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GENERAL L_p-INTERSECTION BODIES

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Abstract. For $0 < p < 1$, Haberl and Ludwig defined symmetric and asymmetric L_p-intersection bodies. In this paper, we introduce general L_p-intersection bodies and study their properties. In particular, we obtain the extremal values of their volume and establish a Brunn-Minkowski type inequality for them.

1. INTRODUCTION

Classical intersection bodies of star bodies were defined by Lutwak (see [23]). During the past two decades, they and their L_p generalizations have received considerable attention (see [5, 6, 14, 15, 16, 20, 22, 23, 28]).

An L_p generalization of intersection bodies was first defined by Haberl and Ludwig. For $0 < p < 1$, Haberl and Ludwig ([8]) defined asymmetric L_p-intersection bodies and gave a characterization using the notion of valuation. They also pointed out that the classical intersection bodies may be obtained as a limit of L_p-intersection bodies as $p \to 1$. Recently, Haberl ([7]) obtained a series of results for L_p-intersection bodies and Berck ([2]) investigated the convexity of L_p-intersection bodies. L_p-intersection bodies are an important concept in the dual L_p Brunn-Minkowski theory. For further results on L_p-intersection bodies, see also [13, 33, 38, 39].

The main aim of this paper is to introduce general L_p-intersection bodies and to determine the extremal values of their volume. Moreover, we establish a Brunn-Minkowski type inequality for them.

If K is a compact star-shaped (about the origin) set in $\mathbb{R}^n$, its radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \longrightarrow [0, +\infty)$, is defined by (see [5])

$$\rho(K, x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \qquad x \in \mathbb{R}^n \setminus \{0\}.$$

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If ρ_K is positive and continuous, K will be called a star body (about the origin). Let $\mathcal{S}_o^n$ denote the set of star bodies (about the origin) in $\mathbb{R}^n$. Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$, where S^{n-1} denotes the unit sphere in $\mathbb{R}^n$.

If $c > 0$ and $K \in \mathcal{S}_o^n$, then $\rho(cK, \cdot) = c\rho(K, \cdot)$.

For $K, L \in \mathcal{S}_o^n$, $p > 0$ and $\lambda, \mu \geq 0$ (not both zero), the L_p-radial combination, $\lambda \circ K \tilde{+}_p \mu \circ L \in \mathcal{S}_o^n$, of K and L is defined by (see [7])

$$\rho(\lambda \circ K \tilde{+}_p \mu \circ L, \cdot)^p = \lambda \rho(K, \cdot)^p + \mu \rho(L, \cdot)^p. \tag{1.1}$$

It follows that $\lambda \circ K = \lambda^{1/p} K$. For $p = 1$, $\lambda \circ K \tilde{+}_p \mu \circ L$ is just the radial linear combination, $\lambda K \tilde{+} \mu L$, of K and L.

Lutwak introduced the following notion of an intersection body of a star body (see [23]): For $K \in \mathcal{S}_o^n$, the intersection body, IK, of K is the star body whose radial function in the direction $u \in S^{n-1}$ is equal to the $(n-1)$-dimensional volume of the section of K by $u^\perp$, the hyperplane orthogonal to u, i.e., for all $u \in S^{n-1}$,

$$\rho(IK, u) = V_{n-1}(K \cap u^\perp),$$

where V_{n-1} denotes $(n-1)$-dimensional volume.

In 2006, Haberl and Ludwig ([8]) defined the asymmetric L_p-intersection body $I_p^+ K$ as follows: For $K \in \mathcal{S}_o^n$, $0 < p < 1$, define

$$\rho_{I_p^+ K}^p(u) = \int_{K \cap u^+} |u \cdot x|^{-p} dx \tag{1.2}$$

for all $u \in S^{n-1}$, where $u^+ = \{x : u \cdot x \geq 0, x \in \mathbb{R}^n\}$ and $u \cdot x$ denotes the standard inner product of u and x. They also define

$$I_p^- K = I_p^+(-K). \tag{1.3}$$

From definitions (1.2) and (1.3), we see that

$$\rho_{I_p^- K}^p(u) = \rho_{I_p^+(-K)}^p(u) = \int_{-K \cap u^+} |u \cdot x|^{-p} dx = \int_{K \cap (-u)^+} |u \cdot x|^{-p} dx. \tag{1.4}$$

Moreover, Haberl and Ludwig ([8]) defined the (symmetric) L_p-intersection body as follows: For $K \in \mathcal{S}_o^n$, $0 < p < 1$, the L_p-intersection body, $I_p K$, of K is the origin-symmetric star body whose radial function is given by

$$\rho_{I_p K}^p(u) = \frac{1}{2} \int_K |u \cdot x|^{-p} dx \tag{1.5}$$

for all $u \in S^{n-1}$. Here for convenience, we add a coefficient $1/2$ in definition (1.5).

