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ON THE INTEGERS OF THE FORM $p + b$

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Abstract. Let B be a subset of positive integers, and $\mathcal{P}$ the set of all positive primes. For a subset A of positive integers, $A(x)$ denotes the number of integers in A not exceeding x. Let $\mathcal{S}$ denote the set of integers of the form $p + b$ with $p \in \mathcal{P}$ and $b \in B$. In this paper, we prove that if $B(x) \gg \log x/ \log\log x$ and $B(cx) \gg B(x)$ for some positive constant $c < 1$, then $\mathcal{S}(x) \gg x/ \log\log x$. This result is best possible in a sense: For any positive integer m, we construct an explicit subset B of positive integers with $B(x) \gg (\log x)^m$ and $B(cx) \gg B(x)$ for any positive constant $c < 1$ such that $\mathcal{S}(x) \ll x/ \log\log x$. We also give an application to the integers of the form $p + 2^{a^2} + 2^{b^2}$, where $p \in \mathcal{P}$ and a, b are integers. Two open problems are posed for further research.

1. INTRODUCTION

Let $\mathbb{N}$ denote the set of all nonnegative integers and $\mathcal{P}$ denote the set of all positive primes. In 1849, Polignac [18] conjectured that every odd number greater than 3 can be represented as the sum of an odd prime and a power of 2. He found a counterexample soon. In 1934, Romanoff [19] proved that the set

$$\{p + 2^a : p \in \mathcal{P}, a \in \mathbb{N}\}$$

has a positive lower density. In 1950, van der Corput [7] proved that there are a positive proportion of positive odd integers not of the form $p + 2^a$ with $p \in \mathcal{P}$ and $a \in \mathbb{N}$. In the same year, using covering congruences, Erdős [10] constructed an arithmetic progression consisting only of odd numbers, no term of which is of the form $p + 2^a$. In recent years, developing the idea of Erdős, many authors study on this subject. One can refer to [1-6, 9, 11, 14, 20-26].

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In [8], Crocker proved that there exist infinitely many odd positive integers x not of the form $p + 2^a + 2^b$. In 2011, Pan [16] proved that

$$\sharp\{n \in [1, x] : n \text{ is odd and not of the form } p^\alpha + 2^a + 2^b\} \gg_\epsilon x^{1-\epsilon}$$

for any $\epsilon > 0$, where $\gg_\epsilon$ means the implied constant only depends on ϵ.

Recently, Pan and Zhang [17] proved the sets $\{p^2 + b^2 + 2^n : p \in \mathcal{P}, b, n \in \mathbb{N}\}$ and $\{b_1^2 + b_2^2 + 2^{n^2} : b_1, b_2, n \in \mathbb{N}\}$ have positive lower densities. Conversely, they also proved that there exists a residue class with an odd modulus that contains no integer of each form.

Throughout this paper, Vinogradov's notation $f(x) \ll g(x)$ (or $g(x) \gg f(x)$) means $f(x) = O(g(x))$. For a subset A of $\mathbb{N}$, $A(x)$ denotes the number of integers in A not exceeding x. Let $\pi(x) = \mathcal{P}(x)$.

A subset B of $\mathbb{N}$ is said to satisfy *c-condition* if $B(cx) \gg B(x)$ for some positive constant $c < 1$.

In this paper, we shall study the sumset

$$\mathcal{S} = \{p + b : \quad p \in \mathcal{P}, b \in B\},$$

where B is a subset of $\mathbb{N}$ with c-condition.

The following theorems are proved.

Theorem 1. *For any subset B of positive integers with c-condition, we have*

$$\frac{x}{\log x} \min\left\{B(x), \frac{\log x}{\log\log x}\right\} \ll \mathcal{S}(x) \ll \frac{x}{\log x} \min\{B(x), \log x\}.$$

From Theorem 1, we have the following corollaries immediately.

Corollary 1. *If $B(x) \ll \log x/\log\log x$ and B satisfies c-condition, then*

$$\frac{x}{\log x} B(x) \ll \mathcal{S}(x) \ll \frac{x}{\log x} B(x).$$

Corollary 2. *Let $Q = \{n : n = p + 2^q,\ p, q \in \mathcal{P}\}$. Then*

$$\frac{x}{\log\log x} \ll Q(x) \ll \frac{x}{\log\log x}.$$

Remark 1. Theorem 1.13 in [5] is a quantitative version of Corollary 2.

