is positive(negative) if $f_i(x^{+j})$ is greater than(less than) $f_i(x)$.

A *path* of $G(F)$ of length $r \geq 1$ is a sequence

$$\{(i_1, s_1, i_2), (i_2, s_2, i_3), \dots, (i_r, s_r, i_{r+1})\}.$$

Let the value of sign $s = \prod_{i=1}^r s_i$. It is a *circuit* if $i_{r+1} = i_1$ and it is an *elementary circuit* if, in addition, the vertices are mutually distinct.

3. ATTRACTIVE CYCLES AND NEGATIVE CIRCUITS

For an attractive cycle, each state in it has at least one successor. So, when the network is inside an attractive cycle, it cannot reach a fixed point. Thus it describes *sustained oscillations*. In this section, we are interested in the relationship between sustained oscillations produced by a attractive cycle and the negative circuits of the interaction graph of the network. Recall that if $G(F)$ has a negative circuit, then it has an elementary negative circuit. So, in order to prove that $G(F)$ has an elementary negative circuit, it is sufficient to prove that $G(F)$ has a negative circuit.

Theorem 3.1. *Let (X, \leq) be a dimensional finite lattice with $\#At(X) = n$. Suppose that $F : X \rightarrow X$ is a map such that $x_F^{+i} \in X$ for all $x \in X$, $1 \leq i \leq n$. If $G(F)$ has no negative circuit, then $\Gamma(F)$ has no attractive cycle.*

Remark 3.1 Let (X, \leq) be a dimensional finite lattice with $\#At(X) = n$. Suppose that $F : X \rightarrow X$ is a map, such that $x_F^{+i} \in X$ for all $x \in X$, $1 \leq i \leq n$. If $G(F)$ has no negative circuit, then F has at least one fixed point.

For $x \in X$, $1 \leq i \leq n$, we set $s_i(x) = 0$ if $f_i(x) = x_i$, $s_i(x) = 1$ if $f_i(x)$ is greater than x_i , and $s_i(x) = -1$ if $f_i(x)$ is less than x_i . In order to establish Theorem 3.1 we shall employ the following lemmas.

Lemma 3.1. *Let (X, \leq) be a dimensional finite lattice with $\#At(X) = n$. Suppose that $F : X \rightarrow X$ is a map, such that $x_F^i \in X$ for all $x \in X$, $1 \leq i \leq n$. If $\exists x \in X$, $\exists j \in N_F(x)$, $\exists i \in N_F(x_F^j)$, and $s_i(x) \neq s_i(x_F^j)$, then (j, s, i) is an arc of $G(F)$, where the value of sign s equals $s_j(x) s_i(x_F^j)$.*

Proof. Since $j \in N_F(x)$ and $i \in N_F(x_F^j)$, we obtain $s_j(x) \neq 0$ and $s_i(x_F^j) \neq 0$. Consider the following conditions:

- (C1) $s_j(x) = s_i(x_F^j) = 1$ and $s_i(x) \leq 0$.
- (C2) $s_j(x) = s_i(x_F^j) = -1$ and $s_i(x) \geq 0$.
- (C3) $s_j(x) = -1$, $s_i(x_F^j) = 1$ and $s_i(x) \leq 0$.
- (C4) $s_j(x) = 1$, $s_i(x_F^j) = -1$ and $s_i(x) \geq 0$.

Now, in order to prove this lemma, it is sufficient to prove the following statements:

- (a) If $\exists x \in X$ and $\exists i, j \in \{1, \dots, n\}$ that satisfies C1 or C2, then (j, s, i) is a positive arc of $G(F)$;
- (b) If $\exists x \in X$ and $\exists i, j \in \{1, \dots, n\}$ that satisfies C3 or C4, then (j, s, i) is a negative arc of $G(F)$.

First, we will prove the statement (a). Since the condition C1 or C2 holds, we obtain $j \neq i$.