Haberl and Ludwig ([8]) pointed out that the classical intersection body, IK, of K is obtained as a limit of the L_p-intersection body of K, more precisely, for all $u \in S^{n-1}$,

$$\rho(IK, u) = \lim_{p \longrightarrow 1^-} 2(1-p)\rho(I_pK, u)^p.$$

In [22], Ludwig introduced a function $\varphi_\tau : \mathbb{R} \longrightarrow [0, +\infty)$ by

$$\varphi_\tau(t) = |t| - \tau t, \tag{1.6}$$

for $\tau \in [-1, 1]$. Using (1.6), we define the general L_p-intersection body with parameter τ as follows: For $K \in \mathcal{S}_o^n$, $0 < p < 1$ and $\tau \in (-1, 1)$, the general L_p-intersection body, $I_p^\tau K \in \mathcal{S}_o^n$, of K is defined by

$$\rho^p_{I_p^\tau K}(u) = i(\tau) \int_K \varphi_\tau^{-p}(u \cdot x) dx \tag{1.7}$$

for all $u \in S^{n-1}$, where

$$i(\tau) = \frac{(1+\tau)^p(1-\tau)^p}{(1+\tau)^p + (1-\tau)^p}. \tag{1.8}$$

From (1.6), (1.7) and (1.8), together with (1.2) and (1.4), we have that for all $u \in S^{n-1}$,

$$\begin{aligned}
\rho^p_{I_p^\tau K}(u) &= i(\tau) \int_K [|u \cdot x| - \tau(u \cdot x)]^{-p} dx \\
&= i(\tau) \left[\int_{K \cap u^+} (1-\tau)^{-p}(u \cdot x)^{-p} dx + \int_{K \cap (-u)^+} (1+\tau)^{-p}(-u \cdot x)^{-p} dx \right] \\
&= \frac{i(\tau)}{(1-\tau)^p} \int_{K \cap u^+} |u \cdot x|^{-p} dx + \frac{i(\tau)}{(1+\tau)^p} \int_{K \cap (-u)^+} |u \cdot x|^{-p} dx \\
&= \frac{(1+\tau)^p}{(1+\tau)^p + (1-\tau)^p} \rho^p_{I_p^+ K}(u) + \frac{(1-\tau)^p}{(1+\tau)^p + (1-\tau)^p} \rho^p_{I_p^- K}(u).
\end{aligned}$$

Now denote by

$$f_1(\tau) = \frac{(1+\tau)^p}{(1+\tau)^p + (1-\tau)^p}, \qquad f_2(\tau) = \frac{(1-\tau)^p}{(1+\tau)^p + (1-\tau)^p}, \tag{1.9}$$

where $\tau \in [-1, 1]$, then

$$\rho^p_{I_p^\tau K}(u) = f_1(\tau)\rho^p_{I_p^+ K}(u) + f_2(\tau)\rho^p_{I_p^- K}(u) \tag{1.10}$$

for all $u \in S^{n-1}$. By (1.2), we see that for all $u \in S^{n-1}$,

$$\rho^p_{I_p^{+1} K} = \lim_{\tau \longrightarrow 1} \rho^p_{I_p^\tau K}(u) = \rho^p_{I_p^+ K}(u) \tag{1.11}$$

and

$$\rho^p_{I_p^{-1}K} = \lim_{\tau \longrightarrow -1} \rho^p_{I_p^\tau K}(u) = \rho^p_{I_p^- K}(u). \tag{1.12}$$

By (1.10), for $K \in \mathcal{S}^n_o$, $0 < p < 1$ and $\tau \in [-1, 1]$, the general L_p-intersection body, $I_p^\tau K$, of K is given by

$$I_p^\tau K = f_1(\tau) \circ I_p^+ K \tilde{+}_p f_2(\tau) \circ I_p^- K. \tag{1.13}$$

