TAIWANESE JOURNAL OF MATHEMATICS
Vol. 18, No. 5, pp. 1389-1409, October 2014
DOI: 10.11650/tjm.18.2014.4195
This paper is available online at http://journal.taiwanmathsoc.org.tw

CHARACTERIZATIONS OF THE MULTIVARIATE WAVE PACKET SYSTEMS

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Abstract. Some characterizations of the multivariate wave packet systems in terms of the Fourier transforms of the wave packet systems' generating functions are discussed in this paper. First, the characterizations of the orthogonal wave packet systems are provided. Secondly, a sufficient condition of the completeness of wave packet system in some special cases is established. Finally, the necessary conditions and sufficient conditions for the wave packet systems to be wave packet Parseval frames with the very general lattices are obtained. Thus, the corresponding known results in Gabor systems and wavelet systems are obtained as some corollaries.

1. Introduction

Today, we are living in a data world. On the one hand, people have to develop the good ways to process all kinds of data. On the other hand, they are faced with analyzing the accuracy of such methods and providing a deeper understanding of the underlying structures. In the late 18th century, the Fourier Transform became the first tool to analyze the data and has achieved the greatest achievements. However, the Fourier Transform has a serious disadvantage: the local perturbation of a function leads to a change of all Fourier coefficients simultaneously. This deficiency led to the birth of applied harmonic analysis, which is nowadays one of the major research areas in applied mathematics and engineering. It exploits methods not only from harmonic analysis, but also from areas such as approximation theory, numerical mathematics and operator theory. Now, applied harmonic analysis plays an important role in engineering such as signal processing, image processing, digital communications, medical imaging, compressed sensing, and so on.

Received December 12, 2013, accepted January 23, 2014.
Communicated by Chin-Cheng Lin.
2010 *Mathematics Subject Classification*: 42C15, 42C40.
Key words and phrases: Gabor system, Wavelet system, The wave packet system.
The work is supported by the National Natural Science Foundation of China (No. 61071189) and the Natural Science Foundation of Henan Province (No. 122300410381).
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Gabor systems were first introduced by Gabor [1] in 1946, with the Gaussian window for the purpose of constructing efficient, time-frequency localized expansions of finite-energy signals. They are generated by modulations and translations of a finite family of functions. Casazza and Christensen [2] gave some important characterizations about Gabor frames for subspaces of $L^2(R)$ in 2001. Then, Shi and Chen[3] established some new necessary conditions for Gabor frames. These conditions are also sufficient for tight frames. Recently, Li et al. [4] presented some new sufficient conditions for Gabor frame via Fourier transform.

Gabor systems can only give the time-frequency content of a signal with a constant frequency and time resolution. This is often not the most desired resolution, which leads to a birth of wavelet analysis. Wavelet systems are obtained by shifting and dilating a finite family of functions. They have attracted considerable interests from the mathematical community and from members of many diverse disciplines since Daubechies and his cooperators [5] combined the theory of the continuous wavelet transform with the theory of frames to define wavelet frames for $L^2(R)$. In 1990, Daubechies [6] obtained the first result on the necessary condition for wavelet frames, and then in 1993, Chui and Shi [7] obtained an improved result.

In 1978, Cordoba and Fefferman [8] introduced wave packet systems by applying certain collections of dilations, modulations and translations to the Gaussian function in the study of some classes of singular integral operators. In paper [9], authors devoted to describe any collections of functions which are obtained by applying the same operations to a finite family of functions. In fact, Gabor systems and wavelet systems are special cases of wave packet systems. Wave packet systems have recently been successfully applied to problems in harmonic analysis and operator theory [10, 11] and attracted people's attention.

The properties of wave packet systems have been investigated by many authors. For example, in [12], authors studied both the continuous and discrete versions of wave packet systems by using a unified approach and gave a classification of the wave packet system to be a Parseval frame. They constructed a very general example of wave packet frame. The paper [13] considered wave packet systems as special cases of generalized shift-invariant systems and presented a sufficient condition for a wave packet system to form a frame. Analogues of the notion of Beurling density to describe completeness properties of wave packet systems via geometric properties of the sets of their parameters is introduced in [14], and the necessary conditions for existence of wave packet frames were obtained and the large families of new, non-standard examples of wave packet frames with prescribed dimensions were provided.

Since both Gabor systems and wavelet systems are some particular examples of wave packet systems, people ask naturally: how do we construct some examples of wave packet systems such that they possess simultaneously both Gabor systems and wavelet systems' advantages and, however, overcome their shortcomings? In need of applications, how do we develop the algorithm as classical multiresolution analysis in the setting of the wave packet systems? So far as we know, few results are known about these problems.

The main goal of this paper is to characterize the multivariate wave packet systems in terms of the Fourier transforms of the wave packet systems' generating functions. Some characterizations of all kinds of the orthogonal wave packet systems with different operator order and with arbitrary expanding matrix dilations are established. Also, a sufficient condition of the completeness of wave packet systems in some special cases is derived. Then, the necessary conditions and sufficient conditions about the wave packet systems to be wave packet Parseval frames with the very general lattices are presented. Thus, the corresponding known results in Gabor systems and wavelet systems are obtained as some corollaries. Of course, our method combines with some techniques in wavelet analysis and time-frequency analysis. We mainly borrow some thoughts in classifying the wavelet frame and Gabor systems in [15,16,17].

Let us now describe the organization of the paper. Section 2 is of a preliminary character: it contains various notations and some facts about the frame and the wave packet system. In Section 3, the characterizations of all kinds of the orthogonal wave packet systems with different operator orders are established . Finally, the necessary conditions and sufficient conditions about the wave packet systems to be the Parseval frames with the very general lattices are obtained.

2. Preliminaries

Some basic notations are listed in this section. Throughout this paper, we use the following notations. R and Z denote the set of real numbers and the set of integers, respectively. $L^2(R^n)$ is the space of all square-integrable functions in n dimensions, and $\cdot$ and $\|\cdot\|$ denote the inner product and norm in $L^2(R^n)$, respectively, and $l^2(Z^n)$ denotes the space of all square-summable sequences. We denote by T^n the n-dimensional torus. By $L^p(T^n)$ we denote the space of all Z^n-periodic functions f (i.e., f is 1-periodic in each variable) such that $\int_{T^n} |f(x)|^p dx < +\infty$.

We use the Fourier transform in the form

$$\hat{f}(\omega) = \int_{R^n} f(x) e^{-2\pi i x\cdot\omega} dx, \tag{2.1}$$

where $\cdot$ denotes the standard inner product in R^n, and it is often omitted when we can understand this from the content. Sometimes, $\hat{f}(\omega)$ is defined by $\mathcal{F}f$.

Let E_n denote the set of all expanding matrices. An expanding matrix means that all of its eigenvalues have magnitude greater than 1. For $A \in E_n$, we denote by A^* the transpose of A. It is obvious that $A^* \in E_n$. Let $GL_n(R)$ denote the set of all $n \times n$ non-singular matrices with real entries. For $B \in GL_n(R)$ we denote by B^{-1} the inverse of B. For the sake of simplicity, we denote $(A^*)^{-1}$ by $A^\sharp$.

Let us recall the definition of frame.

Definition 2.1. Let H be a separable Hilbert space. A sequence $\{f_i\}_{i\in N}$ of elements of H is a frame for H if there exist constants $0 < C \le D < \infty$ such that for all $f \in H$,

$$C\|f\|^2 \le \sum_{i=1}^{\infty} | < f, f_i\rangle|^2 \le D\|f\|^2. \tag{2.2}$$

The numbers C, D are called lower and upper frame bounds, respectively (the largest C and the smallest D for which (2.2) holds are the optimal frame bounds). Those sequences which satisfy only the upper inequality in (2.2) are called Bessel sequences. A frame is tight if $C = D$. If $C = D = 1$, it is called a Parseval frame.

In this paper, we will work with three unitary operators on $L^2(R^n)$. Let $A \in E_n$ and $B, C \in GL_n(R)$. The first one consists of the dilation operator $D_A : L^2(R^n) \to L^2(R^n)$ defined by $(D_A)f(x) = q^{\frac{1}{2}}f(Ax)$ with $q = |detA|$. The second is the shift operator $T_{Bk} : L^2(R^n) \to L^2(R^n)$, $k \in Z^n$, defined by $(T_{Bk}f)(x) = f(x-Bk)$. The final one consists of the modulation operator $E_{Cm} : L^2(R^n) \to L^2(R^n)$, $m \in Z^n$, defined by $(E_{Cm}f)(x) = e^{2i\pi Cm\cdot x}f(x)$.

