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PARALLEL*-RICCI TENSOR OF REAL HYPERSURFACES IN $\mathbb{C}P^2$ AND $\mathbb{C}H^2$

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Abstract. In this paper the idea of studying real hypersurfaces in non-flat complex space forms, whose *-Ricci tensor satisfies geometric conditions is presented. More precisely, three dimensional real hypersurfaces in non-flat complex space forms with parallel *-Ricci tensor are studied. At the end of the paper ideas for further research on *-Ricci tensor are given.

1. INTRODUCTION

A *complex space form* is an n-dimensional Kaehler manifold of constant holomorphic sectional curvature *c*. A complete and simply connected complex space form is complex analytically isometric to

- a complex projective space $\mathbb{C}P^n$ if $c > 0$,
- a complex Euclidean space $\mathbb{C}^n$ if $c = 0$,
- or a complex hyperbolic space $\mathbb{C}H^n$ if $c < 0$.

The symbol $M_n(c)$ is used to denote the complex projective space $\mathbb{C}P^n$ and complex hyperbolic space $\mathbb{C}H^n$, when it is not necessary to distinguish them. Furthermore, since $c \neq 0$ in previous two cases the notion of non-flat complex space form refers to both them.

Let *M* be a real hypersurface in a non-flat complex space form. An almost contact metric structure (φ, ξ, η, g) is defined on *M* induced from the Kaehler metric *G* and the complex structure *J* on $M_n(c)$. The *structure vector field* ξ is called *principal* if $A\xi = \alpha\xi$, where *A* is the shape operator of *M* and $\alpha = \eta(A\xi)$ is a smooth function. A real hypersurface is called *Hopf hypersurface*, if ξ is principal and α is called *Hopf principal curvature*.

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The *Ricci tensor* S of a Riemannian manifold is a tensor field of type (1,1) and is given by

$$g(SX, Y) = trace\{Z \mapsto R(Z, X)Y\}.$$

If the Ricci tensor of a Riemannian manifold satisfies the relation

$$S = \lambda g,$$

where λ is a constant, then it is called *Einstein*.

Real hypersurfaces in non-flat complex space forms have been studied in terms of their Ricci tensor S, when it satisfies certain geometric conditions extensively. Different types of parallelism or invariance of the Ricci tensor are issues of great importance in the study of real hypersurfaces.

In [4] it was proved the non-existence of real hypersurfaces in non-flat complex space forms $M_n(c)$, $n \geq 3$ with parallel Ricci tensor, i.e. $(\nabla_X S)Y = 0$, for any X, $Y \in TM$. In [5] Kim extended the result of non-existence of real hypersurfaces with parallel Ricci tensor in case of three dimensional real hypersurfaces. Another type of parallelism which was studied is that of ξ-parallel Ricci tensor, i.e. $(\nabla_\xi S)Y = 0$ for any $Y \in TM$. More precisely in [6] Hopf hypersurfaces in non-flat complex space forms with constant mean curvature and ξ-parallel Ricci tensor were classified. More details on the study of Ricci tensor of real hypersurfaces are included in Section 6 of [7].

Motivated by Tachibana, who in [9] introduced the notion of **-Ricci tensor* on almost Hermitian manifolds, in [2] Hamada defined the **-Ricci tensor* of real hypersurfaces in non-flat complex space forms by

$$g(S^*X, Y) = \frac{1}{2}(trace\{\varphi \circ R(X, \varphi Y)\}), \quad \text{for } X,\, Y \in TM.$$

The *-Ricci tensor S^* is a tensor field of type (1,1) defined on real hypersurfaces. Taking into account the work that so far has been done in the area of studying real hypersurfaces in non-flat complex space forms in terms of their tensor fields, the following issue raises naturally:

The study of real hypersurfaces in terms of their *-Ricci tensor S^*, when it satisfies certain geometric conditions.

In this paper three dimensional real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ equipped with parallel *-Ricci tensor are studied. Therefore, the following condition is satisfied

$$(\nabla_X S^*)Y = 0, \quad X,\, Y \in TM. \tag{1.1}$$

More precisely the following Theorem is proved.

Main Theorem. *There do not exist real hypersurfaces in $\mathbb{C}P^2$, whose *-Ricci tensor is parallel. In $\mathbb{C}H^2$ only the geodesic hypersphere has parallel *-Ricci tensor with* $\coth(r) = 2$.

The paper is organized as follows: In Section 2 preliminaries relations for real hypersurfaces in non-flat complex space forms are presented. In Section 3 the proof of Main Theorem is provided. Finally, in Section 4 ideas for further research on the above issue are included.

2. Preliminaries

Throughout this paper all manifolds, vector fields etc are assumed to be of class C^∞ and all manifolds are assumed to be connected. Furthermore, the real hypersurfaces M are supposed to be without boundary.