From (1.13), it also follows that

$$I_p^0 K = \frac{1}{2} \circ I_p^+ K \tilde{+}_p \frac{1}{2} \circ I_p^- K = I_p K. \tag{1.14}$$

Our first main result is the determination of the extremal values of the volume of general L_p-intersection bodies:

Theorem 1.1. *If $K \in \mathcal{S}^n_o$, $0 < p < 1$, $\tau \in [-1, 1]$, then*

$$V(I_p K) \le V(I_p^\tau K) \le V(I_p^\pm K). \tag{1.15}$$

If K is not origin-symmetric, there is equality in the left inequality if and only if $\tau = 0$ and equality in the right inequality if and only if $\tau = \pm 1$.

Theorem 1.1 is a dual analogue of a volume inequality of Haberl and Schuster (see [9]) for polars of general L_p projection bodies which in turn is part of a new and rapidly evolving asymmetric L_p Brunn-Minkowski theory that has its origins in the work of Ludwig, Haberl and Schuster (see [3, 4, 7, 8, 9, 10, 11, 21, 22, 25, 26, 30, 31, 32, 33, 34, 35, 36, 37]).

We also establish the following Brunn-Minkowski type inequality for general L_p-intersection bodies with respect to L_q ($q > 0$) radial combinations of star bodies.

Theorem 1.2. *If $K, L \in \mathcal{S}^n_o$, $0 < p < 1$, $q > 0$ and $n-p > q$, then for $\tau \in [-1, 1]$,*

$$V(I_p^\tau(K \tilde{+}_q L))^{\frac{pq}{n(n-p)}} \le V(I_p^\tau K)^{\frac{pq}{n(n-p)}} + V(I_p^\tau L)^{\frac{pq}{n(n-p)}}, \tag{1.16}$$

with equality if and only if K and L are dilates.

Brunn-Minkowski type inequalities for intersection bodies and related operators have been the focus of recent interest. We refer to [1, 17, 18, 19, 27, 29, 41, 40, 42] for further information.

We give the proofs of Theorems 1.1-1.2 in Section 4. In addition, in the Section 3 we prove several properties of general L_p-intersection bodies.

2. L_p-Dual Mixed Volumes

For $p > 0$, the L_p-dual mixed volume is defined as follows (see e.g., [7, 38]): For $K, L \in \mathcal{S}^n_o$,

$$\frac{n}{p}\widetilde{V}_p(K,L)=\lim_{\varepsilon\to 0^+}\frac{V(K\tilde{+}_p\varepsilon\circ L)-V(K)}{\varepsilon}.$$

From this definition, Haberl [7] obtained the following integral representation of L_p-dual mixed volumes. If $K,L\in\mathcal{S}_o^n$, $p>0$, then

$$\widetilde{V}_p(K,L)=\frac{1}{n}\int_{S^{n-1}}\rho_K^{n-p}(u)\rho_L^p(u)dS(u). \tag{2.1}$$

Notice that

$$V(K)=\frac{1}{n}\int_{S^{n-1}}\rho_K^n(u)dS(u), \tag{2.2}$$

thus, by (2.1) and (2.2), we have

$$\widetilde{V}_p(K,K)=V(K). \tag{2.3}$$

The Minkowski inequality for L_p-dual mixed volumes can be stated as follows (see e.g., [7]):

Theorem 2.1. *If $K,L\in\mathcal{S}_o^n$, $p>0$, then for $n>p$,*

$$\widetilde{V}_p(K,L)\le V(K)^{\frac{n-p}{n}}V(L)^{\frac{p}{n}}; \tag{2.4}$$

for $n<p$,

$$\widetilde{V}_p(K,L)\ge V(K)^{\frac{n-p}{n}}V(L)^{\frac{p}{n}}. \tag{2.5}$$

In each case, equality holds if and only if K and L are dilates.

The Brunn-Minkowski inequality with respect to L_p-radial combinations (1.1) can be stated as follows:

Theorem 2.2. *If $K,L\in\mathcal{S}_o^n$, $p>0$ and $\lambda,\mu\ge 0$ (not both zero), then for $n>p$,*

$$V(\lambda\circ K\tilde{+}_p\mu\circ L)^{\frac{p}{n}}\le\lambda V(K)^{\frac{p}{n}}+\mu V(L)^{\frac{p}{n}}, \tag{2.6}$$

with equality if and only if K and L are dilates; for $n<p$, (2.6) is reversed.