Let $P \subset Z$, $Q \subset R^n$ and $S = P \times Q$. Then, we have $S \subset Z \times R^n$. Again, let $\{A_p : A_p \in P\} \subset E_n$ and $B \in GL_n(R)$. For the function $\psi \in L^2(R^n)$, the wave packet system Ψ is defined by

$$\Psi = \left\{\psi_{p,\ \nu,\ m}(x) \quad | \quad D_{A_p}E_\nu T_{Bm}\psi(x), m \in Z^n,\ (p, \nu) \in S\right\}. \tag{2.3}$$

It is easy to see that if $A_p = A^j (j \in Z)$ and $S = Z\times\{0\}$ in (2.3), then we obtain the wavelet systems. Of course, the Gabor systems can be got when the set $\{A_p : A_p \in P\}$ in (2.3) only consists of the elementary matrix E. This simple observation suggests that the wave packet systems provide greater flexibility than the wavelet systems or the Gabor systems.

By changing the order of the operators, we can also define the following one-to-one function systems from $S \times Z^n$ into $L^2(R^n)$:

$$\Psi^1 = \left\{\psi_{p,\ \nu,\ m}(x) \quad | \quad D_{A_p}T_{Bm}E_\nu\psi(x), m \in Z^n,\ (p, \nu) \in S\right\}, \tag{2.4}$$

$$\Psi^2 = \left\{\psi_{p,\ \nu,\ m}(x) \quad | \quad E_\nu D_{A_p}T_{Bm}\psi(x), m \in Z^n,\ (p, \nu) \in S\right\}. \tag{2.5}$$

Then, we will give the definitions of the wave packet frame and the frame wave packet function.

Definition 2.2. We say that the wave packet system Ψ defined by (2.3) is a wave packet frame if it is a frame for $L^2(R^n)$. Then, the function ψ is called a frame wave packet function.

In order to prove results presented in next section, we need some lemmas. At first, we will consider the following set of functions:

$$\Gamma = \left\{ f \in L^2(R^n) : \ \hat{f} \in L^\infty(R^n) \text{ and } \hat{f} \text{ has compact support in } R^n \setminus \{0\} \right\}. \tag{2.6}$$

Lemma 2.1. *Γ is a dense subset of $L^2(R^n)$.*

The following two lemmas come from the book [17].

Lemma 2.2. *Suppose that $\{f_k\}_{k=1}^{+\infty}$ is a family of elements in a Hilbert space H such that there exist constants $0 < C \le D < +\infty$ satisfying (2.2) for all f belonging to a dense subset D of H. Then, the same inequalities (2.2) are true for all $f \in H$; that is, $\{f_k\}_{k=1}^{+\infty}$ is a frame for H.*

Lemma 2.3. *The system $\{\psi(x - Cm)\}_{m \in Z^n}$ is orthogonal if and only if*

$$\sum_{m \in Z^n} |\hat{\psi}(\omega + C^\sharp m)|^2 = c\|\psi\|^2, \ a.e. \ \omega \in R^n, \tag{2.7}$$

where $c = |detC|$.

The following useful fact can be found in [12, Lemma 2.2].

Lemma 2.4. *Let $A \in GL_n(R)$, $y, z \in R^n$ and $f \in L^2(R^n)$. Then the following holds:*

(1) $(T_y f)\hat{} = E_{-y}\hat{f}$, $(E_z f)\hat{} = T_z \hat{f}$, $(D_A f)\hat{} = D_{A^\sharp}\hat{f}$;

(2) $T_y E_z f = e^{-2\pi i z \cdot y} E_z T_y f$, $D_A E_y f = E_{A^* y} D_A f$, $D_A T_y f = T_{A^{-1}y} D_A f$;

(3) $(T_y E_z f)\hat{} = e^{-2\pi i z \cdot y} T_z E_{-y} \hat{f}$;

(4) $(D_A T_y f)\hat{}(\xi) = E_{-A^\sharp y} D_{A^\sharp} \hat{f}(\xi) = |detA|^{-\frac{1}{2}} \hat{f}(A^\sharp \xi) e^{-2\pi i A^{-1} y \cdot \xi}$.

3. The Characterization of the Orthogonal Wave Packet Systems

In this section, we will characterize the orthogonality of all kinds of the wave packet systems with different operator order in terms of the Fourier transforms of the wave packet systems' generating functions. Therefore, some existing results in Gabor system and wavelet system are obtained as some corollaries.

For convenience, we only study the special cases of wave packet systems defined by (2.3).

Theorem 3.1. *Let $A \in E_n$ and $B \in GL_n(R)$. Wave packet system $\{D_A^j E_\nu T_{Bm} \psi(x)\}_{j \in Z, m \in Z^n, \nu \in S}$ is orthogonal if and only if both of the equation*

$$\sum_{m \in Z^n} |\hat{\psi}(\omega + B^\sharp m)|^2 = b\|\psi\|^2, \ a.e. \ \omega \in R^n \tag{3.1}$$

and the equation

$$\sum_{m\in Z^n}\hat{\psi}(A^{*j}(\omega+B^{\sharp}m)-\nu)\overline{\hat{\psi}(\omega+B^{\sharp}m)}=0,\ j\in N,\ a.e.\ \omega\in R^n \tag{3.2}$$

hold, where $b=|detB|$ *and* $\nu\neq 0$.

Proof. ($\Longrightarrow$) We firstly assume that wave packet system $\{D_A^jE_\nu T_{Bm}\psi(x)\}_{j\in Z,m\in Z^n,\nu\in S}$ is orthogonal. In particular, the system $\{D_A^0E_0T_{Bm}\psi(x)\}_{m\in Z^n}$ is also orthogonal. By Lemma 2.3, the equation (3.1) holds.

Furthermore, when $j_1<j_2$, by changing variables, we have

$$\begin{aligned}
0&=\langle D_A^{j_1}E_{\nu_1}T_{Bm_1}\psi(\cdot),\, D_A^{j_2}E_{\nu_2}T_{Bm_2}\psi(\cdot)\rangle\\
&=|detA|^{\frac{j_1+j_2}{2}}\int_{R^n}e^{2i\pi\nu_1A^{j_1}x}\psi(A^{j_1}x-Bm_1)e^{-2i\pi\nu_2A^{j_2}x}\overline{\psi(A^{j_2}x-Bm_2)}dx\\
&=|detA|^{\frac{j_2-j_1}{2}}\int_{R^n}\psi(x)e^{2i\pi(\nu_1-\nu_2A^{j_2-j_1})(x+Bm_1)}\\
&\qquad\overline{\psi(A^{j_2-j_1}x+B(A^{j_2-j_1}m_1-m_2))}dx\\
&=e^{-2i\pi(\nu_2A^{j_2-j_1}-\nu_1)Bm_1}\langle\psi(\cdot),E_{(A^{j_2-j_1}\nu_2-\nu_1)}D_A^{j_2-j_1}T_{B(m_2-A^{j_2-j_1}m_1)}\psi(\cdot)\rangle.
\end{aligned}\tag{3.3}$$

Set $C=e^{-2i\pi(\nu_2A^{j_2-j_1}-\nu_1)Bm_1}$, $\nu=A^{j_2-j_1}\nu_2-\nu_1, j=j_2-j_1$ and $m=(m_2-A^{j_2-j_1}m_1)$, then for $j_1<j_2$,

$$\langle D_A^{j_1}E_{\nu_1}T_{Bm_1}\psi(\cdot),\, D_A^{j_2}E_{\nu_2}T_{Bm_2}\psi(\cdot)\rangle=C\langle\psi(\cdot),\, E_\nu D_A^jT_{Bm}\psi(\cdot)\rangle, \tag{3.4}$$

where $j>0\ \ (j\in N)$, $\nu\in S$, $\nu\neq 0$ and $m\in Z^n$. Therefore, the orthogonality between $D_A^{j_1}E_{\nu_1}T_{Bm_1}\psi(x)$ and $D_A^{j_2}E_{\nu_2}T_{Bm_2}\psi(x)$, for $j_1<j_2$, $\nu_1,\nu_1\in S$, $m_1,m_2\in Z$, can be reduced to the orthogonality between $\psi(x)$ and $E_\nu D_A^jT_{Bm}\psi(x)$, where $j>0$ and $\nu\in S,\nu\neq 0,m\in Z$. It follows from Lemma 2.4 and Plancherel theorem that

$$\langle\psi(\cdot),\, E_\nu D_A^jT_{Bm}\psi(\cdot)\rangle=|detA|^{\frac{j}{2}}\int_{R^n}\hat{\psi}(A^{*j}\omega-\nu)\overline{\hat{\psi}(\omega)}e^{2\pi iBm\omega}d\omega. \tag{3.5}$$

