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THE WEIGHTED POINCARÉ DISTANCE IN THE HALF PLANE

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Abstract. In this paper we introduce the weighted Poincaré distance and the induced distance by the weighted Bloch type space. We prove that the weighted Poincaré distance is identical to the inner distance generated by the induced distance.

1. INTRODUCTION

Let $\mathbb{H} = \{x + iy : y > 0\}$ denote the upper half plane in the complex plane $\mathbb{C}$. Let $z, w \in \mathbb{H}$. Given any distance function d on $\mathbb{H}$ we define the d-length of a curve $\gamma : [a, b] \to \mathbb{H}$ by

$$\ell_d(\gamma) = \sup \left\{ \sum_{j=1}^{N} d(\gamma(t_{j-1}), \gamma(t_j)) : N \in \mathbb{N}, 0 = t_0 < \cdots < t_N = 1 \right\}.$$

Using the d-length of curves we define a new distance, d^i, by

$$d^i(z, w) = \inf\{\ell_d(\gamma) : \gamma(0) = z, \gamma(1) = w\},$$

where γ is a continuous curve in $\mathbb{H}$ (see [2]). Automatically $d \leq d^i$ and if equality holds d is called an inner distance. More generally d^i is referred to as the inner distance generated by d. An inner distance d^i generated by d is inner, i.e. $(d^i)^i = d^i$ (see [3]).

For $0 < \alpha \leq 1$ the weighted Poincaré metric, introduced in [1], is given by

$$ds_\alpha^2 = \frac{dx^2 + dy^2}{y^{2\alpha}}.$$

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Suppose that $\gamma(t)$, $0 \le t \le 1$, is a continuous and piecewise smooth curve in $\mathbb{H}$. We recall that the length of $\gamma(t)$ with respect to the weighted Poincaré distance is defined by

$$L_{p_\alpha}(\gamma) = \int_\gamma ds_\alpha = \int_0^1 \frac{|\gamma'(t)|}{[\mathrm{Im}\gamma(t)]^\alpha} dt.$$

Then the weighted Poincaré distance is defined by

$$p_\alpha(z,w) = \inf\{L_{p_\alpha}(\gamma) : \gamma(0) = z, \gamma(1) = w\},$$

where γ is a continuous and piecewise smooth curve and $z, w \in \mathbb{H}$. Note that p_1 is the classical Poincaré distance.

For each $0 < \alpha \le 1$, we let $\mathcal{B}_\alpha$ denote the space of analytic functions f on $\mathbb{H}$ such that

$$\|f\|_\alpha = \sup\{(\mathrm{Im}\, z)^\alpha |f'(z)| : z \in \mathbb{H}\} < +\infty.$$

Then, it is well-known that $\mathcal{B}_1$ is the Bloch space $\mathcal{B}$ and $\mathrm{Lip}_\alpha = \mathcal{B}_{1-\alpha}$ is the analytic Lipschitz space of order $0 < \alpha < 1$. For z, w in $\mathbb{H}$, we define the induced distance (see [4] and [5]) by

$$d_\alpha(z,w) = \sup\{|f(z) - f(w)| : \|f\|_\alpha \le 1\}.$$

We prove that the weighted Poincaré distance p_α is identical to the inner distance d_α^i generated by the induced distance d_α.

Theorem 1.1. *Let* $0 < \alpha \le 1$. *Then* $d_\alpha^i = p_\alpha$.

By Theorem 1.1, $(d_\alpha^i)^i = d_\alpha^i$ and $d_\alpha \le d_\alpha^i$, we get following corollary.

Corollary 1.2. *Let* $0 < \alpha \le 1$. *Then*

(a) $d_\alpha \le p_\alpha$.

(b) $p_\alpha^i = p_\alpha$.

In the case of $\alpha = 1$, we prove the following :

Theorem 1.3.

$$p_1(z,w) = d_1(z,w) \quad \textit{for} \quad z, w \in \mathbb{H}.$$

In [4], the author proved that $d_1 = p_1$ on the unit disc. We proved the same identity on the upper half plane. We do not know that $d_\alpha = p_\alpha$ for $0 < \alpha < 1$. By Corollary 1.2, we know that $d_\alpha \le p_\alpha$. But authors believe that $d_\alpha = p_\alpha$ for the unit disc and the upper half plane.