Proof. For $n>p$, by (1.1) and (2.1), we have that for any $Q\in\mathcal{S}_o^n$,

$$\widetilde{V}_p(Q,\lambda\circ K\tilde{+}_p\mu\circ L)=\lambda\widetilde{V}_p(Q,K)+\mu\widetilde{V}_p(Q,L).$$

Combining this with inequality (2.4), yields

$$\widetilde{V}_p(Q,\lambda\circ K\tilde{+}_p\mu\circ L)\le V(Q)^{\frac{n-p}{n}}[\lambda V(K)^{\frac{p}{n}}+\mu V(L)^{\frac{p}{n}}].$$

Take $Q=\lambda\circ K\tilde{+}_p\mu\circ L$ and use (2.3), to get (2.6). According to the equality condition of (2.4), we see that equality holds in (2.6) if and only if K and L are dilates.

Similarly, if $n < p$, using (2.5), we obtain the reverse form of (2.6). ■

3. Properties of General L_p-intersection Bodies

In this section, we establish several properties of general L_p-intersection bodies.

Theorem 3.1. *If $K \in \mathcal{S}_o^n$, $0 < p < 1$, then for $\tau \in [-1, 1]$,*

$$I_p^{-\tau} K = I_p^{\tau}(-K) = -I_p^{\tau} K. \tag{3.1}$$

Proof. By (1.2) we have for $u \in S^{n-1}$,

$$\begin{aligned}\rho^p_{-I_p^+ K}(u) = \rho^p_{I_p^+ K}(-u) &= \int_{K \cap (-u)^+} |-u \cdot x|^{-p} dx \\ &= \int_{K \cap (-u)^+} |u \cdot x|^{-p} dx = \rho^p_{I_p^- K}(u).\end{aligned}$$

Thus, by (1.3),

$$I_p^- K = I_p^+(-K) = -I_p^+ K \tag{3.2}$$

and

$$I_p^+ K = I_p^-(-K) = -I_p^- K. \tag{3.3}$$

But by (1.9), we have that

$$f_1(\tau) + f_2(\tau) = 1; \tag{3.4}$$

$$f_1(-\tau) = f_2(\tau), \qquad f_2(-\tau) = f_1(\tau). \tag{3.5}$$

This together with (3.2), (3.3), (3.5) and (1.13), yields

$$\begin{aligned}I_p^{-\tau} K &= f_1(-\tau) \circ I_p^+ K \tilde{+}_p f_2(-\tau) \circ I_p^- K \\ &= f_2(\tau) \circ I_p^-(-K) \tilde{+}_p f_1(\tau) \circ I_p^+(-K) = I_p^{\tau}(-K)\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}I_p^{\tau}(-K) &= f_2(\tau) \circ I_p^-(-K) \tilde{+}_p f_1(\tau) \circ I_p^+(-K) \\ &= f_1(\tau) \circ [-I_p^+ K] \tilde{+}_p f_2(\tau) \circ [-I_p^- K] = -I_p^{\tau} K.\end{aligned} \tag{3.7}$$

Hence, from (3.6) and (3.7), we obtain (3.1). ■

Theorem 3.2. *If* $K \in \mathcal{S}_o^n$, $0 < p < 1$, *then* $I_p^+ K = I_p^- K$ *if and only if* K *is origin-symmetric.*

Proof. If $I_p^+ K = I_p^- K$, then by (3.2) we know that for all $u \in S^{n-1}$,

$$\rho^p_{I_p^+ K}(u) = \rho^p_{I_p^- K}(u) = \rho^p_{I_p^+(-K)}(u). \tag{3.8}$$

But (1.2) gives that

$$\rho^p_{I_p^+ K}(u) = \frac{1}{n-p} \int_{S^{n-1} \cap u^+} |u \cdot v|^{-p} \rho_K^{n-p}(v) dS(v)$$

and

$$\rho^p_{I_p^+(-K)}(u) = \frac{1}{n-p} \int_{S^{n-1} \cap u^+} |u \cdot v|^{-p} \rho_{-K}^{n-p}(v) dS(v).$$

From this and (3.8) we obtain

$$\int_{S^{n-1} \cap u^+} |u \cdot v|^{-p} [\rho_K^{n-p}(v) - \rho_K^{n-p}(-v)] dS(v) = 0. \tag{3.9}$$

Since $K \in \mathcal{S}_o^n$, $\rho_K^{n-p}(v) - \rho_K^{n-p}(-v)$ is continuous on $S^{n-1} \cap u^+$. Hence, if (3.9) holds for all $u \in S^{n-1}$, then (see [7])

$$\rho_K^{n-p}(v) - \rho_K^{n-p}(-v) = 0,$$

i.e., $\rho_K(v) = \rho_{-K}(v)$. This means that K is origin-symmetric.