Corollary 3. *If $B(x) \gg \log x/\log\log x$ and B satisfies c-condition, then*

$$\mathcal{S}(x) \gg \frac{x}{\log\log x}.$$

Let $W = \{2^{a^2} + 2^{b^2} : a, b \in \mathbb{N}\}$. Since

$$W(x/2) \geq \frac{1}{2}|\{2^{a^2} \leq x/4 : a \in \mathbb{N}\}|^2 \gg \log x \gg |\{2^{a^2} \leq x : a \in \mathbb{N}\}|^2 \geq W(x),$$

it follows that W satisfies c-condition and $\log x \ll W(x) \ll \log x$. By Corollary 3 we have the following corollary.

Corollary 4. *Let* $V = \{n : n = p + 2^{a^2} + 2^{b^2},\ p \in \mathcal{P}, a, b \in \mathbb{N}\}$. *Then*

$$V(x) \gg \frac{x}{\log\log x}.$$

The next theorem shows that the lower bounds in Theorem 1 and Corollary 3 are best possible in a sense.

Theorem 2. *For any positive integers* m, *there exists a subset* B *of* $\mathbb{N}$ *such that*

$$B(x) = \frac{1 + o(1)}{m + 1} \left(\frac{\log x}{\log\log x} \right)^{m+1} \tag{1}$$

and

$$\frac{x}{\log\log x} \ll \mathcal{S}(x) \ll \frac{x}{\log\log x}.$$

Remark 2. By (1) we know that the set B in Theorem 2 satisfies c-condition.

Now we pose two problems for further research.

Problem 1. *Does there exist a real number* $\alpha > 0$ *and a subset* B *of* $\mathbb{N}$ *with* c*-condition such that* $B(x) \gg x^{\alpha}$ *and* $\mathcal{S}(x) \ll x/\log\log x$*?*

Problem 2. *Does there exist a positive integer* k *such that the set of positive integers which can be represented as* $p + \sum_{i=1}^{k} 2^{a_i^2}$ *with* $p \in \mathcal{P}$ *and* $a_i \in \mathbb{N}$ *has the positive lower density? If such* k *exists, what is the minimal value of such* k*?*

2. Proofs

In this section, p always denotes a prime.

Lemma 1. (see [15, Theorem 7.3].) *Let* N *be a positive even integer, and let* $\pi_N(x)$ *denote the number of primes* p *up to* x *such that* $p + N$ *is also prime. Then*

$$\pi_N(x) \ll \frac{x}{(\log x)^2} \prod_{p|N} \left(1 + \frac{1}{p}\right).$$

Remark 3. If N is a positive odd integer, then $\pi_N(x) \leq 1$.

Lemma 2. *Let $\Phi(x, y)$ denote the number of positive integers $n < x$ that are not divisible by any prime $p < y$. Then*

$$\Phi(x, y) \le x \prod_{p<y} \left(1 - \frac{1}{p}\right) + 2^y \ll \frac{x}{\log y} + 2^y.$$

Lemma 2 follows from (5.4), (5.5) and (5.7) in [12, Chapter 1].

Proof of Theorem 1. The number of pairs (p, b) with $p \le x,\ p \in \mathcal{P}$ and $b \le x,\ b \in B$ is $\pi(x)B(x)$. So the upper bound is clear.

Now we shall prove

$$\mathcal{S}(x) \gg \frac{x}{\log x} \min\left\{B(x), \frac{\log x}{\log\log x}\right\}.$$

Let $r(N)$ denote the number of solutions of the equation $N = p + b$, where $p \in \mathcal{P}$ and $b \in B$.

First we estimate the upper bound of $\sum_{N\le x} r(N)^2$.

Since $r(N)^2$ is the number of quadruples (p_1, b_1, p_2, b_2) such that

$$p_1 + b_1 = p_2 + b_2 = N, \quad p_1, p_2 \in \mathcal{P},\ b_1, b_2 \in B,$$

it follows that

$$\sum_{N\le x} r(N)^2 = \sharp\{(p_1, b_1, p_2, b_2) : p_1 + b_1 = p_2 + b_2 \le x,\ p_1, p_2 \in \mathcal{P}, b_1, b_2 \in B\}.$$

This value does not exceed the number of solutions of the equation

$$p_2 - p_1 = b_1 - b_2, \quad p_1, p_2 \in \mathcal{P}, b_1, b_2 \in B \tag{2}$$

with $p_1, p_2, b_1, b_2 \le x$.