It follows from Levi theorem that the interchange of the order of summation and integration is valid in the following. Then, we have

$$\begin{aligned}
&\int_{B^\sharp([0,1]^n)}\sum_{s\in Z^n}|\hat{\psi}(A^{*j}(\omega+B^\sharp s)-\nu)\overline{\hat{\psi}(\omega+B^\sharp s)}|d\omega\\
=&\int_{R^n}|\hat{\psi}(A^{*j}\omega-\nu)\overline{\hat{\psi}(\omega)}|d\omega\\
&\leq(\int_{R^n}|\hat{\psi}(A^{*j}\omega-\nu)|^2d\omega)^{\frac{1}{2}}(\int_{R^n}|\overline{\hat{\psi}(\omega)}|^2d\omega)^{\frac{1}{2}}<\infty,
\end{aligned}\tag{3.6}$$

where the inequality is obtained by using Cauchy-Schwarz's inequality. Thus the two series in (3.1) and (3.2) absolutely converge for $a.e.\ \omega \in R^n$, consequently, we can define a function $F_j : R \to C$ by

$$F_j(\omega) = \sum_{s \in Z^n} \hat{\psi}(A^{*j}(\omega + B^\sharp s) - \nu)\overline{\hat{\psi}(\omega + B^\sharp s)},\ a.e.\ \ \omega. \tag{3.7}$$

It is clear to see that $F_j(\omega)$ is $B^\sharp T$-periodic, and the above argument gives that $F_j(\omega) \in L^1(B^\sharp[0,1]^n)$. In fact, we even have $F_j(\omega) \in L^2(B^\sharp[0,1]^n)$ by

$$|F_j(\omega)|^2 \le \sum_{s \in Z^n} |\hat{\psi}(A^{*j}(\omega + B^\sharp s) - \nu)|^2 \sum_{s \in Z^n} |\hat{\psi}(\omega + B^\sharp s)|^2. \tag{3.8}$$

Then, it deduce from the definition of $F_j(\omega)$ that

$$\begin{aligned} &\int_{R^n} \hat{\psi}(A^{*j}\omega - \nu)\overline{\hat{\psi}(\omega)} e^{2\pi i B m \omega} d\omega \\ &= \sum_{s \in Z^n} \int_{B^\sharp([0,1]^n)} \hat{\psi}(A^{*j}(\omega + B^\sharp s) - \nu)\overline{\hat{\psi}(\omega + B^\sharp s)} e^{2\pi i B m \omega} d\omega \\ &= \int_{B^\sharp([0,1]^n)} F_j(\omega) e^{2\pi i B m \omega} d\omega. \end{aligned} \tag{3.9}$$

Combining with (3.3)-(3.5) and (3.9), we find

$$0 = \int_{B^\sharp([0,1]^n)} F_j(\omega) e^{2\pi i B m \omega} d\omega. \tag{3.10}$$

That is to say that, for any $j \in N$, the all Fourier coefficients of the functions $F_j(\omega)$ are 0. This shows that $F_j(\omega) = 0,\ a.e.\ \ \omega \in R^n$. So the equation (3.2) holds.

($\Longleftarrow$) Suppose that the equations (3.1) and (3.2) hold. Then it derive from (3.1) and Lemma 2.3 that the system $\{D_A^0 E_0 T_{Bm}\psi(x)\}_{m \in Z}$ is orthogonal. For fixed $j \in Z, \nu \in S$, since

$$\begin{aligned} &\langle D_A^j E_\nu T_{Bm_1}\psi(\cdot),\ D_A^j E_\nu T_{Bm_2}\psi(\cdot)\rangle \\ &= \langle D_A^0 E_0 T_{Bm_1}\psi(\cdot),\ D_A^0 E_0 T_{Bm_2}\psi(\cdot)\rangle = \delta_{m_1,m_2}, \end{aligned} \tag{3.11}$$

the system $\{D_A^j E_\nu T_{Bm}\psi(x)\}_{m \in Z}$ is also orthogonal.

Notice that (3.2) holds, hence according to the proof of (3.4), (3.5) and (3.9), we obtain that for $j_1 < j_2,\ j := j_2 - j_1$ and a.e. $\ \omega \in R^n$,

$$\begin{aligned} &\langle D_A^{j_1} E_{\nu_1} T_{Bm_1}\psi(\cdot),\ D_A^{j_2} E_{\nu_2} T_{Bm_2}\psi(\cdot)\rangle \\ &= C|detA|^{\frac{j}{2}} \int_{B^\sharp([0,1]^n)} F_j(\omega) e^{2\pi i B m \omega} d\omega = 0. \end{aligned} \tag{3.12}$$

If $j_1 > j_2$, then it follows from (3.12) that

$$\begin{aligned} &\langle D_A^{j_1} E_{\nu_1} T_{Bm_1}\psi(\cdot),\, D_A^{j_2} E_{\nu_2} T_{Bm_2}\psi(\cdot)\rangle \\ &= \overline{\langle D_A^{j_2} E_{\nu_2} T_{Bm_2}\psi(\cdot),\, D_A^{j_1} E_{\nu_1} T_{Bm_1}\psi(\cdot)\rangle} = 0. \end{aligned} \tag{3.13}$$

Combining with (3.11)-(3.13), we obtain that the wave packet system $\{D_A^j E_\nu T_{Bm}\psi(x)\}_{j\in Z, m\in Z^n, \nu\in S}$ is orthogonal. Therefore, the proof of Theorem 3.1 is completed.

Remark 3.1. In particular, let A be the elementary matrix E and $S=CZ^n$ in Theorem 3.1. Then, we obtain Theorem 2.1 in [13], which characterizes the orthogonality of Gabor systems in terms of the Fourier transforms of the functions, that is

Corollary 3.1. *Let $B, C \in GL_n(R)$. Then, the Gabor system $\{E_{Ck} T_{Bm}\psi(x)\}_{k,m\in Z^n}$ is orthogonal if and only if both of the equation*

$$\sum_{m\in Z^n} |\hat{\psi}(\omega + B^\sharp m)|^2 = b\|\psi\|^2,\ a.e.\ \omega \in R^n \tag{3.14}$$

and the equation

$$\sum_{m\in Z^n} \hat{\psi}(\omega + B^\sharp m - Ck)\overline{\hat{\psi}(\omega + B^\sharp m)} = 0,\ \ a.e.\ \ \omega \in R^n \tag{3.15}$$

hold for every $k \neq 0$.

Remark 3.2. It is immediate to see that if $S = \{0\}$ in Theorem 3.1, then we obtain a sufficient and necessary condition of the orthogonal wavelet system as the following, which is the generalization of the equalities (1.1) and (1.2) in [6, Chapter 3, page 124].

Corollary 3.2. *Let A be an arbitrary matrix, $B \in GL_n(R)$ and $\psi \in L^2(R^n)$. Then the wavelet system $\{D_A^j T_{Bm}\psi(x)\}_{j\in Z, m\in Z^n}$ is orthogonal if and only if both of the equation*

$$\sum_{m\in Z^n} |\hat{\psi}(\omega + B^\sharp m)|^2 = b\|\psi\|^2,\ a.e.\ \omega \in R^n \tag{3.16}$$

and the equation

$$\sum_{m\in Z^n} \hat{\psi}(A^{*j}(\omega + B^\sharp m))\overline{\hat{\psi}(\omega + B^\sharp m)} = 0,\ \ j \in N,\ \ a.e.\ \omega \in R^n \tag{3.17}$$

hold.

Remark 3.3. Furthermore, we can classify the orthonormality of wave packet systems.