Let M be a real hypersurface immersed in a non-flat complex space form $(M_n(c), G)$ with complex structure J of constant holomorphic sectional curvature c. Let N be a locally defined unit normal vector field on M and $\xi = -JN$ the structure vector field of M.

For a vector field X tangent to M the following relation holds

$$JX = \varphi X + \eta(X)N,$$

where φX and $\eta(X)N$ are the tangential and the normal component of JX respectively. The Riemannian connections $\overline{\nabla}$ in $M_n(c)$ and ∇ in M are related for any vector fields X, Y on M by

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N,$$

where g is the Riemannian metric induced from the metric G.

The shape operator A of the real hypersurface M in $M_n(c)$ with respect to N is given by

$$\overline{\nabla}_X N = -AX.$$

The real hypersurface M has an almost contact metric structure (φ, ξ, η, g) induced from the complex structure J on $M_n(c)$, where φ is the *structure tensor* and it is a tensor field of type (1,1). Moreover, η is an 1-form on M such that

$$g(\varphi X, Y) = G(JX, Y), \qquad \eta(X) = g(X, \xi) = G(JX, N).$$

Furthermore, the following relations hold

$$\begin{gathered}\varphi^2 X = -X + \eta(X)\xi, \qquad \eta\circ\varphi = 0, \qquad \varphi\xi = 0, \qquad \eta(\xi) = 1,\\ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \varphi Y) = -g(\varphi X, Y).\end{gathered}$$

Since J is complex structure implies $\nabla J = 0$. The last relation leads to

$$\nabla_X \xi = \varphi AX, \qquad (\nabla_X \varphi)Y = \eta(Y)AX - g(AX, Y)\xi. \tag{2.1}$$

The ambient space $M_n(c)$ is of constant holomorphic sectional curvature c and this results in the Gauss and Codazzi equations to be given respectively by

$$\begin{aligned} R(X,Y)Z &= \frac{c}{4}[g(Y,Z)X - g(X,Z)Y + g(\varphi Y,Z)\varphi X \\ &\quad -g(\varphi X,Z)\varphi Y - 2g(\varphi X,Y)\varphi Z] \\ &\quad +g(AY,Z)AX - g(AX,Z)AY, \\ (\nabla_X A)Y - (\nabla_Y A)X &= \frac{c}{4}[\eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X,Y)\xi] \end{aligned} \tag{2.2}$$

where R denotes the Riemannian curvature tensor on M and X, Y, Z are any vector fields on M.

The tangent space $T_P M$ at every point $P \in M$ can be decomposed as

$$T_P M = span\{\xi\} \oplus \mathbb{D},$$

where $\mathbb{D} = \ker \eta = \{X \in T_P M : \eta(X) = 0\}$ and is called *holomorphic distribution*. Due to the above decomposition the vector field $A\xi$ can be written

$$A\xi = \alpha\xi + \beta U, \tag{2.3}$$

where $\beta = |\varphi\nabla_\xi\xi|$ and $U = -\frac{1}{\beta}\varphi\nabla_\xi\xi \in \ker(\eta)$ provided that $\beta \neq 0$.

Since the ambient space $M_n(c)$ is of constant holomorphic sectional curvature c following similar calculations to those in Theorem 2 in [3] and taking into account relation (2.2), it is proved that the **-Ricci tensor* S^* of M is given by

$$S^* = -[\frac{cn}{2}\varphi^2 + (\varphi A)^2]. \tag{2.4}$$

3. Proof of Main Theorem

Let *M* be a non-Hopf hypersurface in $\mathbb{C}P^2$ or $\mathbb{C}H^2$, i.e. $M_2(c)$. Then the following relations hold on every non-Hopf three-dimensional real hypersurface in $M_2(c)$.

Lemma 3.1. *Let M be a real hypersurface in $M_2(c)$. Then the following relations hold on M*

$$AU = \gamma U + \delta\varphi U + \beta\xi, \quad A\varphi U = \delta U + \mu\varphi U, \tag{3.1}$$

$$\nabla_U\xi = -\delta U + \gamma\varphi U, \quad \nabla_{\varphi U}\xi = -\mu U + \delta\varphi U, \quad \nabla_\xi\xi = \beta\varphi U, \tag{3.2}$$

$$\nabla_U U = \kappa_1\varphi U + \delta\xi, \quad \nabla_{\varphi U}U = \kappa_2\varphi U + \mu\xi, \quad \nabla_\xi U = \kappa_3\varphi U, \tag{3.3}$$

$$\nabla_U\varphi U = -\kappa_1 U - \gamma\xi, \quad \nabla_{\varphi U}\varphi U = -\kappa_2 U - \delta\xi, \quad \nabla_\xi\varphi U = -\kappa_3 U - \beta\xi, \tag{3.4}$$

where γ, δ, μ, κ_1, κ_2, κ_3 are smooth functions on M and $\{U, \varphi U, \xi\}$ is an orthonormal basis of M.