The organization of paper is as follows: In Section 2, we find geodesics for the weighted Poincaré metric and compute the weighted Poincaré distance. In Section 3, we characterize the weighted Bloch function by the weighted Poincaré distance p_α. In Section 4, we prove several lemmas for main results. We also introduce the sufficient and necessary condition for weighted Bloch functions using the induced distance d_α. Finally, we prove Theorem 1.1 and Theorem 1.3 in Section 5.

2. The Weighted Poincaré Distance

Let $0 < \alpha \leq 1$. The weighted Poincaré metric (see [1]) is given by

$$ds_\alpha^2 = \frac{dx^2 + dy^2}{y^{2\alpha}}.$$

The parametric equation for geodesics is here:

$$\begin{cases} \ddot{x} = \dfrac{2\alpha}{y}\dot{x}\dot{y} \\ \ddot{y} = \dfrac{\alpha}{y}(\dot{y}^2 - \dot{x}^2). \end{cases}$$

Another equivalent differential system is the following

$$\begin{cases} \dfrac{\dot{x}}{y^{2\alpha}} = C_1 \\ \dfrac{\dot{x}^2 + \dot{y}^2}{y^{2\alpha}} = C_2. \end{cases}$$

For a simple calculation, we give $C_1 = C_2 = 1$ (the standard geodesic in $\mathbb{H}$). Then

$$\begin{cases} \dfrac{\dot{x}}{y^{2\alpha}} = 1 \\ \dfrac{\dot{x}^2 + \dot{y}^2}{y^{2\alpha}} = 1, \quad x(1) = 0 \end{cases}$$

and so

$$\frac{\dot{x}^2 + \dot{y}^2}{\dot{x}^2} = \frac{1}{y^{2\alpha}}.$$

Thus we have

$$\begin{cases} \dfrac{dy}{dx} = -\dfrac{\sqrt{1 - y^{2\alpha}}}{y^\alpha} & \text{if} \quad x > 0 \\ \dfrac{dy}{dx} = \dfrac{\sqrt{1 - y^{2\alpha}}}{y^\alpha} & \text{if} \quad x < 0. \end{cases}$$

For example, if $\alpha = \frac{1}{2}$, then

$$\begin{aligned} x(y) &= \int_y^1 \sqrt{\frac{t}{1-t}}dt \\ &= \sqrt{y(1-y)} + \cos^{-1}(\sqrt{y}) \\ &= \sqrt{y(1-y)} + \sin^{-1}(\sqrt{1-y}). \end{aligned}$$

Now let $0 < \alpha < 1$. First, we recall the hypergeometric function $F(a, b; c|z)$. Let $a \in \mathbb{R}$ and $c > b > 0$. Then

$$B(b, c-b)F(a, b; c|z) = \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt, \quad |z| < 1,$$

where B is the Beta function.

We see that, if $x > 0$,

$$\begin{aligned} x(y) &= \int_y^1 \frac{t^\alpha}{\sqrt{1-t^{2\alpha}}}dt \\ &= \int_0^1 \frac{t^\alpha}{\sqrt{1-t^{2\alpha}}}dt - \int_0^y \frac{t^\alpha}{\sqrt{1-t^{2\alpha}}}dt \\ &= \mathrm{I}_1 + \mathrm{I}_2. \end{aligned}$$

We calculate the first term I_1. We have

$$\begin{aligned} \mathrm{I}_1 &= \int_0^1 \frac{t^\alpha}{\sqrt{1-t^{2\alpha}}}dt \\ &= \frac{1}{2\alpha}\int_0^1 t^{-\frac{1}{2}+\frac{1}{2\alpha}}(1-t)^{-\frac{1}{2}}dt \\ &= \frac{1}{2\alpha}B\left(\frac{1+\alpha}{2\alpha}, \frac{1}{2}\right). \end{aligned}$$

Now, we calculate the second term I_2. We have

$$\begin{aligned} \mathrm{I}_2 &= -\int_0^y \frac{t^\alpha}{\sqrt{1-t^{2\alpha}}}dt \\ &= -\frac{1}{2\alpha}\int_0^{y^{2\alpha}} t^{-\frac{1}{2}+\frac{1}{2\alpha}}(1-t)^{-\frac{1}{2}}dt \\ &= -\frac{y^{1+\alpha}}{1+\alpha}\mathrm{F}\left(\frac{1}{2}, \frac{1+\alpha}{2\alpha}; \frac{3}{2}+\frac{1}{2\alpha}\middle| y^{2\alpha}\right). \end{aligned}$$