Corollary 3.3. *Let $A \in E_n$ and $B \in GL_n(R)$. Wave packet system $\{D_A^j E_\nu T_{Bm}\psi(x)\}_{j\in Z, m\in Z^n, \nu\in S}$ is orthonormal if and only if both of the equality*

$$\sum_{m\in Z^n} |\hat{\psi}(\omega + B^\sharp m)|^2 = b, \; a.e. \;\; \omega \in R^n \tag{3.18}$$

and the equality

$$\sum_{m\in Z^n} \hat{\psi}(A^{*j}(\omega + B^\sharp m) - \nu)\overline{\hat{\psi}(\omega + B^\sharp m)} = 0, \;\; j \in N, \;\; a.e. \;\; \omega \in R^n \tag{3.19}$$

hold, where $b = |detB|$ and $\nu \neq 0$.

Remark 3.4. In the following, we will classify the wave packet systems Ψ^1 and Ψ^2.

For wave packet systems Ψ^1, by Lemma 2.4, we get

$$D_A^j T_{Bm} E_\nu \psi(x) = e^{-2\pi i Bm\cdot\nu} D_A^j E_\nu T_{Bm}\psi(x). \tag{3.20}$$

Then, it deduces from Theorem 3.1 and (3.20) that

Corollary 3.4. *Let $A \in E_n$ and $B \in GL_n(R)$. Wave packet system $\{D_A^j T_{Bm} E_\nu \psi(x)\}_{j\in Z, m\in Z^n, \nu\in S}$ is orthogonal if and only if both of the equation*

$$\sum_{m\in Z^n} |\hat{\psi}(\omega + B^\sharp m)|^2 = b\|\psi\|^2, \; a.e. \;\; \omega \in R^n \tag{3.21}$$

and the equation

$$\sum_{m\in Z^n} \hat{\psi}(A^{*j}(\omega + B^\sharp m) - \nu)\overline{\hat{\psi}(\omega + B^\sharp m)} = 0, \;\; j \in N, \;\; a.e. \;\; \omega \in R^n \tag{3.22}$$

hold, where $b = |detB|$ and $\nu \neq 0$.

For wave packet systems Ψ^2, it follows from (2) in Lemma 2.4 that

$$E_\nu D_{A_p} T_{Bm}\psi(x) = D_{A_p} E_{A^\sharp\nu} T_{Bm}\psi(x). \tag{3.23}$$

Thus from theorem 3.1 and (3.23), we have

Corollary 3.5. *Let $A \in E_n$ and $B \in GL_n(R)$. Wave packet system $\{E_\nu D_A^j T_{Bm}\psi(x)\}_{j\in Z, m\in Z^n, \nu\in S}$ is orthogonal if and only if both of the equation*

$$\sum_{m\in Z^n} |\hat{\psi}(\omega + B^\sharp m)|^2 = b\|\psi\|^2, \; a.e. \;\; \omega \in R^n \tag{3.24}$$

and the equation

$$\sum_{m\in Z^n} \hat{\psi}(A^{*j}(\omega + B^\sharp m) - A^\sharp \nu)\overline{\hat{\psi}(\omega + B^\sharp m)} = 0,\ j \in N,\ a.e.\ \omega \in R^n \tag{3.25}$$

hold, where $b = |detB|$ *and* $\nu \neq 0$.

In what follows, we will consider the completeness of wave packet systems in some special cases. In the same way, we use some techniques of wavelet theory. In order to make the problems more simpler, we pose the condition $e^{-2\pi i BmCk} = 1$ in the theorem.

Theorem 3.2. *Let* $A \in E_n$ *and* $B, C \in GL_n(R)$. *Suppose that the function* $\psi \in L^2(R^n)$ *satisfies the equalities*

$$\sum_{j\in Z}\sum_{k\in Z^n} |\hat{\psi}(A^{\sharp j}\omega - Ck)|^2 = b\ \ a.e.\ \omega \in R^n \tag{3.26}$$

and

$$\sum_{r=0}^{+\infty}\sum_{k\in Z^n} \hat{\psi}(A^{*r}\omega - Ck)\overline{\hat{\psi}(A^{*r}(\omega + B^\sharp \alpha) - Ck)} = 0,\ \ a.e.\ \omega \in R^n, \tag{3.27}$$

where $b = |detB|$ *and* $\alpha \in Z^n/A^*Z^n$. *Then the wave packet system* $\{D_A^j E_{Ck} T_{Bm}\psi(x)\}_{j\in Z, k, m\in Z^n}$ *is complete in* $L^2(R^n)$ *when* $e^{-2\pi i BmCk} = 1$.

Proof. Let $W_{j,k} = \overline{\text{span}\{D_A^j E_{Ck} T_{Bm}\psi(x) : m \in Z^n\}}$, and $Q_{j,k}$ denote the orthogonal projection onto the space $W_{j,k}$. Then, for every $f \in L^2(R^n)$, we have

$$Q_{j,k}f(x) = \sum_{m\in Z^n} \langle f, D_A^j E_{Ck} T_{Bm}\psi\rangle D_A^j E_{Ck} T_{Bm}\psi(x). \tag{3.28}$$

In order to prove the completeness of wave packet system $\{D_A^j E_{Ck} T_{Bm}\psi(x)\}_{j\in Z, k, m\in Z^n}$, it is sufficient to show that for any $f \in L^2(R^n)$,

$$\sum_{j\in Z}\sum_{k\in Z^n} (Q_{j,k}f)\hat{}(\omega) = \hat{f}(\omega). \tag{3.29}$$

By Lemmas 2.1 and 2.2, it suffices to prove that above equality holds for $f \in \Gamma$ defined by (2.6). For $f \in \Gamma$, we assert

$$(Q_{j,k}f)\hat{}(\omega) = \frac{1}{b}\sum_{s\in Z^n} \hat{f}(\omega + A^{*j}B^\sharp s)\overline{\hat{\psi}(A^{\sharp j}\omega + B^\sharp s - Ck)}\hat{\psi}(A^{\sharp j}\omega - Ck). \tag{3.30}$$

By Plancherel theorem and Lemma 2.4, we obtain

$$
\begin{aligned}
&\langle f, D_A^j E_{Ck} T_{Bm}\psi\rangle \\
&= |detA|^{-\frac{j}{2}} \int_{R^n} \hat{f}(\omega)\overline{\hat{\psi}(A^{\sharp j}\omega - Ck)} e^{2\pi i Bm(A^{\sharp j}\omega - Ck)} d\omega \\
&= |detA|^{\frac{j}{2}} \int_{B^{\sharp}([0,1]^n)} \left(\sum_{s\in Z^n} \hat{f}(A^{*j}(\omega + B^{\sharp}s))\overline{\hat{\psi}(\omega + B^{\sharp}s - Ck)} \right) e^{2\pi i Bm\omega} d\omega,
\end{aligned}
\tag{3.31}
$$

where the final equation use $e^{-2\pi i BmCk} = 1$. That is to say, the sequences $\{\langle f, D_A^j E_{Ck} T_{Bm}\psi\rangle\}_{m\in Z^n}$ is evidently the Fourier coefficients of the $B^{\sharp}([0,1]^n)-$periodic function

$$
|detA|^{\frac{j}{2}} \sum_{s\in Z^n} \hat{f}(A^{*j}(\omega + B^{\sharp}s))\overline{\hat{\psi}(\omega + B^{\sharp}s - Ck)}.
\tag{3.32}
$$

Thus, we have

$$
\begin{aligned}
&\sum_{s\in Z^n} \hat{f}(\omega + A^{*j}B^{\sharp}s)\overline{\hat{\psi}(A^{\sharp j}\omega + B^{\sharp}s - Ck)} \\
&= \frac{b}{|detA|^{\frac{j}{2}}} \sum_{m\in Z^n} \langle \hat{f}, (D_A^j E_{Ck} T_{Bm}\psi)\widehat{\ }\rangle e^{-2\pi i BmA^{\sharp j}\omega}.
\end{aligned}
\tag{3.33}
$$

Multiplying both sides of (3.33) by $\hat{\psi}(A^{\sharp j}\omega - Ck)$ and again using $e^{-2\pi i BmCk} = 1$, we have

$$
\begin{aligned}
&\sum_{s\in Z^n} \hat{f}(\omega + A^{*j}B^{\sharp}s)\overline{\hat{\psi}(A^{\sharp j}\omega + B^{\sharp}s - Ck)}\hat{\psi}(A^{\sharp j}\omega - Ck) \\
&= b\sum_{m\in Z^n} \langle \hat{f}, (D_A^j E_{Ck} T_{Bm}\psi)\widehat{\ }\rangle (D_A^j E_{Ck} T_{Bm}\psi)\widehat{\ }(\omega).
\end{aligned}
\tag{3.34}
$$

It deduces from (3.28) and (3.34) that the assertion holds.