Conversely, if K is origin-symmetric, i.e., $K = -K$, then by (3.2), we get

$$I_p^+ K = I_p^+(-K) = I_p^- K. \qquad \blacksquare$$

Theorem 3.3. *If* $K \in \mathcal{S}_o^n$, $0 < p < 1$, $\tau \in [-1, 1]$ *and* $\tau \neq 0$, *then*

$$I_p^\tau K = I_p^{-\tau} K \iff I_p^+ K = I_p^- K. \tag{3.10}$$

Proof. From (1.10) and (3.5), we have that for all $u \in S^{n-1}$,

$$\begin{aligned} \rho^p_{I_p^{-\tau} K}(u) &= f_1(-\tau)\rho^p_{I_p^+ K}(u) + f_2(-\tau)\rho^p_{I_p^- K}(u) \\ &= f_2(\tau)\rho^p_{I_p^+ K}(u) + f_1(\tau)\rho^p_{I_p^- K}(u). \end{aligned} \tag{3.11}$$

Hence, by (3.4) and (3.11), if $I_p^+ K = I_p^- K$, then for all $u \in S^{n-1}$,

$$\rho^p_{I_p^\tau K}(u) = \rho^p_{I_p^{-\tau} K}(u).$$

This gives $I_p^{\tau}K = I_p^{-\tau}K$.

Conversely, if $I_p^{\tau}K = I_p^{-\tau}K$, then (1.10) and (3.11) yield that

$$[f_1(\tau) - f_2(\tau)]\rho^p_{I_p^+K}(u) = [f_1(\tau) - f_2(\tau)]\rho^p_{I_p^-K}(u),$$

for all $u \in S^{n-1}$. Since $f_1(\tau) - f_2(\tau) \neq 0$ when $\tau \neq 0$, we conclude that $I_p^+K = I_p^-K$. ∎

From Theorem 3.2 and (3.10), we obtain that

Corollary 3.1. *If $K \in \mathcal{S}_o^n$, $0 < p < 1$, $\tau \in [-1, 1]$ and $\tau \neq 0$, then $I_p^{\tau}K = I_p^{-\tau}K$ if and only if K is origin-symmetric.*

In addition, using (1.10), (1.14) and Theorem 3.2, we have the following result.

Theorem 3.4. *If $K \in \mathcal{S}_o^n$, $0 < p < 1$, $\tau \in [-1, 1]$ and $\tau \neq 0$, then K is origin-symmetric if and only if $I_p^{\tau}K = I_pK$.*

Proof. From (1.14), we know that for all $u \in S^{n-1}$,

$$\rho^p_{I_pK}(u) = \frac{1}{2}\rho^p_{I_p^+K}(u) + \frac{1}{2}\rho^p_{I_p^-K}(u). \tag{3.12}$$

If K is origin-symmetric, then according to Theorem 3.2 and (3.12), we have

$$I_pK = I_p^+K = I_p^-K.$$

Similarly, for origin-symmetric star bodies, from (1.10), (3.4) and Theorem 3.2, we know that

$$I_p^{\tau}K = I_p^+K = I_p^-K.$$

From this, if K is origin-symmetric, then $I_p^{\tau}K = I_pK$.