For the proof of the above Lemma see [8]

Let M be a real hypersurface in $M_2(c)$, i.e. $\mathbb{C}P^2$ or $\mathbb{C}H^2$, whose *-Ricci tensor satisfies relation (1.1), which is more analytically written

$$\nabla_X(S^*Y) = S^*(\nabla_X Y), \quad X, Y \in TM. \tag{3.5}$$

We consider the open subset $\mathcal{N}$ of *M* such that

$$\mathcal{N} = \{P \in M : \beta \neq 0, \text{ in a neighborhood of } P\}.$$

In what follows we work on the open subset $\mathcal{N}$.

On $\mathcal{N}$ relation (2.3) and relations (3.1)-(3.4) of Lemma 3.1 hold. So relation (2.4) for $X \in \{U, \varphi U, \xi\}$ taking into account $n = 2$ and relations (2.3) and (3.1) yields

$$\begin{aligned} S^*\xi &= \beta\mu U - \beta\delta\varphi U, \quad S^*U = (c + \gamma\mu - \delta^2)U \text{ and} \\ S^*\varphi U &= (c + \gamma\mu - \delta^2)\varphi U. \end{aligned} \tag{3.6}$$

The inner product of relation (3.5) for $X = Y = \xi$ with ξ due to the first and the third of (3.6), the first of (2.1) for $X = \xi$ and the third of relations (3.3) and (3.4) implies

$$\delta = 0. \tag{3.7}$$

Moreover, the inner product of relation (3.5) for $X = \varphi U$ and $Y = \xi$ with ξ because of (3.7), the first of (2.1) for $X = \varphi U$, the first and the second of (3.6) and the second of (3.3) results in

$$\mu = 0.$$

Finally, the inner product of relation (3.5) for $X = \xi$ and $Y = \varphi U$ with ξ taking into account $\mu = \delta = 0$, the first and the third of (3.6) and the last relation of (3.4) leads to

$$c = 0,$$

which is a contradiction. So the open subset $\mathcal{N}$ is empty and we lead to the following Proposition.

Proposition 3.2. *Every real hypersurface in $M_2(c)$ whose *-Ricci tensor is parallel, is a Hopf hypersurface.*

Since M is a Hopf hypersurface, the structure vector field ξ is an eigenvector of the shape operator, i.e. $A\xi = \alpha\xi$. Due to Theorem 2.1 in [7] α is constant. We consider a point $P \in M$ and choose a unit principal vector field $W \in \mathbb{D}$ at *P*, such that $AW = \lambda W$ and $A\varphi W = \nu\varphi W$. Then $\{W, \varphi W, \xi\}$ is a local orthonormal basis and the following relation holds (Corollary 2.3 [7])

$$\lambda\nu = \frac{\alpha}{2}(\lambda + \nu) + \frac{c}{4}. \tag{3.8}$$

The first of relation (2.1) and relation (2.4) for $X \in \{W, \varphi W, \xi\}$ because of $A\xi = \alpha\xi$, $AW = \lambda W$ and $A\varphi W = \nu\varphi W$ implies respectively

$$\nabla_W \xi = \lambda\varphi W \ \text{ and } \ \nabla_{\varphi W}\xi = -\nu W \tag{3.9}$$

$$S^*\xi = 0, \quad S^*W = (c+\lambda\nu)W \ \text{ and } \ S^*\varphi W = (c+\lambda\nu)\varphi W. \tag{3.10}$$

Relation (3.5) for $X = W$ and $Y = \xi$ because of the first of (3.9) and the first and third relation of (3.10) yields

$$\lambda(c+\lambda\nu) = 0.$$

Suppose that $(c+\lambda\nu) \neq 0$ then the above relation results in $\lambda = 0$. Moreover, relation (3.5) for $X = \varphi W$ and $Y = \xi$ because of the second of (3.9) and the first and second relation of (3.10) yields

$$\nu = 0.$$

Substitution of $\lambda = \nu = 0$ in (3.8) results in $c = 0$, which is a contradiction. So relation $c = -\lambda\nu$ holds. The last one implies $\lambda\nu \neq 0$ since $c \neq 0$.

Let $\lambda \neq \nu$ then $\lambda = -\frac{c}{\nu}$. Substitution of the last one in (3.8) leads to

$$2\alpha\nu^2 + 5c\nu - 2\alpha c = 0. \tag{3.11}$$

In case of $\mathbb{C}P^2$ we have that $c = 4$ and from equation (3.11) there is always a solution for ν. So ν is constant and λ will be also constant. Therefore, the real hypersurface has three distinct constant eigenvalues. The latter results in M being a real hypersurface of type (B), i.e. a tube of radius r over complex quadric. Substitution of the eigenvalues of type (B) in $\lambda\nu = -c$ leads to a contradiction. So no real hypersurface in $\mathbb{C}P^2$ has parallel *-Ricci tensor (eigenvalues can be found in [7]).