To clarify the meaning of the series in the above equations, we point out that the interchange of the order of summation and integration is valid, since we have assumed that function $\hat{f}$ is compactly supported, consequently, the sums over s, m are finite. By (3.26) and (3.30), we can write the projections $Q_{j,k}$:

$$
\begin{aligned}
&\sum_{j\in Z}\sum_{k\in Z^n} (Q_{j,k}f)\widehat{\ }(\omega) \\
&= \frac{1}{b}\sum_{j\in Z}\sum_{k\in Z^n} \hat{f}(\omega)|\hat{\psi}(A^{\sharp j}\omega - Ck)|^2 \\
&+\frac{1}{b}\sum_{j\in Z}\sum_{k\in Z^n}\sum_{s\neq 0} \hat{f}(\omega + A^{*j}B^{\sharp}s)\overline{\hat{\psi}(A^{\sharp j}\omega + B^{\sharp}s - Ck)}\hat{\psi}(A^{\sharp j}\omega - Ck) \\
&= \hat{f}(\omega) + \frac{1}{b}\sum_{j\in Z}\sum_{s\neq 0} \hat{f}(\omega + A^{*j}B^{\sharp}s)\left(\sum_{k\in Z^n} \overline{\hat{\psi}(A^{\sharp j}\omega + B^{\sharp}s - Ck)}\hat{\psi}(A^{\sharp j}\omega - Ck)\right).
\end{aligned}
\tag{3.35}
$$

In the following, we will calculate the second term of the last equality in (3.35). It is not difficult to see that there exists only one non-negative integer r such that $s = A^{*r}\alpha$ if $s = (s_1, s_2, \cdots, s_n) \neq (0, 0, \cdots, 0)$, where $\alpha \in Z^n/A^*Z^n$. Set $\aleph =: Z^n/A^*Z^n$, then we have

$$
\begin{aligned}
&\int_{R^n} \frac{1}{b}\overline{\hat{f}(\omega)} \sum_{s\neq 0}\sum_{j\in Z} \\
&\qquad \left(\hat{f}(\omega + A^{*j}B^{\sharp}s) \sum_{k\in Z^n} \hat{\psi}(A^{\sharp j}\omega - Ck)\overline{\hat{\psi}(A^{\sharp j}\omega + B^{\sharp}s - Ck)}\right) d\omega \\
&= \int_{R^n} \frac{1}{b}\overline{\hat{f}(\omega)} \sum_{r=0}^{+\infty}\sum_{\alpha\in\aleph}\sum_{j\in Z} \\
&\qquad \left(\hat{f}(\omega + A^{*j}B^{\sharp}A^{*r}\alpha) \sum_{k\in Z^n} \hat{\psi}(A^{\sharp j}\omega - Ck)\overline{\hat{\psi}(A^{\sharp j}\omega + B^{\sharp}A^{*r}\alpha - Ck)}\right) d\omega \\
&= \int_{R^n} \frac{1}{b}\overline{\hat{f}(\omega)} \sum_{\alpha\in\aleph}\sum_{p\in Z} \\
&\qquad \left(\hat{f}(\omega + A^{*p}B^{\sharp}\alpha) \sum_{r=0}^{+\infty}\sum_{k\in Z^n} \hat{\psi}(A^{*r}A^{\sharp p}\omega - Ck)\overline{\hat{\psi}(A^{*r}(A^{\sharp p}\omega + B^{\sharp}\alpha) - Ck)}\right) d\omega \\
&= 0.
\end{aligned} \tag{3.36}
$$

So, it follows from (2.27) and (3.36) that (3.29) holds. The proof of Theorem 3.2 is finished.

4. The Necessary Condition and Sufficient Condition of the Wave Packet Parseval Frame

In this section, we will give the necessary condition and sufficient condition for the wave packet system to be a Parseval frame with the very general lattices. Then, we obtain the corresponding known results in Gabor system and wavelet system as some corollaries.

First, we establish a lemma as follows.

Lemma 4.1. *Suppose that wave packet system $\{D_{A_p}E_{\nu}T_{Bm}\psi(x)\}_{m\in Z^n,\,(p,\,\nu)\in S}$ is defined by (2.3), then for any $f \in \Gamma$,*

$$
\begin{aligned}
&\sum_{(p,\,\nu)\in S}\sum_{m\in Z^n} |\langle f,\, D_{A_p}E_{\nu}T_{Bm}\psi\rangle|^2 \\
&= \sum_{(p,\,\nu)\in S} \frac{q_p}{b}\int_{B^{\sharp}([0,1]^n)} |\sum_{s\in Z^n} \hat{f}(A_p^*(\omega + B^{\sharp}s + \nu))\overline{\hat{\psi}(\omega + B^{\sharp}s)}|^2 d\omega,
\end{aligned} \tag{4.1}
$$

where $b = |detB|$ and $q_p = |detA_p|$.

Proof. Let $f \in \Gamma$, then $\hat{f} \in C_c(R)$ and $\hat{f}$ have compact support. Put $q_p = |detA_p|$, then by Lemma 2.3 and Plancherel theorem, we have

$$
\begin{aligned}
&\sum_{(p,\nu)\in S}\sum_{m\in Z^n} |\langle f,\, D_{A_p}E_\nu T_{Bm}\psi\rangle|^2 \\
&= \sum_{(p,\nu)\in S}\sum_{m\in Z^n} |\langle \hat{f},\, D_{A_p^\sharp}T_\nu E_{-Bm}\hat{\psi}\rangle|^2 \qquad (4.2)\\
&= \sum_{p\in P} q_p \sum_{\nu\in Q}\sum_{m\in Z^n} |\int_{R^n} \hat{f}(A_p^*(\omega+\nu))\overline{\hat{\psi}(\omega)}e^{2\pi iBm\omega}d\omega|^2.
\end{aligned}
$$

We affirm:

$$
\begin{aligned}
&\sum_{p\in P} q_p \sum_{\nu\in Q}\sum_{m\in Z^n} |\int_{R^n} \hat{f}(A_p^*(\omega+\nu))\overline{\hat{\psi}(\omega)}e^{2\pi iBm\omega}d\omega|^2 \qquad (4.3)\\
&= \sum_{(p,\nu)\in S} \frac{q_p}{b}\int_{B^\sharp([0,1]^n)} |\sum_{s\in Z^n} \hat{f}(A_p^*(\omega+B^\sharp s+\nu))\overline{\hat{\psi}(\omega+B^\sharp s)}|^2 d\omega.
\end{aligned}
$$

Now, we prove (4.3). Fix $(p,\, \nu) \in S$. The interchange of the order of summation and integration is valid, since we have assumed that function $\hat{f}$ is compactly supported, and, consequently the sum over m is finite.