Thus, we have

$$x(y) = \frac{1}{2\alpha}B\left(\frac{1+\alpha}{2\alpha}, \frac{1}{2}\right) - \frac{y^{1+\alpha}}{1+\alpha}\mathrm{F}\left(\frac{1}{2}, \frac{1+\alpha}{2\alpha}; \frac{3}{2}+\frac{1}{2\alpha}\middle| y^{2\alpha}\right). \tag{2.1}$$

Similarly, we can calculate the case x <0. Then, we get

$$x(y) = -\frac{1}{2\alpha}B\left(\frac{1+\alpha}{2\alpha}, \frac{1}{2}\right) + \frac{y^{1+\alpha}}{1+\alpha}\mathrm{F}\left(\frac{1}{2}, \frac{1+\alpha}{2\alpha}; \frac{3}{2} + \frac{1}{2\alpha}\middle| y^{2\alpha}\right).$$

Now we calculate the weighted Poincaré distance by using the above results. We define

$$D_\delta(z) = \delta z \quad \text{for} \quad \delta \geq 0$$

and

$$T_t(z) = z + t \quad \text{for} \quad t \in \mathbb{R}.$$

Then it is easy to see that

$$L_{p_\alpha}(D_\delta(\gamma)) = \delta^{1-\alpha} L_{p_\alpha}(\gamma)$$

and

$$L_{p_\alpha}(T_t(\gamma)) = L_{p_\alpha}(\gamma).$$

Note that if γ_g is a geodesic, so are $D_\delta(\gamma)$ and $T_t(\gamma)$. By these homogeneity properties, we only need to know the distance between the point i and any other point of the standard geodesic of $\mathbb{H}$.

Let γ_g be the standard geodesic in $\mathbb{H}$ connecting two points i and $x + iy$ where $x \neq 0$ and $0 \leq y \leq 1$.

For example, when $\alpha = \frac{1}{2}$, we have

$$\begin{aligned} L_{p_{\frac{1}{2}}}(\gamma_g) &= \int_y^1 \frac{1}{\sqrt{t}\sqrt{1-t}}dt \\ &= 2\cos^{-1}(\sqrt{y}) \\ &= 2\sin^{-1}(\sqrt{1-y}) \\ &= 2\left(x - \sqrt{y(1-y)}\right). \end{aligned}$$

Thus

$$p_{\frac{1}{2}}(i, x+iy) = 2\left(x - \sqrt{y(1-y)}\right).$$

Now for $0 < \alpha < 1$ the length of γ_g is calculated by

$$\begin{aligned} L_{p_\alpha}(\gamma_g) &= \int_{\gamma_g} \frac{\sqrt{dx^2 + dy^2}}{y^\alpha} \\ &= \int_y^1 \frac{1}{t^\alpha\sqrt{1-t^{2\alpha}}}dt \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \frac{1}{t^\alpha \sqrt{1-t^{2\alpha}}} dt - \int_0^y \frac{1}{t^\alpha \sqrt{1-t^{2\alpha}}} dt \\
&= \mathrm{J}_1 + \mathrm{J}_2.
\end{aligned}
$$

We calculate the first term J_1. We have

$$\mathrm{J}_1 = \frac{1}{2\alpha} B\left(\frac{1-\alpha}{2\alpha}, \frac{1}{2}\right).$$

Now, we calculate the second term J_2. We have

$$
\begin{aligned}
\mathrm{J}_2 &= -\int_0^y \frac{1}{t^\alpha \sqrt{1-t^{2\alpha}}} dt \\
&= -\frac{1}{1-\alpha^2} y^{1-\alpha} \left[(1+\alpha)\sqrt{1-y^{2\alpha}} + y^{2\alpha} \mathrm{F}\left(\frac{1}{2}, \frac{1+\alpha}{2\alpha}; \frac{3}{2} + \frac{1}{2\alpha} \middle| y^{2\alpha}\right)\right].
\end{aligned}
$$