Conversely, if $I_p^{\tau}K = I_pK$, then from (1.10) and (3.12) we have that for all $u \in S^{n-1}$,

$$f_1(\tau)\rho^p_{I_p^+K}(u) + f_2(\tau)\rho^p_{I_p^-K}(u) = \frac{1}{2}\rho^p_{I_p^+K}(u) + \frac{1}{2}\rho^p_{I_p^-K}(u).$$

This together with (3.4), yields

$$\left[f_1(\tau) - \frac{1}{2}\right]\rho^p_{I_p^+K}(u) = \left[f_1(\tau) - \frac{1}{2}\right]\rho^p_{I_p^-K}(u). \tag{3.13}$$

But $\tau \neq 0$ gives $f_1(\tau) - \frac{1}{2} \neq 0$. Thus, from (3.13), we obtain for all $u \in S^{n-1}$,

$$\rho^p_{I_p^+K}(u) = \rho^p_{I_p^-K}(u),$$

that is, $I_p^+K = I_p^-K$. This and Theorem 3.2 yield that K is an origin-symmetric star body. ∎

4. Proofs of Theorems 1.1-1.2

We first give the proof of Theorem 1.1:

Proof of Theorem 1.1. By (3.1), we have that for $K \in \mathcal{S}_o^n$, $0 < p < 1$ and $\tau \in [-1, 1]$,

$$V(I_p^\tau K) = V(I_p^{-\tau} K).$$

This together with inequality (2.6), yields that for $\tau \in [-1, 1]$, the function $V(I_p^\tau K)$ is convex and symmetric. Therefore,

$$V(I_p K) \le V(I_p^\tau K) \le V(I_p^\pm K).$$

This yields inequalities (1.15).

From the equality condition of (2.6), we see that equality holds in the right inequality of (1.15) if and only if $I_p^+ K$ and $I_p^- K$ are dilates. Hence, $I_p^+ K = c I_p^- K$ for some $c > 0$. Using $V(I_p^+ K) = V(I_p^- K)$, we see that $c = 1$. This gives $I_p^+ K = I_p^- K$. Thus, from Theorem 3.2, we see that if K is not origin-symmetric, then equality holds in the right inequality of (3.1) if and only if $\tau = \pm 1$.

From Theorem 3.4, we see that if K is not origin-symmetric, then equality holds in the left inequality of (1.15) if and only if $\tau = 0$. ■

In order to complete the proof of Theorem 1.2, we require the following lemma:

Lemma 4.1. *If* $K, L \in \mathcal{S}_o^n$, $0 < p < 1$, $q > 0$, $n - p > q$ *and* $\tau \in [-1, 1]$, *then for all* $u \in S^{n-1}$,

$$\rho_{I_p^\tau (K \tilde{+}_q L)}^{\frac{pq}{n-p}}(u) \le \rho_{I_p^\tau K}^{\frac{pq}{n-p}}(u) + \rho_{I_p^\tau L}^{\frac{pq}{n-p}}(u), \tag{4.1}$$

with equality if and only if K *and* L *are dilates.*

Proof. Since $q > 0$ and $n - p > q$, we have $(n-p)/q > 1$. From definition (1.7), a transformation to polar coordinates, and the Minkowski integral inequality (see [12]), we obtain for $\tau \in (-1, 1)$,

$$\begin{aligned}
\rho_{I_p^\tau (K \tilde{+}_q L)}^{\frac{pq}{n-p}}(u) &= \left[i(\tau) \int_{K \tilde{+}_q L} \varphi_\tau^{-p}(u \cdot x) dx \right]^{\frac{q}{n-p}} \\
&= \left[i(\tau) \int_{K \tilde{+}_q L} [|u \cdot x| - \tau (u \cdot x)]^{-p} dx \right]^{\frac{q}{n-p}} \\
&= \left[\frac{i(\tau)}{n-p} \int_{S^{n-1}} [|u \cdot v| - \tau (u \cdot v)]^{-p} \rho_{K \tilde{+}_q L}^{n-p}(v) dS(v) \right]^{\frac{q}{n-p}}
\end{aligned}$$

$$
\begin{aligned}
&= \left[\frac{i(\tau)}{n-p}\int_{S^{n-1}}[|u\cdot v|-\tau(u\cdot v)]^{-p}(\rho_K^q(v)+\rho_L^q(v))^{\frac{n-p}{q}}dS(v)\right]^{\frac{q}{n-p}}\\
&\le \left[\frac{i(\tau)}{n-p}\int_{S^{n-1}}[|u\cdot v|-\tau(u\cdot v)]^{-p}\rho_K^{n-p}(v)dS(v)\right]^{\frac{q}{n-p}}\\
&\quad+\left[\frac{i(\tau)}{n-p}\int_{S^{n-1}}[|u\cdot v|-\tau(u\cdot v)]^{-p}\rho_L^{n-p}(v)dS(v)\right]^{\frac{q}{n-p}}\\
&= \rho_{I_p^\tau K}^{\frac{pq}{n-p}}(u)+\rho_{I_p^\tau L}^{\frac{pq}{n-p}}(u)
\end{aligned}
$$

for all $u \in S^{n-1}$. This gives (4.1). From the equality condition of the Minkowski integral inequality, we see that equality holds in (4.1) if and only if K and L are dilates.