Similar to (3.6)-(3.9), we can define the function $F_p : R \to C$ by

$$
F_p(\omega) = \sum_{s\in Z^n} \hat{f}(A_p^*(\omega+B^\sharp s+\nu))\overline{\hat{\psi}(\omega+B^\sharp s)},\ a.e.\ \omega \qquad (4.4)
$$

and prove that $F_p(\omega) \in L^1(B^\sharp[0,1]^n)$. Also, we can obtain $F_p(\omega) \in L^2(B^\sharp[0,1]^n)$. Then, according to the definition of $F_p(\omega)$, we have

$$
\begin{aligned}
&\int_{R^n} \hat{f}(A_p^*(\omega+\nu))\overline{\hat{\psi}(\omega)}e^{2\pi iBm\omega}d\omega \\
&= \int_{B^\sharp([0,1]^n)} \left(\sum_{s\in Z^n} \hat{f}(A_p^*(\omega+B^\sharp s+\nu))\overline{\hat{\psi}(\omega+B^\sharp s)}\right) e^{2\pi iBm\omega}d\omega \qquad (4.5)\\
&= \int_{B^\sharp([0,1]^n)} F_p(\omega)e^{2\pi iBm\omega}d\omega.
\end{aligned}
$$

Parseval's equality shows that

$$
\sum_{m\in Z^n} |\int_{B^\sharp([0,1]^n)} F_p(\omega)e^{2\pi iBm\omega}d\omega|^2 = \frac{1}{b}\int_{B^\sharp([0,1]^n)} |F_p(\omega)|^2 d\omega. \qquad (4.6)
$$

Combining (4.5),(4.6) and the definition of $F_p(\omega)$, we get

$$
\begin{aligned}
&\sum_{m\in Z^n} |\int_{R^n} \hat{f}(A_p^*(\omega+\nu))\overline{\hat{\psi}(\omega)}e^{2\pi iBm\omega}d\omega|^2 \qquad (4.7)\\
&= \frac{1}{b}\int_{B^\sharp([0,1]^n)} |\sum_{s\in Z^n} \hat{f}(A_p^*(\omega+B^\sharp s+\nu))\overline{\hat{\psi}(\omega+B^\sharp s)}|^2 d\omega.
\end{aligned}
$$

So, (4.3) holds.

At last, comparing to (4.2) and (4.3), we find that (4.1) holds.

Based on Lemma 4.1, we will establish a sufficient condition for the wave packet system to be a Parseval frame.

Theorem 4.1. *The wave packet system* $\{D_{A_p}E_\nu T_{Bm}\psi(x)\}_{m\in Z^n,\,(p,\nu)\in S}$ *defined by (2.3) is a wave packet Parseval frame for* $L^2(R^n)$ *if the two following equalities*

$$\sum_{(p,\nu)\in S}|\hat{\psi}(A_p^\sharp\omega-\nu)|^2=b,\quad a.e.\ \ \omega\in R^n \tag{4.8}$$

and

$$\sum_{\nu\in Q}\hat{\psi}(A_p^\sharp\omega-\nu)\overline{\hat{\psi}(A_p^\sharp\omega+B^\sharp s-\nu)=0},\quad a.e.\ \ \omega\in R^n \tag{4.9}$$

hold for every $s\in Z^n$ and $s\neq 0$, where $b=|detB|$.

Proof. In the same way, let $f\in\Gamma$. Notice that function $\hat{f}$ in the series $\sum\limits_{(p,\nu)\in S}\sum\limits_{m\in Z^n}|\langle f,\,D_{A_p}E_\nu T_{Bm}\psi\rangle|^2$ is compactly supported, hence the sum over m is finite, consequently, the interchange of the order of summation and integration is valid.

By Lemma 2.4, Lemma 4.1 and Plancherel theorem, we can write

$$\begin{aligned}
&\sum_{(p,\nu)\in S}\sum_{m\in Z^n}|\langle f,\,D_{A_p}E_\nu T_{Bm}\psi\rangle|^2\\
&=\sum_{(p,\nu)\in S}\frac{q_p}{b}\int_{B^\sharp([0,1]^n)}|\sum_{s\in Z^n}\hat{f}(A_p^*(\omega+B^\sharp s+\nu))\overline{\hat{\psi}(\omega+B^\sharp s)}|^2d\omega\\
&=\sum_{(p,\nu)\in S}\frac{q_p}{b}\int_{B^\sharp([0,1]^n)}(\sum_{s\in Z^n}\hat{f}(A_p^*(\omega+B^\sharp s+\nu))\overline{\hat{\psi}(\omega+B^\sharp s)})\times\\
&\quad(\sum_{m\in Z^n}\overline{\hat{f}(A_p^*(\omega+B^\sharp m+\nu))}\hat{\psi}(\omega+B^\sharp m))d\omega\\
&=\sum_{(p,\nu)\in S}\frac{q_p}{b}\int_{R^n}\overline{\hat{f}(A_p^*(\omega+\nu)}\hat{\psi}(\omega)(\sum_{s\in Z^n}\hat{f}(A_p^*(\omega+B^\sharp s+\nu))\overline{\hat{\psi}(\omega+B^\sharp s)}d\omega\\
&=\sum_{(p,\nu)\in S}\frac{q_p}{b}\int_{R^n}|\hat{f}(A_p^*\omega)\hat{\psi}(\omega-\nu)|^2d\omega\\
&\quad+\sum_{(p,\nu)\in S}\frac{q_p}{b}\int_{R^n}\overline{\hat{f}(A_p^*\omega)}\hat{\psi}(\omega-\nu)(\sum_{s\neq 0}\hat{f}(A_p^*(\omega+B^\sharp s)\overline{\hat{\psi}(\omega+B^\sharp s-\nu)})d\omega.
\end{aligned}\tag{4.10}$$

For future reference, we introduce the following notations:

$$I(f)=\sum_{(p,\nu)\in S}\sum_{m\in Z^n}|\langle f,\,D_{A_p}E_\nu T_{Bm}\psi\rangle|^2, \tag{4.11}$$

$$I_1(f) = \sum_{(p,\nu)\in S} \frac{q_p}{b} \int_{R^n} |\hat{f}(A_p^*(\omega))\hat{\psi}(\omega-\nu)|^2 d\omega \tag{4.12}$$

and

$$\begin{aligned} &I_2(f) \\ &= \sum_{(p,\nu)\in S} \frac{q_p}{b} \int_{R^n} \overline{\hat{f}(A_p^*\omega)}\hat{\psi}(\omega-\nu)(\sum_{s\neq 0} \hat{f}(A_p^*(\omega+B^\sharp s)\overline{\hat{\psi}(\omega+B^\sharp s-\nu)})d\omega. \end{aligned} \tag{4.13}$$

Obviously, from (4.11)-(4.13), we have the following decomposition

$$I(f) = I_1(f) + I_2(f). \tag{4.14}$$

By $f \in \Gamma$ and the inequality

$$|\hat{\psi}(\omega-\nu)||\bar{\hat{\psi}}(\omega+B^\sharp s-\nu))| \leq \frac{1}{2}(|\hat{\psi}(\omega-\nu)|^2 + |\hat{\psi}(\omega+B^\sharp s-\nu))|^2), \tag{4.15}$$

we have

$$\sum_{(p,\nu)\in S} \int_{R^n} \overline{\hat{f}(A_p^*(\omega))}\hat{\psi}(\omega-\nu)(\sum_{s\neq 0} \hat{f}(A_p^*(\omega+B^\sharp s))\overline{\hat{\psi}(\omega+B^\sharp s-\nu)})d\omega < \infty. \tag{4.16}$$

Thus, we can change the orders of integration and summation in the expression (4.13). Using the equations (4.8) (4.9) and (4.10), we get

$$\begin{aligned} &\sum_{(p,\nu)\in S} \sum_{m\in Z^n} |\langle f,\, D_{A_p}E_\nu T_{Bm}\psi\rangle|^2 \\ &= \int_{R^n} \sum_{p\in P} \frac{q_p}{b} |\hat{f}(A_p^*\omega)|^2 (\sum_{\nu\in Q} |\hat{\psi}(\omega-\nu)|^2) d\omega \\ &\quad + \int_{R^n} \sum_{p\in P} \sum_{s\neq 0} \frac{q_p}{b} \overline{\hat{f}(A_p^*\omega)} \hat{f}(A_p^*(\omega+B^\sharp s))(\sum_{\nu\in Q} \hat{\psi}(\omega-\nu)\overline{\hat{\psi}(\omega+B^\sharp s-\nu)})d\omega \\ &= \int_{R^n} \frac{1}{b} |\hat{f}(\omega)|^2 (\sum_{p\in P} \sum_{\nu\in Q} |\hat{\psi}(A_p^\sharp\omega-\nu)|^2) d\omega \\ &\quad + \int_{R^n} \sum_{s\neq 0} \frac{1}{b} \overline{\hat{f}(\omega)} (\sum_{p\in P} \hat{f}(\omega+A_p^*B^\sharp s) \sum_{\nu\in Q} \hat{\psi}(A_p^\sharp\omega-\nu)\overline{\hat{\psi}(A_p^\sharp\omega+B^\sharp s-\nu)})d\omega \\ &= \int_{R^n} |\hat{f}(\omega)|^2 d\omega = \|f\|^2. \end{aligned} \tag{4.17}$$

So, it follows from Lemmas 2.1 and 2.2 that the wave packet system $\{D_{A_p}E_\nu T_{Bm}\psi(x)\}_{m\in Z^n,\,(p,\nu)\in S}$ defined by (2.3) is a Parseval frame. The proof of Theorem 4.1 is completed.