By (2.1), we have

$$\mathrm{J}_2 = -\frac{1}{2\alpha} B\left(\frac{1-\alpha}{2\alpha}, \frac{1}{2}\right) + \frac{1}{1-\alpha}\left(x - y^{1-\alpha}\sqrt{1-y^{2\alpha}}\right).$$

Thus we have

$$L_{p_\alpha}(\gamma_g) = \frac{1}{1-\alpha}\left(x - y^{1-\alpha}\sqrt{1-y^{2\alpha}}\right).$$

Hence

$$p_\alpha(i, x+iy) = \frac{1}{1-\alpha}\left(x - y^{1-\alpha}\sqrt{1-y^{2\alpha}}\right).$$

If γ_g is the geodesic in $\mathbb{H}$ connecting two points i and iy where $0 < y < 1$, then $\gamma_g(t) = it, y \le t \le 1$ and the length of γ_g is calculated by

$$L_{p_\alpha}(\gamma_g) = \int_y^1 \frac{dt}{t^\alpha} = \frac{1}{1-\alpha}(1 - y^{1-\alpha})$$

or

$$p_\alpha(i, iy) = \frac{1}{1-\alpha}(1 - y^{1-\alpha}).$$

3. Weighted Bloch Spaces

We recall that for each $0 < \alpha \le 1$ the weighted Bloch space $\mathcal{B}_\alpha$ is the space of analytic functions f on $\mathbb{H}$ such that

$$\|f\|_\alpha = \sup\{(\operatorname{Im} z)^\alpha |f'(z)| : z \in \mathbb{H}\} < +\infty.$$

Theorem 3.1. *Suppose* $0 < \alpha \leq 1$, *and* f *is analytic on* $\mathbb{H}$. *Then* f *is in* $\mathcal{B}_\alpha$ *if and only if there exists a constant* $C > 0$ *such that*

$$|f(z) - f(w)| \leq Cp_\alpha(z, w), \quad z, w \in \mathbb{H}.$$

Furthermore, we have

$$\|f\|_\alpha = \sup\left\{\frac{|f(z) - f(w)|}{p_\alpha(z, w)} : z \neq w\right\}$$

for all $f \in \mathcal{B}_\alpha$.

Proof. The proof is as in ([4], Theorem 19). First assume that

$$|f(z) - f(w)| \leq Cp_\alpha(z, w), \quad z, w \in \mathbb{H}.$$

We may assume that C is the smallest constant satisfying the above condition. Fix $z \in \mathbb{H}$ and let $\gamma(s)$ be the geodesic (parametrized by arc-length) in the underlying weighted Poincaré metric that starts at z. Since $p_\alpha(\gamma(0), \gamma(s)) = s$, we have

$$|f(z) - f(w)| \leq Cs, \quad 0 < s < \epsilon.$$

Dividing both sides by s and then letting $s \to 0$ in the above inequality, we obtain

$$|f'(z)||\gamma'(0)| \leq C.$$

By the minimal length property of geodesics,

$$s = p_\alpha(\gamma(0), \gamma(s)) = \int_0^s \frac{|\gamma'(t)|}{[\mathrm{Im}\gamma(t)]^\alpha} dt, \quad 0 < s < \epsilon.$$

Then

$$\begin{aligned} 1 &= \lim_{s\to 0} \frac{1}{s} \int_0^s \frac{|\gamma'(t)|}{[\mathrm{Im}\gamma(t)]^\alpha} dt \\ &= \frac{|\gamma'(0)|}{(\mathrm{Im}\, z)^\alpha}. \end{aligned}$$

It follows that $(\mathrm{Im}\, z)^\alpha |f'(z)| \leq C$ and hence $f \in \mathcal{B}_\alpha$ with

$$\sup\{(\mathrm{Im}\, z)^\alpha |f'(z)| : z \in \mathbb{H}\} \leq \sup\left\{\frac{|f(z) - f(w)|}{p_\alpha(z, w)} : z \neq w\right\}.$$