If $b_1 = b_2$, then $p_1 = p_2$. Hence, the number of solutions of (2) in this case is at most

$$\pi(x)B(x) \ll \frac{x}{\log x} B(x).$$

Now, fix b_1 and b_2 such that $b_1 - b_2 \ne 0$. By Lemma 1 and Remark 3, we have

$$\sharp\{(p_1, p_2) \in \mathcal{P} \times \mathcal{P} : p_2 - p_1 = b_1 - b_2, p_1, p_2 \le x\} \ll \frac{x}{(\log x)^2} \prod_{p \mid b_1 - b_2} \left(1 + \frac{1}{p}\right).$$

For any positive integer h, by $\phi(n) \gg n/\log\log n$ (see [13, Theorem 328]), we have

$$\prod_{p\mid h} \left(1 + \frac{1}{p}\right) \le \prod_{p\mid h} \left(1 - \frac{1}{p}\right)^{-1} = \frac{h}{\phi(h)} \ll \log\log h.$$

Hence

$$\begin{aligned}
&\sum_{N\le x} r(N)^2\\
&\ll \frac{x}{\log x}B(x) + \frac{x}{(\log x)^2}\sum_{\substack{b_2<b_1\le x\\ b_1,b_2\in B}}\ \prod_{p\mid b_1-b_2}\left(1+\frac{1}{p}\right)\\
&\ll \frac{x}{\log x}B(x) + \frac{x}{(\log x)^2}B(x)^2\log\log x\\
&\ll \frac{x\log\log x}{(\log x)^2}B(x)\cdot\max\left\{\frac{\log x}{\log\log x}, B(x)\right\}.
\end{aligned}$$

Next we estimate the lower bound of $\sum_{N\le x} r(N)$.
Since B satisfies c-condition, it follows that

$$\begin{aligned}
\sum_{N\le x} r(N) &= \sharp\{(p,b) : p+b\le x, p\in\mathcal{P}, b\in B\}\\
&\ge \sharp\{p\in\mathcal{P} : p\le(1-c)x\}\cdot\sharp\{b\in B : b\le cx\}\\
&\gg \frac{x}{\log x}B(x).
\end{aligned}$$

Therefore, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\frac{x^2}{(\log x)^2}B(x)^2 &\ll \left(\sum_{N\le x} r(N)\right)^2 \le \mathcal{S}(x)\sum_{N\le x} r(N)^2\\
&\le \mathcal{S}(x)\frac{x\log\log x}{(\log x)^2}B(x)\max\left\{\frac{\log x}{\log\log x}, B(x)\right\}.
\end{aligned}$$

Hence

$$\mathcal{S}(x) \gg \frac{x}{\log x}\min\left\{B(x), \frac{\log x}{\log\log x}\right\}.$$

This completes the proof of Theorem 1. ∎

Proof of Theorem 2. Let

$$B = \bigcup_{j=1}^{\infty}\{kp_1p_2\cdots p_j :\ 1\le k\le j^m,\ p_{j+1}\nmid k\},$$

where p_j is the jth prime.

For any real number $x\ge p_1p_2\cdots p_{m+1}$, there exists a positive integer $t\ge m+1$ such that

$$p_1p_2\cdots p_t \le x < p_1p_2\cdots p_{t+1}. \tag{3}$$

For $i = 1, 2, \ldots, t-m$, we have

$$i^m \cdot p_1 p_2 \cdots p_i \le (t-m)^m \cdot p_1 p_2 \cdots p_{t-m} \le p_1 p_2 \cdots p_t \le x.$$

Hence,

$$C := \bigcup_{j=1}^{t-m} \{k p_1 p_2 \cdots p_j : \ 1 \le k \le j^m, \ p_{j+1} \nmid k\} \subseteq B \cap [1, x].$$

Clearly, we have

$$\big|\big(B \cap [1,x]\big) \setminus C\big| \le (t-m+1)^m + (t-m+2)^m + \cdots + t^m \ll t^m.$$

Now we estimate the cardinality of C. Noting that the set C is the union of disjoint sets, we have

$$|C| = \sum_{j=1}^{t-m} \left(j^m - \left\lfloor \frac{j^m}{p_{j+1}} \right\rfloor \right).$$

