

A SHARP MAXIMAL INEQUALITY FOR A GEOMETRIC BROWNIAN MOTION

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Abstract. Let $X = (X_t)_{t \geq 0}$ be a geometric Brownian motion with drift μ and volatility $\sigma > 0$, and let $Y = (Y_t)_{t \geq 0}$ be the associated maximum process of X . Under certain conditions, we prove a sharp maximal inequality for the geometric Brownian motion. The method of proof is essentially based on explicit forms of the following optimal stopping problem: Find a stopping time τ^* , if it exists, such that

$$\Phi(x, y) := \sup_{\tau} \mathbf{E}^{x, y} \left[Y_{\tau} - c \int_0^{\tau} X_s^{\theta} ds \right], \quad c, \theta > 0$$

where the supremum is taken over all stopping times τ for the process X .

The present result complements and extends a similar result proved by Graversen and Peskir [1].

1. INTRODUCTION

The original motivation of the present paper is the maximal inequality for a geometric Brownian motion proved by Graversen and Peskir [1], see also Peskir [2]. The result is derived from explicit forms of an optimal stopping problem with a constant observation cost. For a related class of optimal stopping problems, but with zero observation costs, see for instance Shepp and Shiryaev ([3],[4]). In the present paper, we prove a sharp maximal inequality which complements and extends a result by Graversen and Peskir [1].

The next result is proved in Graversen and Peskir ([1], pg 870).

Theorem 1.1. *Let $B = (B_t)_{t \geq 0}$ be a standard Brownian motion, and let $\mu < 0$ and $\sigma > 0$ be given and fixed. Then, the inequality*

$$\mathbf{E} \left(\max_{0 \leq t \leq \tau} \exp \left(\sigma B_t + \left(\mu - \frac{\sigma^2}{2} \right) t \right) \right) \leq 1 - \frac{\sigma^2}{2\mu} + \frac{\sigma^2}{2\mu} \exp \left(-\frac{(\sigma^2 - 2\mu)^2}{2\sigma^2} \mathbf{E}(\tau) - 1 \right)$$

holds whenever τ is a stopping time for B .

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It is essential to point out that the proof of the above result is based on optimal estimates of solutions of a certain first order nonlinear differential equation and explicit forms of the optimal stopping problem (2), in the special case when $\theta \equiv 0$. In the present paper, we shall prove a similar type of result in the case when $\theta > 0$ is strictly positive. This is the subject of the next section.

Let $X = (X_t)_{t \geq 0}$ be a geometric Brownian motion with drift μ and volatility $\sigma > 0$, and let $Y = (Y_t)_{t \geq 0}$ be the associated maximum process of X , given respectively by

$$(1) \quad \begin{aligned} dX_t &= \mu X_t dt + \sigma X_t dB_t; & X(0) &= x, \\ Y_t &= \max\{y, \max_{0 \leq u \leq t} X_u\}, \end{aligned}$$

initially starting at (x, y) such that $0 < x \leq y$ and where B_t is a standard Brownian motion.

Consider the following optimal stopping problem:

Problem 1.1. Find a stopping time τ^* , if it exists, such that

$$(2) \quad \Phi(x, y) := \sup_{\tau} \mathbf{E}^{x, y} \left[Y_{\tau} - c \int_0^{\tau} X_s^{\theta} ds \right], \quad c, \theta > 0$$

where the supremum is taken over all stopping times τ for the process X .

2. MAIN RESULTS

The following two results play an important role in the proof of our main result. The proofs of these results follow similarly as in Graverson and Peskir [1], with minor modifications. However, these results are not contained in [1].