If it satisfies C1, then $f_j(x) > x_j$. Next we consider the sequence $x_j = p_0, p_1, \dots, p_d = f_j(x)$ such that $p_k = p_{k-1} + 1$, $k = 1, \dots, d$. Setting

$$y^k = (x_1, \dots, x_{j-1}, p_k, x_{j+1}, \dots, x_n) (k = 0, \dots, d).$$

Since $x_F^j \in X$ and $p_k \leq f_j(x)$, we obtain $y^k \in X$. Since $s_i(x_F^j) = 1$ and $j \neq i$, we have $f_i(y^d) = f_i(x_F^j) > (x_F^j)_i = x_i = (y^d)_i$. Consider the smallest $0 \leq t \leq d$ such that $f_i(y^t) > (y^t)_i$. Since $s_i(x) \leq 0$ and $j \neq i$, $f_i(y^0) = f_i(x) \leq x_i = (y^0)_i$, we obtain $t > 0$ and $f_i(y^{t-1}) \leq (y^{t-1})_i$. By $j \neq i$, we deduce that $f_i(y^t) > x_i$ and $f_i(y^{t-1}) \leq x_i$, hence $f_i(y^{t-1}) < f_i(y^t)$, so $f_i(y^{t-1}) \neq f_i(y^t)$. Since $y^t = (y^{t-1})^{+j}$, we have $f_{ij}(y^{t-1}) = 1$, and thus, by $f_i(y^{t-1}) < f_i(y^t)$, we obtain (j, s, i) is a positive arc of $G(F)$.

If it satisfies C2, then $f_j(x) < x_j$. We consider the sequence $f_j(x) = p_0, p_1, \dots, p_d = x_j$, such that $p_k = p_{k-1} + 1$, $k = 1, \dots, d$. Setting

$$y^k = (x_1, \dots, x_{j-1}, p_k, x_{j+1}, \dots, x_n) (k = 0, \dots, d).$$

Since $x_F^{\sim j} \in X$ and $p_k \leq x_j$, we obtain $y^k \in X$. Since $s_i(x) \geq 0$ and $j \neq i$, we have $f_i(y^d) = f_i(x) \geq x_i = (y^d)_i$. Consider the smallest $0 \leq t \leq d$ such that $f_i(y^t) \geq (y^t)_i$. Since $s_i(x_F^{\sim j}) = -1$, $f_i(y^0) = f_i(x_F^{\sim j}) < (x_F^{\sim j})_i = (y^0)_i$, hence $t > 0$ and $f_i(y^{t-1}) < (y^{t-1})_i$. By $j \neq i$, we deduce that $f_i(y^t) \geq x_i$ and $f_i(y^{t-1}) < x_i$, hence $f_i(y^{t-1}) < f_i(y^t)$, so $f_i(y^{t-1}) \neq f_i(y^t)$. Since $y^t = (y^{t-1})^{+j}$, we have $f_{ij}(y^{t-1}) = 1$, and thus, by $f_i(y^{t-1}) < f_i(y^t)$, we obtain (j, s, i) is a positive arc of $G(F)$. Therefore the statement (a) is proved.

Next, we will prove the statement (b). We suppose it satisfies C3 (the other case being similar). Suppose $j = i$, otherwise it is similar to the proof of the statement (a). Since $s_i(x) = s_j(x) = -1$, $f_i(x) < x_i$. We consider the sequence $x_i = p_0, p_1, \dots, p_d = f_i(x)$, such that $p_k = p_{k-1} - 1$, $k = 1, \dots, d$. Setting

$$y^k = (x_1, \dots, x_{i-1}, p_k, x_{i+1}, \dots, x_n) (k = 0, \dots, d).$$

So $(y^0)_i = x_i > f_i(x) = (y^d)_i$ and, by $s_i(x_F^{\sim j}) = s_i(x_F^{\sim i}) = 1$, we have $f_i(y^d) = f_i(x_F^{\sim i}) > (x_F^{\sim i})_i = f_i(x) = (y^d)_i$. Consider the smallest $0 \leq t \leq d$ such that $f_i(y^t) > (y^t)_i$. Since $f_i(y^0) = f_i(x) = (y^d)_i$, we have $t > 0$, and $f_i(y^{t-1}) \leq (y^d)_i$. We deduce that $f_i(y^t) > f_i(y^{t-1})$, so $f_i(y^t) \neq f_i(y^{t-1})$. Since $y^{t-1} = (y^t)^{+i}$, $f_{ii}(y^t) = 1$, and thus, by $f_i(y^t) > f_i(y^{t-1})$, (i, s, i) is a negative arc of $G(F)$, and the statement (b) is proved. ■