If $\tau = \pm 1$, then by (1.11) and (1.12), (4.1) is also true. ∎

Proof of Theorem 1.2. From $0 < p < 1$, $q > 0$ and $n - p > q$, we see that $n(n-p)/pq > 1$. Using (4.1) and the Minkowski integral inequality (see [12]), we obtain

$$
\begin{aligned}
V(I_p^\tau(K\tilde{+}_qL))^{\frac{pq}{n(n-p)}} &= \left[\frac{1}{n}\int_{S^{n-1}}\rho^n_{I_p^\tau(K\tilde{+}_qL)}(u)dS(u)\right]^{\frac{pq}{n(n-p)}}\\
&= \left[\frac{1}{n}\int_{S^{n-1}}[\rho^{\frac{pq}{n-p}}_{I_p^\tau(K\tilde{+}_qL)}(u)]^{\frac{n(n-p)}{pq}}dS(u)\right]^{\frac{pq}{n(n-p)}}\\
&\le \left[\frac{1}{n}\int_{S^{n-1}}[\rho^{\frac{pq}{n-p}}_{I_p^\tau K}(u)+\rho^{\frac{pq}{n-p}}_{I_p^\tau L}(u)]^{\frac{n(n-p)}{pq}}dS(u)\right]^{\frac{pq}{n(n-p)}}\\
&\le \left[\frac{1}{n}\int_{S^{n-1}}\rho^n_{I_p^\tau K}(u)dS(u)\right]^{\frac{pq}{n(n-p)}}+\left[\frac{1}{n}\int_{S^{n-1}}\rho^n_{I_p^\tau L}(u)dS(u)\right]^{\frac{pq}{n(n-p)}}\\
&= V(I_p^\tau K)^{\frac{pq}{n(n-p)}}+V(I_p^\tau L)^{\frac{pq}{n(n-p)}}.
\end{aligned}
$$

Hence, we obtain (1.16), and equality holds in (1.16) if and only if K and L are dilates. ∎

If $\tau = 0$ in Theorem 1.2, then the following Brunn-Minkowski inequality for L_p-intersection bodies follows.

Corollary 4.1. *If $K, L \in \mathcal{S}_o^n$, $0 < p < 1$, $q > 0$ and $n - p > q$, then*

$$
V(I_p(K\tilde{+}_qL))^{\frac{pq}{n(n-p)}} \le V(I_pK)^{\frac{pq}{n(n-p)}}+V(I_pL)^{\frac{pq}{n(n-p)}},
$$

with equality if and only if K and L are dilates.

Taking $q = 1$ in Corollary 4.1, and noting that $n \ge 2$ and $0 < p < 1$ imply that $n - p > 1$, we also have

Corollary 4.2. *If $K, L \in \mathcal{S}_o^n$, $0 < p < 1$ and $n \geq 2$, then*

$$V(I_p(K \tilde{+} L))^{\frac{p}{n(n-p)}} \leq V(I_pK)^{\frac{p}{n(n-p)}} + V(I_pL)^{\frac{p}{n(n-p)}}, \tag{4.2}$$

with equality if and only if K and L are dilates.

Inequality (4.2) is due to Yuan and Sum (see [39]). Since

$$\rho(IK, u) = \lim_{p \to 1^-} 2(1-p)\rho(I_pK, u)^p,$$

we can let $p \to 1$ in (4.2), to obtain

Corollary 4.3. *If $K, L \in \mathcal{S}_o^n$, $n \geq 2$, then*

$$V(I(K \tilde{+} L))^{\frac{1}{n(n-1)}} \leq V(IK)^{\frac{1}{n(n-1)}} + V(IL)^{\frac{1}{n(n-1)}}, \tag{4.3}$$

with equality if and only if K and L are dilates.

Inequality (4.3) can be found in [38, 39] and is the Brunn-Minkowski inequality for the classical intersection bodies.

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