Lemma 2.1. Let $n := 1 - \frac{2\mu}{\sigma^2}$ and $0 < \theta < 1$ be given and fixed, where $\mu < 0$ and $\sigma > 0$. If $g_*(y)$ is a maximal nonnegative solution of the first order nonlinear differential equation

$$(3) \quad g'(y) = \beta \frac{g(y)^{1+n-\theta}}{y^n - g(y)^n}$$

under the condition $0 < g(y) < y$, then

$$(4) \quad \frac{y}{\left[1 + \beta \left(\frac{n+\theta}{n-1}\right)y\right]^{1/n}} \leq g_*(y) \leq \left(\beta \frac{n-\theta}{n-1}\right)^{\frac{1}{\theta-n}} y^{\frac{n-1}{n-\theta}}$$

for all $y > \left(\beta \frac{n-\theta}{n-1}\right)^{\frac{1}{\theta-1}}$ with $\beta = \frac{1}{D\theta\left(\frac{\theta}{n-1}\right)}$ and $D = \frac{2c}{\sigma^2\theta(\theta-1)+2\mu\theta}$.

Proof. Using the well-known method of upper and lower solutions, we prove the existence of a maximal nonnegative solution $g^*(y)$ of the problem (3) under the restriction that $0 < g(y) < y$. Let $p = \beta(n + \theta)$ and $z(y) = g(y)^{n+\theta}$, then the problem (3) reduces to

$$(5) \quad z'(y) = p \frac{z(y)^{\frac{2n}{n+\theta}}}{y^n - z(y)^{\frac{n}{n+\theta}}}$$

for all $0 < z(y)^{\frac{1}{n+\theta}} < y$.

Assume that the lower solution is of the form

$$u(y)^{\frac{n}{n+\theta}} = \alpha(y)y^{n-1},$$

where $\alpha(\cdot)$ is an increasing strictly positive function to be determined. It now follows that

$$(6) \quad \begin{aligned} u'(y) - p \frac{u(y)^{\frac{2n}{n+\theta}}}{y^n - u(y)^{\frac{n}{n+\theta}}} &= \frac{d}{dy} (\alpha(y)y^{n-1})^{\frac{n+\theta}{n}} - p \frac{\alpha^2(y)y^{2n-2}}{y^n - \alpha(y)y^{n-1}} \\ &\geq \alpha(y)(n-1)y^{n-2} - p \frac{\alpha(y)^2 y^{2n-2}}{y^n - \alpha(y)y^{n-1}} \end{aligned}$$

which is a lower solution of the problem (3) by choosing $\alpha(y) = \frac{1}{\beta \left(\frac{n+\theta}{n-1} \right) + \frac{1}{y}}$ for all $y > 0$.

It now remains to construct an explicit upper solution $v(y)$ for the problem (5) such that $u(y) \leq v(y)$ for all $y > 0$. Using a comparison argument, it follows that

$$(7) \quad \begin{aligned} z'(y) - p \frac{z(y)^{\frac{2n}{n+\theta}}}{y^n - z(y)^{\frac{n}{n+\theta}}} &= v'(y) - p \frac{v(y)^{\frac{2n}{n+\theta}}}{y^n - v(y)^{\frac{n}{n+\theta}}} \\ &\leq v'(y) - p \frac{v(y)^{\frac{2n}{n+\theta}}}{y^n} \end{aligned}$$

which is an upper solution by choosing $v(y) = \left(\beta \frac{n-\theta}{n-1} \right)^{\frac{n+\theta}{\theta-n}} y^{(1-n)\frac{n+\theta}{\theta-n}}$ for all $y > 0$.

To this end, using similar arguments as in [1] it can be easily shown that there exists a maximal solution $g^*(y)$ between $u(y)$ and $v(y)$ for all $y > 0$. This completes the proof.