On the other hand, if f is in $\mathcal{B}_\alpha$, then

$$C = \sup\{(\mathrm{Im}\, z)^\alpha |f'(z)| : z \in \mathbb{H}\} < +\infty$$

and hence

$$|f'(z)| \le \frac{C}{(\operatorname{Im} z)^\alpha}$$

for all $z \in \mathbb{H}$. If $\gamma(t)$, $0 \le t \le 1$, is a smooth curve from z to w, the fundamental theorem of calculus shows that

$$\begin{aligned} |f(z) - f(w)| &= \left| \int_0^1 \frac{d}{dt} f(\gamma(t)) dt \right| \\ &\le \int_0^1 |f'(\gamma(t))||\gamma'(t)| dt \\ &\le C \int_0^1 \frac{|\gamma'(t)|}{[\operatorname{Im}\gamma(t)]^\alpha} dt \\ &= C L_{p_\alpha}(\gamma). \end{aligned}$$

It is easy to see that this also holds if γ is continuous but only piecewise smooth. Taking the infimum over all piecewise smooth curves connecting z to w, we conclude that

$$|f(z) - f(w)| \le C p_\alpha(z, w), \quad z, w \in \mathbb{H}.$$

This completes the proof. ■

4. Induced Distances from Weighted Bloch Spaces

For $0 < \alpha \le 1$ and z, w in $\mathbb{H}$, we define

$$d_\alpha(z, w) = \sup\{|f(z) - f(w)| : \|f\|_\alpha \le 1\}.$$

Lemma 4.1. *Let $0 < \alpha \le 1$. For a fixed z in $\mathbb{H}$ we define*

$$f_z(w) = (2i)^\alpha \int_i^w \frac{d\zeta}{(2\zeta - z - \bar{z})^\alpha}, \quad w \in \mathbb{H}.$$

Then $f \in \mathcal{B}_\alpha$ and $\|f_z\|_\alpha = 1$.

Proof. Let $z, w \in \mathbb{H}$. Since $f_z'(w) = (2i)^\alpha \frac{1}{(2w - z - \bar{z})^\alpha}$, we have

$$\begin{aligned} (\operatorname{Im} w)^\alpha |f_z'(w)| &= (\operatorname{Im} w)^\alpha \left| \frac{2i}{2w - z - \bar{z}} \right|^\alpha \\ &= (\operatorname{Im} w)^\alpha \frac{1}{|w - \operatorname{Re} z|^\alpha} \le 1 \end{aligned}$$

and $(\operatorname{Im} z)^\alpha |f_z'(z)| = 1$. Thus we have

$$\|f_z\|_\alpha = \sup_{w \in \mathbb{H}} (\operatorname{Im} w)^\alpha |f_z'(w)| = 1.$$ ■

Lemma 4.2. *For $0 < \alpha \leq 1$, d_α is a distance on $\mathbb{H}$.*

Proof. Let $z, w \in \mathbb{H}$. Suppose that $d_\alpha(z,w) = 0$. We define

$$f_i(w) = (2i)^\alpha \int_i^w \frac{d\zeta}{(2\zeta)^\alpha}.$$

By Lemma 4.1, it follows that $\| f_i \|_\alpha = 1$ and

$$f_i(z) - f_i(w) = (2i)^\alpha \int_w^z \frac{d\zeta}{(2\zeta)^\alpha}.$$

By the definition of d_α, for $\alpha = 1$, we have

$$\begin{aligned} |f_i(z) - f_i(w)| &= \left| \int_w^z \frac{d\zeta}{\zeta} \right| \\ &= |\mathrm{Log}(z) - \mathrm{Log}(w)| \\ &\leq d_1(z,w) = 0, \end{aligned}$$

where Log is the principal branch of logarithm. When $0 < \alpha < 1$, we have

$$\begin{aligned} |f_i(z) - f_i(w)| &= \left| \int_w^z \frac{d\zeta}{\zeta^\alpha} \right| \\ &= \frac{1}{1-\alpha} \left| z^{1-\alpha} - w^{1-\alpha} \right| \\ &\leq d_\alpha(z,w) = 0. \end{aligned}$$

Hence we have $z = w$. ∎

Proposition 4.3. *Let $z, w \in \mathbb{H}$. Then*

(a) $d_\alpha(D_\delta(z), D_\delta(w)) = \delta^{1-\alpha} d_\alpha(z,w), \quad 0 < \delta < 1.$

(b) $d_\alpha(T_t(z), T_t(w)) = d_\alpha(z,w), \quad t \in \mathbb{R}.$

Proof. Since (b) is clear, we prove only (a).
Note that

$$d_\alpha(\delta z, \delta w) = \sup\{|f(\delta z) - f(\delta w)| : \|f\|_\alpha \leq 1\}.$$