Lemma 3.2. *Let (X, \leq) be a dimensional finite lattice with $\#At(X) = n$. Suppose that $F : X \rightarrow X$ is a map, such that $x_F^{\sim i} \in X$ for all $x \in X$, $1 \leq i \leq n$. Let $\{x^1, \dots, x^r\}$ be an elementary trajectory of $\Gamma(F)$ with length $r > 1$, and let $i \in N_F(x^r)$. For $1 \leq p < r$, if $s_i(x^p) \neq s_i(x^r)$ then there exists $j \in N_F(x^1)$, such that $G(F)$ has a path from j to i with sign s , where the value of sign s equals $s_j(x^1) s_i(x^r)$.*

Proof. We proceed by induction.

Case $r = 2$. Let $j = \varphi(1)$. Then $j \in N_F(x^1)$ such that $x^2 = (x^1)_F^{\sim j} \neq x^1$; hence $s_j(x^1) \neq 0$. Following the conditions of this lemma, we have $i \in N_F(x^2) = N_F((x^1)_F^{\sim j})$ and $s_i(x^1) \neq s_i(x^2) = s_i((x^1)_F^{\sim j})$, and thus, by Lemma 3.1, (j, s, i) is an arc of $G(F)$, where the value of sign s equals $s_j(x^1) s_i((x^1)_F^{\sim j}) = s_j(x^1) s_i(x^2)$.

Case $r > 2$. Let $k = \varphi(r - 1)$. Then $k \in N_F(x^{r-1})$ such that $x^r = (x^{r-1})_F^{\sim k} \neq x^{r-1}$; hence $s_k(x^{r-1}) \neq 0$. Following the conditions of this lemma, we have $i \in N_F(x^r)$ and $s_i(x^{r-1}) \neq s_i(x^r) = s_i((x^{r-1})_F^{\sim k})$, and thus, by Lemma 3.1, (k, s_{ki}, i) is an arc of $G(F)$, where the value of sign s_{ki} equals $s_k(x^{r-1}) s_i((x^{r-1})_F^{\sim k}) = s_k(x^{r-1}) s_i(x^r)$.

Now, we consider the smallest $1 \leq p < r$ such that $s_k(x^p) = s_k(x^{r-1}) \neq 0$.

First, suppose that $p = 1$. Then $s_k(x^1) = s_k(x^{r-1})$, hence $s_k(x^1) \neq 0$; hence $k \in N_F(x^1)$ and this arc (k, s_{ki}, i) is a path from k to i , where the value of sign s_{ki} equals $s_k(x^{r-1}) s_i(x^r) = s_k(x^1) s_i(x^r)$. So that the lemma holds.

Next, suppose that $p > 1$. Then, by the choice of p , for all $1 \leq m < p$, we have $s_k(x^m) \neq s_k(x^{r-1})$. Thus the trajectory $\{x^1, \dots, x^p\}$ satisfies the conditions of the lemma for $k \in N_F(x^p)$. Since $p < r$, by induction hypothesis, there exists $j \in N_F(x^1)$ such that $G(F)$ has a path from j to k of sign s_{jk} , where the value of sign s_{jk} equals $s_j(x^1) s_k(x^p)$; hence $s_j(x^1) \neq 0$. Since $G(F)$ contains an arc from k to i with sign s_{ki} , we deduce that $G(F)$ contains a path from j to i of sign s , where the value of sign s equals $s_{jk}s_{ki} = s_j(x^1)s_k(x^p)s_k(x^{r-1})s_i(x^r)$, and since $s_k(x^p) = s_k(x^{r-1}) \neq 0$, we deduce that the value of sign s equals $s_j(x^1)s_i(x^r)$, and the lemma is proved. ■