In the next result, we shall give explicit forms for the optimal stopping problem (2) subject to the process given in (1). Assuming that the supremum in (2) is attained, it follows from the general theory of optimal stopping problems that

$$(8) \quad \tau^* = \inf\{t > 0 : X_t \leq g_*(Y_t)\}$$

is the exit time of the process (X, Y) given in (1), where $y \mapsto g_*(y)$ is an optimal stopping boundary to be determined. It turns out that the problem of solving the optimal stopping problem (2) reduces to that of solving a free boundary problem: Find $g_*(y)$ and $\Phi(x, y)$ such that

$$(9) \quad L_X \Phi(x, y) = cx^\theta \text{ for } g_*(y) < x < y$$

which satisfies the following boundary conditions:

$$(10) \quad \Phi(x, y)|_{x=g_*(y)+} = y \quad (\text{instantaneous stopping}),$$

$$(11) \quad \Phi_x(x, y)|_{x=g_*(y)+} = 0 \quad (\text{smooth fit}),$$

$$(12) \quad \Phi_y(x, y)|_{x=y-} = 0 \quad (\text{normal reflection}),$$

where L_X is the infinitesimal generator associated with the process X .

Let $0 < \theta < 1$ and $n := 1 - \frac{2\mu}{\sigma^2}$ be fixed, then it follows immediately that (9) admits a general solution of the form

$$(13) \quad \Phi(x, y) = N(y) + M(y)x^n + \frac{cx^\theta}{\frac{1}{2}\sigma^2\theta(\theta - 1) + \mu\theta}$$

where $y \mapsto N(y)$ and $y \mapsto M(y)$ are unknown functions to be determined.

Now using the boundary conditions (10) and (11), we have

$$(14) \quad N(y) = y - \left(1 - \frac{\theta}{n}\right) Dg_*(y)^\theta \quad \text{and} \quad M(y) = -\frac{\theta}{n} Dg_*(y)^{\theta-n}$$

where $D = 2c/(\sigma^2\theta(\theta - 1) + 2\mu\theta)$ with $\mu < 0$ and $\sigma > 0$.

It follows that

$$(15) \quad \Phi(x, y) = y - \left(1 - \frac{\theta}{n}\right) Dg_*(y)^\theta + Dx^\theta - \frac{\theta}{n} Dg_*(y)^{\theta-n} x^n$$

using $N(y)$ and $M(y)$ given in (14), which is the value function given in (17).

Finally, using the boundary condition (12) in (15) it turns out that $y \mapsto g_*(y)$ satisfies the nonlinear differential equation (3). The optimality of the stopping time τ^* and value function $\Phi(x, y)$ can be established using a standard verification theorem.

We have proved the following result.

Lemma 2.2. *Let $X = (X_t)_{t \geq 0}$ be a geometric Brownian motion with drift $\mu < 0$ and volatility $\sigma > 0$ and $Y = (Y_t)_{t \geq 0}$ be the associated maximum process of X given in (1) respectively, and let $0 < \theta < 1$ and $n := 1 - \frac{2\mu}{\sigma^2}$. Then, the optimal stopping problem*

$$(16) \quad \Phi(x, y) := \sup_{\tau} \mathbf{E}^{x,y} \left[Y_\tau - c \int_0^\tau X_s^\theta ds \right], \quad c > 0$$

is solved by

(a) the value function $\Phi(x, y)$ of the form

$$(17) \quad \Phi(x, y) = \begin{cases} y + D \left(x^\theta - \left(1 - \frac{\theta}{n}\right) g_*(y)^\theta - \frac{\theta}{n} g_*(y)^{\theta-n} x^n \right) & , \text{ if } g_*(y) < x \leq y \\ y & , \text{ if } 0 < x \leq g_*(y) \end{cases}$$

(b) the optimal stopping time τ^* of the form

$$(18) \quad \tau^* = \inf\{t > 0 : X_t \leq g_*(Y_t)\},$$

where $D = 2c/(\sigma^2\theta(\theta - 1) + 2\mu\theta)$ and g_* is a maximal (strictly increasing) nonnegative solution of the nonlinear differential equation (3) such that (4) holds. ■

Remark 2.1. In the above result, it should be noted that the characterization of the optimal stopping boundary g_* as a maximal solution is originally due to Peskir [2].

Our main result is stated and proved in the theorem that follows.