For $f \in \mathcal{B}_\alpha$ with $\|f\|_\alpha \leq 1$ let $f_\delta(z) = f(\delta z)$. Then

$$\begin{aligned} \|f_\delta\|_\alpha &= \sup_{z=x+iy} y^\alpha |f_\delta'(z)| \\ &= \sup y^\alpha |f'(\delta z)| \delta \\ &= \delta^{1-\alpha} \sup (\delta y)^\alpha |f'(\delta z)| \\ &= \delta^{1-\alpha} \|f\|_\alpha. \end{aligned}$$

Hence $\|f_\delta\|_\alpha \le \delta^{1-\alpha}$ so that

$$\left\| \frac{1}{\delta^{1-\alpha}} f_\delta \right\|_\alpha \le 1.$$

Now we have

$$\begin{aligned} d_\alpha(z,w) &= \sup\{|f(z) - f(w)| : \|f\|_\alpha \le 1\} \\ &\ge \frac{1}{\delta^{1-\alpha}} |f(\delta z) - f(\delta w)| \end{aligned}$$

so that

$$\delta^{1-\alpha} d_\alpha(z,w) \ge d_\alpha(\delta z, \delta w). \tag{4.1}$$

For the converse, we note that

$$\left\| \delta^{1-\alpha} f_{\frac{1}{\delta}} \right\| \le 1.$$

Thus we have

$$\begin{aligned} d_\alpha(\delta z, \delta w) &\ge \left| \delta^{1-\alpha} f_{\frac{1}{\delta}}(\delta z) - \delta^{1-\alpha} f_{\frac{1}{\delta}}(\delta w) \right| \\ &= \delta^{1-\alpha} |f(z) - f(w)| \end{aligned}$$

so that

$$d_\alpha(\delta z, \delta w) \ge \delta^{1-\alpha} d_\alpha(z,w). \tag{4.2}$$

By (4.1) and (4.2), we get the required result. ■

Lemma 4.4. *For $0 < \alpha \le 1$ and $z, w \in \mathbb{H}$, we have*

$$\lim_{z,w \to z_0} \frac{d_\alpha(z,w)}{|z-w|} = \frac{1}{(\operatorname{Im} z_0)^\alpha}.$$

Proof. By the definition of d_α, we have

$$\frac{d_\alpha(z,w)}{|z-w|} \ge \frac{|f(z) - f(w)|}{|z-w|}$$

for all $\|f\|_\alpha \le 1$ and all z, w in $\mathbb{H}$. Let γ be a simple closed curve in $\mathbb{H}$ containing z_0 inside of γ. By the Cauchy integral formula, we have

$$\begin{aligned} \lim_{z,w \to z_0} \left| \frac{f(z) - f(w)}{z - w} \right| &= \lim_{z,w \to z_0} \left| \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)(\zeta - w)} d\zeta \right| \\ &= \left| \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^2} d\zeta \right| \\ &= |f'(z_0)|. \end{aligned}$$

Thus, letting $z, w \to z_0$, we obtain

$$\liminf_{z,w\to z_0} \frac{d_\alpha(z,w)}{|z-w|} \geq |f'(z_0)|$$

for $\|f\|_\alpha \leq 1$.

Let

$$f_{z_0}(w) = (2i)^\alpha \int_i^w \frac{d\zeta}{(2\zeta - z_0 - \bar{z}_0)^\alpha}, \quad w \in \mathbb{H}.$$

Then $\|f_{z_0}\|_\alpha = 1$ and

$$|f'_{z_0}(z_0)| = \frac{1}{(\operatorname{Im} z_0)^\alpha}.$$

Thus

$$\liminf_{z,w\to z_0} \frac{d_\alpha(z,w)}{|z-w|} \geq \frac{1}{(\operatorname{Im} z_0)^\alpha}.$$

It remains to show that

$$\limsup_{z,w\to z_0} \frac{d_\alpha(z,w)}{|z-w|} \leq \frac{1}{(\operatorname{Im} z_0)^\alpha}.$$