Remark 4.1. Let A be the elementary matrix E in Theorem 4.1. Then we obtain a sufficient condition of the Gabor Parseval frame as follows.

Corollary 4.1. *Let* $B, C \in GL_n(R)$. *Then, the Gabor system* $\{E_{Ck}T_{Bm}\psi(x)\}_{k,m\in Z^n}$ *is a Parseval frame for* $L^2(R^n)$ *if the two following equalities*

$$\sum_{k\in Z^n} |\hat{\psi}(\omega - Ck)|^2 = b, \quad a.e. \;\; \omega \in R^n$$

and

$$\sum_{k\in Z^n} \hat{\psi}(\omega - Ck)\overline{\hat{\psi}(\omega + B^\sharp s - Ck)} = 0, \quad a.e. \;\; \omega \in R^n$$

hold for every $s \in Z^n$ *and* $s \neq 0$, *where* $b = |detC|$.

Also, if $P = \{A^j : j \in Z, A \in GL_n(R)\}$ and $Q = \{0\}$ in Theorem 4.1, then we obtain the sufficient of the wavelet frame, that is

Corollary 4.2. *Let* A *be an arbitrary matrix and* $\psi \in L^2(R^n)$. *Then the wavelet system* $\{D_A^j T_{Bm}\psi(x)\}_{j\in Z, m\in Z^n}$ *is a Parseval wavelet frame if the two following equalities*

$$\sum_{j\in Z} |\hat{\psi}(A^j\omega)|^2 = b, \quad a.e. \;\; \omega \in R^n$$

and

$$\hat{\psi}(A^\sharp\omega)\overline{\hat{\psi}(A^{\sharp j}\omega + B^\sharp s)} = 0, \quad j \in N, \; a.e. \;\; \omega \in R^n$$

hold for every $s \in Z^n$ *and* $s \neq 0$, *where* $b = |detB|$.

Applying some techniques in [6, 7], we can also provide a necessary condition for the wave packet system to be a Parseval frame in $L^2(R^n)$.

Theorem 4.2. *If the wave packet system* $\{D_{A_p}E_\nu T_{Bm}\psi(x)\}_{m\in Z^n, (p,\nu)\in S}$ *defined by* (*2.3*) *is a Parseval frame for* $L^2(R^n)$, *then the equalities* (*4.8*) *and*

$$\sum_{(p,\nu)\in S} \hat{\psi}(A_p^\sharp\omega - \nu)\overline{\hat{\psi}(A_p^\sharp\omega + B^\sharp s - \nu) = 0}, \quad a.e. \;\; \omega \in R^n \tag{4.18}$$

hold for every $s \in Z^n$ and $s \neq 0$, where $b = |detB|$.

Proof. Now we assume that

$$\sum_{(p,\,\nu)\in S} \sum_{m\in Z^n} |\langle f, \, D_{A_p}E_\nu T_{Bm}\psi\rangle|^2 = \|f\|^2 \tag{4.19}$$

for all $f \in L^2(R^n)$. In particular, (4.19) holds for all functions $f \in \Gamma$.

It is obvious that for $f \in \Gamma$, $I(f) < \infty$ if and only if $I_1(f) < \infty$. Now taking $f = \chi_C$, where C is any compact subset in R^n, we see that $I_1(f) < \infty$ for all $f \in \Gamma$, if and only if $\tau(\omega) := \sum_{(p,\nu)\in S} |\hat{\psi}(A_p^\sharp \omega - \nu)|^2$ is locally integrable in R^n. Thus, by our assumption, the function $\tau(\omega)$ is locally integrable, consequently, almost every point of the function τ is a Lebesgue point. Choose $\omega_0 \in R$ to be a Lebesgue point of the function $\tau(\omega)$. Then we have

$$\sum_{(p,\nu)\in S} |\hat{\psi}(A_p^\sharp \omega_0 - \nu)|^2 = \lim_{\epsilon\longrightarrow 0} \int_{|\omega-\omega_0|\langle\epsilon} \frac{1}{|B(\epsilon)|} \sum_{(p,\nu)\in S} |\hat{\psi}(A_p^\sharp \omega - \nu)|^2 d\omega, \tag{4.20}$$

where $B(\epsilon)$ denotes the ball of radius $\epsilon > 0$ about the origin. For small enough ϵ, we define f_ϵ by

$$\hat{f}_\epsilon(\omega) = \frac{1}{\sqrt{|B(\epsilon)|}} \chi_{B(\epsilon)}(\omega - \omega_0). \tag{4.21}$$

Clearly,

$$\|f_\epsilon\|^2 = \|\hat{f}_\epsilon\|^2 = 1. \tag{4.22}$$

By the decomposition (4.14), we have

$$I(f_\epsilon) = I_1(f_\epsilon) + I_2(f_\epsilon). \tag{4.23}$$

Thus, by using (4.20)-(4.23) and the definition of I_1, we find

$$1 = \frac{1}{|B(\epsilon)|} \int_{B(\epsilon)} \tau(\omega - \omega_0) d\omega + I_2(f_\epsilon). \tag{4.24}$$

If we can show that $\lim_{\epsilon\longrightarrow 0} I_2(f_\epsilon) = 0$, then since ω_0 is a Lebesgue point of τ, we have

$$1 = \lim_{\epsilon\longrightarrow 0}\Big(\frac{1}{|B(\epsilon)|} \int_{B(\epsilon)} \tau(\omega - \omega_0) d\omega + I_2(f_\epsilon)\Big) = \tau(\omega_0) + \lim_{\epsilon\longrightarrow 0} I_2(f_\epsilon) = \tau(\omega_0). \tag{4.25}$$

That is to say, $\tau(\omega) = 1, \quad a.e. \ \ \omega \in R^n$, which completes the proof of (4.8).

Now, we devote to calculating $\lim_{\epsilon\longrightarrow 0} I_2(f_\epsilon)$.

$$\begin{aligned}
&|I_2(f_\epsilon)| \\
&\le \sum_{(p,\nu)\in S} \frac{q_p}{b} \int_{R^n} |\overline{\hat{f}_\epsilon(A_p^*(\omega))}||\hat{\psi}(\omega-\nu)| \sum_{s\neq 0} |\hat{f}_\epsilon(A_p^*(\omega + B^\sharp s)||\hat{\psi}(\omega + B^\sharp s - \nu))| d\omega \\
&= \int_{R^n} \sum_{p\in P} \sum_{s\neq 0} \frac{1}{b} |\hat{f}_\epsilon(\omega)||\hat{f}_\epsilon(\omega + A_p^* B^\sharp s)| \Big(\sum_{\nu\in Q} |\hat{\psi}(A_p^\sharp \omega - \nu)||\hat{\psi}(A_p^\sharp \omega + B^\sharp s - \nu))| d\omega \\
&\le \int_{R^n} \sum_{p\in P} \sum_{s\neq 0} \frac{1}{b} |\hat{f}_\epsilon(\omega) \hat{f}_\epsilon(\omega + A_p^* B^\sharp s)| \Big(\sum_{\nu\in Q} (|\hat{\psi}(A_p^\sharp \omega - \nu)|^2 + |\hat{\psi}(A_p^\sharp \omega + B^\sharp s - \nu))|^2\Big) d\omega \\
&= \int_{R^n} \sum_{p\in P} \sum_{s\neq 0} \sum_{\nu\in Q} \frac{1}{b} |\hat{f}_\epsilon(\omega)||\hat{f}_\epsilon(\omega + A_p^* B^\sharp s)||\hat{\psi}(A_p^\sharp \omega - \nu)|^2 d\omega \\
&\quad + \int_{R^n} \sum_{p\in P} \sum_{s\neq 0} \sum_{\nu\in Q} \frac{1}{b} |\hat{f}_\epsilon(\omega)||\hat{f}_\epsilon(\omega + A_p^* B^\sharp s)||\hat{\psi}(A_p^\sharp \omega + B^\sharp s - \nu))|^2 d\omega.
\end{aligned} \tag{4.26}$$

Since ϵ is small enough, for every $s \neq 0$, we have

$$|\hat{f}_\epsilon(\omega)||\hat{f}_\epsilon(\omega + A_p^* B^\sharp s)| = 0. \tag{4.27}$$

Thus, for small enough ϵ, $I_2(f_\epsilon) = 0$.