Lemma 3.3. *Let (X, \leq) be a dimensional finite lattice with $\#At(X) = n$. Suppose that $F : X \rightarrow X$ is a map, such that $x_F^i \in X$ for all $x \in X, 1 \leq i \leq n$. Let A be an attractive cycle of $\Gamma(F)$. If there exists $x \in A$ such that $N_F(x) = \{i\}$, then $G(F)$ has a negative circuit.*

Proof. Let $x^1 = x$, then $s_i(x^1) \neq 0$. Suppose that $s_i(x^1) = 1$ (the other case being similar). Let $x^2 = (x^1)_F^i$, then $(x^2)_i = f_i(x^1) > (x^1)_i$ and (x^1, x^2) is an arc of $\Gamma(F)$. Since $x^1 \in A$, we have $x^2 \in A$. Since A is an attractive cycle, we deduce that $\Gamma(F)$ has an elementary trajectory $\{x^2, \dots, x^r\}$ from x^2 to $x^r = x^1$, all the vertices of which belong to A .

Assume on the contrary that $s_i(x^p) = 0$ or 1 for all $1 < p < r$. Then $(x^p)_i \leq f_i(x^p)$. Suppose $\varphi(p) \neq i$, then $(x^p)_i = (x^{p+1})_i$; otherwise, $f_i(x^p) = (x^{p+1})_i$. Hence $(x^p)_i \leq (x^{p+1})_i$ for all $1 < p < r$, and we deduce that $(x^2)_i \leq (x^r)_i = (x^1)_i$. But this is impossible if $s_i(x^1) = 1$.

Thus there exists a smallest $1 < p < r$ such that $s_i(x^p) = -1$. Then, $\{x^1, \dots, x^p\}$ is an elementary trajectory with $i \in N_F(x^p)$ and by the choice of p , we have $s_i(x^m) \neq s_i(x^p)$ for all $1 \leq m < p$. So, according to Lemma 3.2, there exists $j \in N_F(x^1)$ and $G(F)$ has a path from j to i with sign s , where the value of sign s equals $s_j(x^1) s_i(x^p)$. Since i is the unique one in $N_F(x^1)$, we have $j = i$ and consequently, $G(F)$ has a path from i to itself, and thus creating a circuit of sign s , where the value of sign s equals $s_i(x^1) s_i(x^p)$. Since $s_i(x^1) = 1$ and $s_i(x^p) = -1$, we deduce that $s_i(x^1) s_i(x^p) = -1$. Thus $G(F)$ has a negative circuit. ■

Lemma 3.4. *Let (X, \leq) be a dimensional finite lattice with $\#At(X) = n$. Suppose that $F : X \rightarrow X$ is a map, such that $x_F^i \in X$ for all $x \in X, 1 \leq i \leq n$. Let A be an attractive cycle of $\Gamma(F)$. If there are at least two different numbers i and j in $N_F(x)$ for all $x \in A$, then there exists $H : X \rightarrow X$, such that $\Gamma(H)$ contains an attractive cycle B which is strictly included in A , and the elementary circuit of $G(H)$ is an elementary circuit of $G(F)$ with the same sign.*

Proof. Let $y \in A$, by conditions of this lemma, there exists a number $j \in N_F(y)$.

Consider the map $H : X \rightarrow X$, defined by: for any $x \in X$

$$H(x) = (f_1(x), \dots, f_{j-1}(x), x_j, f_{j+1}(x), \dots, f_n(x)).$$

For any $x \in A$, if there is a number $i \in N_H(x)$, then by $h_j(x) = x_j$, we obtain $i \neq j$, hence $h_i(x) = f_i(x)$, so that $x_F^i = x_H^i$. Since A is an attractive cycle of $\Gamma(F)$, we have $x_H^i = x_F^i \in A$. So A is a trap domain of $\Gamma(H)$. By the definition of an attractor, we obtain that $\Gamma(H)$ contains one attractor $B \subseteq A$.

Let $x \in B$. Then $x \in A$. By conditions of this lemma, there exists a number $i \neq j$ such that $i \in N_F(x)$. We deduce that $x_i \neq f_i(x) = h_i(x)$ so $i \in N_H(x)$. Since $x \in B$ and B is an attractor of $\Gamma(H)$, $x_H^i \in B$. Thus B is an attractor of cardinality at least two, so B is an attractive cycle of $\Gamma(H)$.