Let $z, w \in \mathbb{H}$ with $z \neq w$. Then

$$\begin{aligned}
|f(z) - f(w)| &= \left| \int_0^1 \frac{d}{dt} f(tz + (1-t)w)dt \right| \\
&\leq |z-w| \int_0^1 |f'(tz + (1-t)w)|dt \\
&\leq |z-w| \int_0^1 \frac{1}{|\operatorname{Im}[tz + (1-t)w]|^\alpha} dt \\
&\leq |z-w| \frac{1}{(\min\{\operatorname{Im} z, \operatorname{Im} w\})^\alpha}
\end{aligned}$$

for all $f \in \mathcal{B}_\alpha$ with $\|f\|_\alpha \leq 1$. Taking the supremum over all such f, we get

$$d_\alpha(z,w) \leq \frac{|z-w|}{(\min\{\operatorname{Im} z, \operatorname{Im} w\})^\alpha}.$$

Letting $z, w \to z_0$, we obtain

$$\limsup_{z,w\to z_0} \frac{d_\alpha(z,w)}{|z-w|} \leq \frac{1}{(\operatorname{Im} z_0)^\alpha}.$$

This completes the proof. ∎

We can characterize functions in $\mathcal{B}_\alpha$ by using the distance d_α as following.

Theorem 4.5. *Let $0 < \alpha \le 1$ and f is analytic on $\mathbb{H}$. Then f is in $\mathcal{B}_\alpha$ if and only if there exists a constant $C > 0$ such that*

$$|f(z) - f(w)| \le \mathrm{C} d_\alpha(z, w), \quad z, w \in \mathbb{H}.$$

Furthermore, we have

$$\|f\|_\alpha = \sup\left\{\frac{|f(z) - f(w)|}{d_\alpha(z, w)} : z \neq w\right\}$$

for all $f \in \mathcal{B}_\alpha$.

Proof. First of all, if we put $M = \sup\left\{\frac{|f(z)-f(w)|}{d_\alpha(z,w)} : z \neq w\right\}$, we show that

$$M \le \sup\{(\operatorname{Im} z)^\alpha |f'(z)| : z \in \mathbb{H}\}.$$

Let

$$F(z) = \frac{f(z)}{\|f\|_\alpha}.$$

Then $\|F\|_\alpha = 1$. By the definition of d_α, it follows that

$$\begin{aligned} |F(z) - F(w)| &= \frac{1}{\|f\|_\alpha} |f(z) - f(w)| \\ &\le d_\alpha(z, w). \end{aligned}$$

Thus we have

$$M = \sup\left\{\frac{|f(z) - f(w)|}{d_\alpha(z, w)}\right\} \le \sup\{(\operatorname{Im} u)^\alpha |f'(u)|\}.$$

On the other hand, for any $z \in \mathbb{H}$, we clearly have

$$M \ge \lim_{w \to z} \frac{|f(z) - f(w)|}{d_\alpha(z, w)} = \lim_{w \to z} \frac{|f(z) - f(w)|}{|z - w|} \frac{|z - w|}{d_\alpha(z, w)}.$$

Applying Lemma 4.4, we obtain $M \ge (\operatorname{Im} z)^\alpha |f'(z)|$ for all $z \in \mathbb{H}$.
It follows that

$$\sup\left\{\frac{|f(z) - f(w)|}{d_\alpha(z, w)} : z \neq w\right\} \ge \sup\{(\operatorname{Im} z)^\alpha |f'(z)| : z \in \mathbb{H}\},$$

which completes the proof of Theorem 4.5. ∎

5. Comparison Between p_α and d_α

We will prove Theorem 1.1.

Proof. In order to prove Theorem 1.1, we will prove that

$$\ell_{d_\alpha}(\gamma) = L_{p_\alpha}(\gamma)$$

for all C^1 curve $\gamma : [0,1] \to \mathbb{H}$.