In case of $\mathbb{C}H^2$ we have that $c = -4$ and from equation (3.11) there is a solution for ν if $0 \leq \alpha^2 \leq \frac{25}{4}$. If $\alpha = 0$ equation (3.11) implies $c\nu = 0$, which is impossible. So there is a solution for ν if $0 < \alpha^2 \leq \frac{25}{4}$ and ν will be constant. The latter results in that λ is also constant and so the real hypersurface is of type (B), i.e. a tube of radius r around totally geodesic $\mathbb{R}H^n$. Substitution of the eigenvalues of type (B) in $\lambda\nu = -c$ leads to a contradiction and this completes the proof of our Main Theorem (eigenvalues can be found in [1]).

In case $\lambda = \nu$ then $c + \lambda^2 = 0$, which results in $c < 0$. So M is locally congruent to a real hypersurface of type (A) in $\mathbb{C}H^2$. In this case only the geodesic hypersphere satisfies the above relation and we obtain $\coth(r) = 2$ and the *-Ricci tensor vanishes identically.

4. Discussion-open Problems

In this paper three dimensional real hypersurfaces in non-flat complex space forms with parallel *-Ricci tensor are studied and the non-existence of them is proved. Therefore, a question which raises in a natural way is

Are there real hypersurfaces in non-flat complex space forms of dimension greater than three with parallel *-Ricci tensor?

Generally, the next step in the study of real hypersurfaces in non-flat complex space forms is to study them when a tensor field P type (1,1) of them satisfies other types of parallelism such as the $\mathbb{D}$-parallelism or ξ-parallelism. The first one implies that P is parallel in the direction of any vector field X orthogonal to ξ, i.e. $(\nabla_X P)Y = 0$, for any $X \in \mathbb{D}$, and the second one implies that P is parallel in the direction of the structure vector ξ, i.e. $(\nabla_\xi P)Y = 0$. So the questions which should be answered are the following

Are there real hypersurfaces in non-flat complex space forms whose *-Ricci tensor satisfies the condition of $\mathbb{D}$-parallelism or ξ-parallelism?

Finally, other types of parallelism play important role in the study of real hypersurfaces is that of semi-parallelism and pseudo-parallelism. A tensor field P of type (1, *s*) is said to be *semi-parallel* if it satsfies $R \cdot P = 0$, where R is the Riemannian curvature tensor and acts as a derivation on P. Moreover, P is said to be *pseudo-parallel* if there exists a function L such that $R(X,Y) \cdot P = L\{(X \wedge Y) \cdot P\}$, where $(X \wedge Y)Z = g(Y,Z)X - g(Z,X)Y$. So the questions are:

Are there real hypersurfaces in non-flat complex space forms with semi-parallel or pseudo-parallel *-Ricci tensor?

The importance of answering the above question lays in the fact that the class of real hypersurfaces with parallel $*$-Ricci tensor is included in the class of real hypersurfaces with semi-parallel *-Ricci tensor. Furthermore, the last one is included in the class of real hypersurfaces with pseudo-parallel *-Ricci tensor.

References

1. J. Berndt, Real hypersurfaces with constant principal curvatures in complex hyperbolic space, *J. Reine Angew. Math.*, **395** (1989), 132-141.

2. T. Hamada, Real hypersurfaces of complex space forms in terms of Ricci *-tensor, *Tokyo J. Math.*, **25** (2002), 473-483.

3. T. Ivey and P. J. Ryan, The *-Ricci tensor for hypersurfaces in $\mathbb{C}P^n$ and $\mathbb{C}H^n$, *Tokyo J. Math.*, **34** (2011), 445-471.

4. U.-H. Ki, Real hypersurfaces with parallel Ricci tensor of a complex space form, *Tsukuba J. Math.*, **13** (1989), 73-81.

5. U. K. Kim, Nonexistence of Ricci-parallel real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$", *Bull. Korean Math. Soc.*, **41** (2004), 699-708.

6. M. Kimura and S. Maeda, On real hypersurfaces of a complex projective space II, *Tsukuba J. Math.*, **15** (1991), 547-561.

7. R. Niebergall and P. J. Ryan, Real hypersurfaces in complex space forms, *Math. Sci. Res. Inst. Publ.*, **32** (1997), 233-305.

8. K. Panagiotidou and Ph. J. Xenos, Real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ whose structure Jacobi operator is Lie $\mathbb{D}$-parallel, *Note Mat.*, **32** (2012), 89-99.

9. S. Tachibana, On almost-analytic vectors in almost Kählerian manifolds, *Tohoku Math. J.*, **11** (1959), 247-265.

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