In the following, we turn to prove (4.18). According to (4.14) and (4.19), for all $f \in \Gamma$, we easily deduce $I_2(f) = 0$, i.e.

$$\int_{R^n} \sum_{p\in P} \sum_{s\neq 0} \sum_{\nu\in Q} \frac{1}{b} \overline{\hat{f}(\omega)} \hat{f}(\omega + A_p^* B^\sharp s)(\hat{\psi}(A_p^\sharp \omega - \nu)\overline{\hat{\psi}(A_p^\sharp \omega + B^\sharp s - \nu)})d\omega = 0. \tag{4.28}$$

By the polarization identity, for any $f, g \in \Gamma$, we get

$$\int_{R^n} \sum_{p\in P} \sum_{s\neq 0} \sum_{\nu\in Q} \frac{1}{b} \overline{\hat{f}(\omega)} \hat{g}(\omega + A_p^* B^\sharp s)(\hat{\psi}(A_p^\sharp \omega - \nu)\overline{\hat{\psi}(A_p^\sharp \omega + B^\sharp s - \nu)})d\omega = 0. \tag{4.29}$$

Similar to discussion of above function $\tau(\omega)$, the function

$$\mu(\omega) := \sum_{\nu\in Q} \hat{\psi}(A_p^\sharp \omega - \nu)\overline{\hat{\psi}(A_p^\sharp \omega + B^\sharp s - \nu)} \tag{4.30}$$

is locally integrable. Hence, almost every point of the function $\nu(\omega)$ is a lebesgue point in R^n. Fix $s_0 \neq 0$. Let ω_0 ($\omega_0 \neq 0$ and $\omega_0 + s_0 \neq 0$) be a Lebesgue point of $\mu(\omega)$, and f_ϵ be defined by (4.21) and h be defined as the following

$$\hat{h}_\epsilon(\omega) = \frac{1}{\sqrt{|B(\epsilon)|}} \chi_{B(\epsilon)}(\omega - \omega_0 - A_p^* B^\sharp s_0). \tag{4.31}$$

Then, we have

$$\hat{f}_\epsilon(\omega - \omega_0)\hat{h}_\epsilon(\omega - \omega_0 + A_p^* B^\sharp s_0) = \frac{1}{\sqrt{|B(\epsilon)|}} \chi_{B(\epsilon)}(\omega - \omega_0). \tag{4.32}$$

Let $f = f_\epsilon$ and $g = h_\epsilon$ in (4.29), then we get

$$\begin{aligned} 0 &= \int_{R^n} \sum_{p\in P} \sum_{s\neq 0} \sum_{\nu\in Q} \frac{1}{b} \overline{\hat{f}_\epsilon(\omega)} \hat{g}_\epsilon(\omega + A_p^* B^\sharp s)(\hat{\psi}(A_p^\sharp \omega - \nu)\overline{\hat{\psi}(A_p^\sharp \omega + B^\sharp s - \nu)})d\omega \\ &= \frac{1}{b\sqrt{|B(\epsilon)|}} \int_{B(\epsilon)} \sum_{p\in P} \sum_{\nu\in Q} \hat{\psi}(A_p^\sharp \omega - \nu - \omega_0)\overline{\hat{\psi}(A_p^\sharp \omega + B^\sharp s_0 - \nu - \omega_0)}d\omega \\ &+ \int_{R^n} \sum_{p\in P} \sum_{s\neq 0, s_0} \sum_{\nu\in Q} \frac{1}{b} \overline{\hat{f}_\epsilon(\omega)} \hat{g}_\epsilon(\omega + A_p^* B^\sharp s)(\hat{\psi}(A_p^\sharp \omega - \nu)\overline{\hat{\psi}(A_p^\sharp \omega + B^\sharp s - \nu)})d\omega. \end{aligned} \tag{4.33}$$

It is obvious that for small enough ϵ,

$$\int_{R^n} \sum_{p\in P} \sum_{s\neq 0, s_0} \sum_{\nu\in Q} \frac{1}{b} \overline{\hat{f}_\epsilon(\omega)\hat{g}_\epsilon(\omega + A_p^* B^\sharp s)}(\hat{\psi}(A_p^\sharp \omega - \nu)\overline{\hat{\psi}(A_p^\sharp \omega + B^\sharp s - \nu)})d\omega = 0. \tag{4.34}$$

Therefore, by the fact that ω_0 is a Lebesgue point of $\mu(\omega)$,

$$\begin{aligned}0 &= \lim_{\epsilon\to 0}\frac{1}{b\sqrt{|B(\epsilon)|}}\int_{B(\epsilon)}\sum_{p\in P}\sum_{\nu\in Q}\hat{\psi}(A_p^{\sharp}\omega-\nu-\omega_0)\overline{\hat{\psi}(A_p^{\sharp}\omega+B^{\sharp}s_0-\nu-\omega_0)}d\omega\\ &= \frac{1}{b}\sum_{p\in P}\sum_{\nu\in Q}\hat{\psi}(A_p^{\sharp}\omega-\nu)\overline{\hat{\psi}(A_p^{\sharp}\omega+B^{\sharp}s_0-\nu)}d\omega,\end{aligned} \tag{4.35}$$

for all s_0 and $a.e.$ $\omega\in R^n$. So, we get (4.18). The proof of Theorem 4.2 is ended.

Remark 4.2. If A is the elementary matrix E in Theorem 4.2, then, we obtain the necessary condition of the Gabor frames as follows.

Corollary 4.3. *Let $B, C\in GL_n(R)$. If the Gabor system $\{E_{Ck}T_{Bm}\psi(x)\}_{k,m\in Z^n}$ is a Gabor Parseval frame for $L^2(R^n)$, then the two following equalities*

$$\sum_{k\in Z^n}|\hat{\psi}(\omega-Ck)|^2=b,\quad a.e.\ \ \omega\in R^n$$

and

$$\sum_{k\in Z^n}\hat{\psi}(\omega-Ck)\bar{\hat{\psi}}(\omega+B^{\sharp}s-Ck)=0,\quad a.e.\ \ \omega\in R^n$$

hold for every $s\in Z^n$ and $s\neq 0$, where $b=|detB|$.

Also, if $P=\{A^j : j\in Z, A\in GL_n(R)\}$ and $Q=\{0\}$ in Theorem 4.2, then we obtain the following necessary condition of the wavelet frames.

Corollary 4.4. *Let A be an arbitrary matrix, $B\in GL_n(R)$ and $\psi\in L^2(R^n)$. If the wavelet system $\{D_A^jT_{Bm}\psi(x)\}_{j\in Z,m\in Z^n}$ is a Parseval wavelet frame, then the two following equalities*

$$\sum_{j\in Z}|\hat{\psi}(A^j\omega)|^2=b,\quad a.e.\ \ \omega\in R^n$$

and

$$\sum_{j\in Z}\hat{\psi}(A^{\sharp}\omega)\overline{\hat{\psi}(A^{\sharp j}\omega+B^{\sharp}s)}=0,\quad a.e.\ \ \omega\in R^n$$

hold for every $s\in Z^n$ and $s\neq 0$, where $b=|detB|$.

Remark 4.3. Comparing with corollary 4.1 and corollary 4.3, we obtain the following characterization of the Gabor Parseval frame, which is the case of single generator of theorem 3.1 in [13].

Corollary 4.5. *Let $B, C\in GL_n(R)$. Then, the Gabor system $\{E_{Ck}T_{Bm}\psi(x)\}_{k,m\in Z^n}$ is a Gabor Parseval frame for $L^2(R^n)$ if and only if the two following equalities*

$$\sum_{k\in Z^n}|\hat{\psi}(\omega-Ck)|^2=b,\quad a.e.\ \ \omega\in R^n$$

and

$$\sum_{k\in Z^n} \hat{\psi}(\omega - Ck)\bar{\hat{\psi}}(\omega + B^\sharp s - Ck) = 0, \quad a.e. \;\; \omega \in R^n$$

hold for every $s \in Z^n$ *and* $s \neq 0$*, where* $b = |detC|$.

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