Let γ be a C^1 curve in the half plane $\mathbb{H}$. Then, by Lemma 4.4, for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left| \frac{d_\alpha(\gamma(t),\gamma(t'))}{|\gamma(t)-\gamma(t')|} - \frac{1}{[\mathrm{Im}\gamma(t)]^\alpha} \right| \le \epsilon, \quad 0 \le t, t' \le 1, \quad |t-t'| \le \delta.$$

Multiplying by $|\gamma(t)-\gamma(t')|$, we get

$$\left| d_\alpha(\gamma(t),\gamma(t')) - \frac{|\gamma(t)-\gamma(t')|}{[\mathrm{Im}\gamma(t)]^\alpha} \right| \le C\epsilon, \quad 0 \le t, t' \le 1, \quad |t-t'| \le \delta,$$

where C is the Euclidean length of γ. Since γ is C^1, by the mean value inequality, we arrive the following:

$$\left| d_\alpha(\gamma(t),\gamma(t')) - \frac{|\gamma'(t)|}{[\mathrm{Im}\gamma(t)]^\alpha}|t-t'| \right| \le 2C\epsilon, \quad 0 \le t, t' \le 1, \quad |t-t'| \le \delta.$$

We take $0 = t_0 < \cdots < t_N = 1$ with $t_j - t_{j-1} \le \delta, j = 1, \ldots, N$. Then

$$\left| \sum_{j=1}^N d_\alpha(\gamma(t_{j-1}),\gamma(t_j)) - \sum_{j=1}^N \frac{|\gamma'(t_{j-1})|}{[\mathrm{Im}\gamma(t_{j-1})]^\alpha}(t_j - t_{j-1}) \right| \le 2C\epsilon.$$

This implies that

$$|\ell_{d_\alpha}(\gamma) - L_{p_\alpha}(\gamma)| \le 2C\epsilon.$$

Since ϵ is arbitrary, we get $\ell_{d_\alpha}(\gamma) = L_{p_\alpha}(\gamma)$. This implies that $d_\alpha^i = p_\alpha$. ■

The Bloch space $\mathcal{B}$ on $\mathbb{H}$ is defined to be the space of analytic functions f on $\mathbb{H}$ such that

$$\|f\|_{\mathcal{B}} = \sup\{\mathrm{Im}(z)|f'(z)| : z \in \mathbb{H}\} < +\infty.$$

An important property of the Bloch space is its invariance under Mŏbius transformations. Thus the induced distance d_1 is also invariant under Mŏbius transformations.

Now, we will prove Theorem 1.3.

Proof. By Theorem 1.1, we have

$$d_1(z,w) \le d_1^i(z,w) = p_1(z,w) \quad \text{for} \quad z, w \in \mathbb{H}.$$

Now we prove that

$$p_1(i, iy) \le d_1(i, iy) \quad \text{for} \quad y > 0.$$

We know that

$$p_1(i, iy) = |\ln(y)|.$$

We take

$$f(z) = \mathrm{Log}\left(\frac{z}{i}\right), \quad z \in \mathbb{H}.$$

Then f is analytic in $\mathbb{H}$ and

$$f'(z) = \frac{1}{z}.$$

Thus

$$|f'(z)| = \frac{1}{|z|} \leq \frac{1}{\mathrm{Im}(z)}, \quad z \in \mathbb{H}$$

and

$$\|f\|_{\mathcal{B}} \leq 1.$$

Now

$$|f(i) - f(iy)| = |\ln(y)| = p_1(i, iy).$$

Hence we get

$$p_1(i, iy) \leq d_1(i, iy).$$

We know that p_1 and d_1 are invariant under Mӧbius transformations. Thus

$$p_1(z, w) \leq d_1(z, w).$$

Hence we get

$$p_1(z, w) = d_1(z, w), \quad z, w \in \mathbb{H}. \qquad \blacksquare$$

By Theorem 1.3, we conjecture that $p_\alpha = d_\alpha$ for all $0 < \alpha < 1$.

Define

$$f_\alpha(z) = \frac{1}{1-\alpha}\left\{1 - \left(\frac{z}{i}\right)^{1-\alpha}\right\}.$$

Then $\|f_\alpha\|_\alpha \leq 1$ and

$$|f_\alpha(i) - f_\alpha(iy)| = \frac{1}{1-\alpha}(1 - y^{1-\alpha}) = p_\alpha(i, iy).$$

Thus we have $d_\alpha(i, iy) = p_\alpha(i, iy)$. However, p_α and d_α are not invariant under Mӧbious transformations. We don't know that $d_\alpha(i, x + iy) = p_\alpha(i, x + iy)$ for any point $x + iy$ in the standard geodesic of $\mathbb{H}$. Thus the following problem is open. The same problem for the unit disc model is still open (see [4]).

Problem 5.1. Let $0 < \alpha < 1$. Then $d_\alpha = p_\alpha$.

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