Theorem 2.1. Let $X = (X_t)_{t \geq 0}$ be a geometric Brownian motion with drift $\mu < 0$ and volatility $\sigma > 0$, $n := 1 - \frac{2\mu}{\sigma^2}$ and let $0 < \theta < 1$ with $n > 2\theta$. Then, the inequality

$$\begin{aligned} & \mathbf{E}^x \left[\max_{0 \leq t \leq \tau} X_t \right] \\ & \leq \left(1 - \frac{1}{2}((\theta - 1)\sigma^2 + 2\mu) \left(1 - \frac{\theta}{n - \theta} \right) \left(\frac{n + \theta}{n - 1} \right) \mathbf{E}^x \left[\int_0^\tau X_s^\theta ds \right] \right) x \\ & \quad - \frac{1}{\theta} \left(1 - \frac{\theta}{n - \theta} \right) \left(\frac{n + \theta}{n - 1} \right) x^{\theta+1} + \frac{n + \theta}{\theta(n - 1)} \left(\frac{n}{(n + \theta) \left(1 - \frac{\theta}{n - \theta} \right)} \right)^{\frac{\theta}{\theta - n}} x^{\frac{n\theta}{n - \theta} + 1} \end{aligned}$$

holds for all stopping times τ for the process $X = (X_t)$ with finite expectation.

Proof. Let $\Phi(x, y)$ be given by (17) for $g_*(y) < x \leq y$, and let $\Psi(\beta)$ be a real-valued convex function defined by

$$\begin{aligned} \Psi(\beta) := & -D \left(\frac{n - \theta}{n - 1} \right)^{\theta/\theta - n} \beta^{\theta/\theta - n} x^{(n-1)\theta/n - \theta} \\ & - \frac{\theta}{n} \left(\frac{n + \theta}{n - 1} \right) \left(\frac{n - \theta}{n - 1} \right)^{\theta/\theta - n} \beta^{(\theta/\theta - n) + 1} x^{((n-1)\theta/n - \theta) + 1} D. \end{aligned}$$

Consider the following convex minimization problem

$$\inf_{\beta > 0} \Psi(\beta)$$

with a unique positive minimizer

$$\beta^* = \frac{\frac{n}{n-\theta}}{\frac{n+\theta}{n-1} \left(1 - \frac{\theta}{n-\theta}\right) x},$$

where $2\theta < n$ and $x > 0$.

It follows that

$$\begin{aligned} \mathbf{E}^{x,x} [\max_{0 \leq t \leq \tau} X_t] &\leq c \mathbf{E}^{x,x} \left[\int_0^\tau X_s^\theta ds \right] + \Phi(x, x) \\ (19) \qquad \qquad \qquad &\leq c \mathbf{E}^{x,x} \left[\int_0^\tau X_s^\theta ds \right] + x + Dx^\theta + \Psi(\beta), \end{aligned}$$

where the first inequality follows from (16) and the second inequality follows using Lemma 2.1.

Now taking the infimum with respect to β on both sides of the inequality (19), we have

$$\begin{aligned} \mathbf{E}^{x,x} [\max_{0 \leq t \leq \tau} X_t] &\leq c \mathbf{E}^{x,x} \left[\int_0^\tau X_s^\theta ds \right] + x + Dx^\theta + \inf_{\beta > 0} \Psi(\beta) \\ (20) \qquad \qquad \qquad &= c_* \mathbf{E}^{x,x} \left[\int_0^\tau X_s^\theta ds \right] + x + D_* x^\theta + \Psi(\beta_*), \end{aligned}$$

where the constants c_* and D_* follow from the identity $\frac{1}{D_* \theta (\frac{\theta}{n} - 1)} = \beta_*$. The desired result now follows. ■

3. CONCLUSION

Under certain conditions, a maximal inequality for a geometric Brownian motion is proved. The result complements and extends the one in Graversen and Peskir [1].

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