Suppose, by contradiction, that $B = A$. Since $y \in A = B$ and $j \in N_F(y)$, we have $y_F^j \in B$. Since B is an attractive cycle of $\Gamma(H)$, we deduce that $\Gamma(H)$ has a trajectory $\{y = x^1, \dots, x^r = y_F^j\}$. Since $h_j(x) = x_j$ for all $x \in X$, we have $(x^1)_j = (x^2)_j = \dots = (x^r)_j$. So $y_j = (y_F^j)_j$, a contradiction. Therefore $B \neq A$.

If (k, s_1, i) is an arc of an elementary circuit of $G(H)$, then by the definition of elementary circuit, $k \neq i$ and $\exists x \in X$ such that $h_{ik}(x) = 1$. Thus $\exists z \in x^{+k}$ such that $h_i(x) \neq h_i(z)$. Suppose, by contradiction, that $i = j$ then $h_i(z) = h_j(z) = z_j$. Since $k \neq j$, we obtain $z_j = x_j = h_j(x) = h_i(x)$ as a contradiction. Therefore, $i \neq j$. We deduce that $f_i(x) = h_i(x) \neq h_i(z) = f_i(z)$, hence $f_{ik}(x) = 1$. Then it is clear that (k, s_2, i) is an arc of $G(F)$. Since $h_i(z) > h_i(x)$ if and only if $f_i(z) > f_i(x)$, we have $s_1 = s_2$. Thus we complete this proof. ■

Proof of Theorem 3.1. If there exists an attractive cycle of $\Gamma(F)$, denoted by A and there exists $x \in A$ such that i is the unique one in $N_F(x)$, then by Lemma 3.3, $G(F)$ has a negative circuit.

Otherwise, there are at least two different numbers i and j in $N_F(x)$ for all $x \in A$, then by Lemma 3.4, there exists $H_1 : X \rightarrow X$ such that $\Gamma(H_1)$ contains an attractive cycle B_1 which is strictly included in A , and the elementary circuit of $G(H)$ is an elementary circuit of $G(F)$ with the same sign.

If there exists $x \in B_1$ such that i is the unique one in $N_{H_1}(x)$, then by Lemma 3.3, $G(H_1)$ has a negative circuit, hence $G(H_1)$ has an elementary negative circuit. In the proof of Lemma 3.4, we obtain that it is an elementary negative circuit of $G(F)$.

If there are two different numbers i and j in $N_{H_1}(x)$ for all $x \in B_1$, by the same process of analysis, we obtain a sequence of sets: $A = B_0 \supset B_1 \supset B_2 \supset \dots$, and a sequence of maps from X to itself: $F = H_0, H_1, H_2, \dots$, such that B_i is an attractive cycle of $\Gamma(H_i)$ and there are two different numbers k_i and j_i in $N_{H_i}(x)$ for all $x \in B_i$ and for $i = 0, 1, 2, \dots$. Since X is finite, there exists a number r , such that B_r is the minimal subset of X , in which B_r is an attractive cycle of $\Gamma(H_r)$ and there are two different numbers k_r and j_r in $N_{H_r}(x)$ for all $x \in B_r$. By Lemma 3.4, there exists $H_{r+1} : X \rightarrow X$ such that $\Gamma(H_{r+1})$ contains an attractive cycle B_{r+1} which is

strictly included in B_r . By the minimality of B_r , there exists at most one number i_r in $N_{H_{r+1}}(x)$. Since B_{r+1} is an attractive cycle of $\Gamma(H_{r+1})$, we obtain that i_r is the unique one in $N_{H_{r+1}}(x)$. By Lemma 3.3, $G(H_{r+1})$ has a negative circuit, hence it has an elementary negative circuit C and then, C is an elementary negative circuit of $G(H_r)$, $G(H_{r-1})$, \dots , and $G(H_0) = G(F)$. Therefore, $G(F)$ has a negative circuit. ■

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