Since $p_{j+1} \ge j$ and

$$\frac{(t-m)^{m+1}}{m+1} = \int_0^{t-m} x^m dx \le \sum_{j=1}^{t-m} j^m \le \int_1^{t-m+1} x^m dx < \frac{(t-m+1)^{m+1}}{m+1},$$

we have

$$|C| = \sum_{j=1}^{t-m} j^m + O(\sum_{j=1}^{t-m} j^{m-1}) = \frac{1+o(1)}{m+1}(t-m)^{m+1} = \frac{1+o(1)}{m+1} t^{m+1}.$$

Thus

$$B(x) = \big|\big(B \cap [1,x]\big) \setminus C\big| + |C| = \frac{1+o(1)}{m+1} t^{m+1}.$$

By [13, Theorems 6 and 420] and [13, Theorem 8], we have

$$\sum_{p \le x} \log p = (1+o(1))x, \quad p_n = (1+o(1)) n \log n. \tag{4}$$

Hence, by (3) and (4), we have

$$\log x \ge \sum_{p \le p_t} \log p = (1+o(1)) p_t = (1+o(1)) t \log t$$

and

$$\log x < \sum_{p \le p_{t+1}} \log p = (1+o(1)) p_{t+1} = (1+o(1)) t \log t.$$

It follows that

$$t=\frac{(1+o(1))\log x}{\log\log x}.$$

Hence, we have

$$B(x)=\frac{1+o(1)}{m+1}\left(\frac{\log x}{\log\log x}\right)^{m+1}.$$

It is clear that B satisfies c-condition. By Theorem 1, we have $\mathcal{S}(x)\gg x/\log\log x$. Next we prove that

$$\mathcal{S}(x)\ll\frac{x}{\log\log x}.$$

For any integer h with $1\le h\le t$, let

$$B_h=\bigcup_{j=1}^{h}\{kp_1p_2\cdots p_j:\ 1\le k\le j^m,\ p_{j+1}\nmid k\},$$

$$\mathcal{S}_1(x)=\sharp\{n\le x:\ n=p+b,\ p\in\mathcal{P},\ b\in B_h\}$$

and

$$\mathcal{S}_2(x)=\sharp\{n\le x:\ n=p+b,\ p\in\mathcal{P},\ b\in B\setminus B_h\}.$$

Clearly, we have $\mathcal{S}(x)\le\mathcal{S}_1(x)+\mathcal{S}_2(x)$ and

$$\mathcal{S}_1(x)\le\pi(x)|B_h|\le\pi(x)\cdot\sum_{j=1}^{h}j^m\ll\frac{x}{\log x}h^{m+1}.$$

Suppose that $n=p+b$ with $p\in\mathcal{P}$ and $b\in B\setminus B_h$. If $(n,p_1p_2\cdots p_h)>1$, then, by $n=p+b$ and $p_1p_2\cdots p_h\mid b$ for any $b\in B\setminus B_h$, we have $p=p_i$ for some i with $1\le i\le h$. By Lemma 2, we have

$$\begin{aligned}\mathcal{S}_2(x)&\le\sharp\{n\le x:\ n=p+b,(n,p_1p_2\cdots p_h)>1,\ p\in\mathcal{P},\ b\in B\setminus B_h\}\\&\quad+\sharp\{n\le x:\ n=p+b,(n,p_1p_2\cdots p_h)=1,\ p\in\mathcal{P},\ b\in B\setminus B_h\}\\&\le\sharp\{n\le x:n=p+b,p\in\{p_1,p_2,\dots,p_h\},\ b\in B,b\le x\}\\&\quad+\sharp\{n\le x:(n,p_1p_2\cdots p_h)=1\}\\&\le hB(x)+\frac{x}{\log p_{h+1}}+2^{p_{h+1}}\\&\ll h(\log x)^{m+1}+\frac{x}{\log h}+2^{p_{h+1}}.\end{aligned}$$

Thus,

$$\mathcal{S}(x)\le\mathcal{S}_1(x)+\mathcal{S}_2(x)\ll\frac{x}{\log x}h^{m+1}+\frac{x}{\log h}+2^{p_{h+1}}.$$

Taking

$$h=\left(\frac{\log x}{\log\log x}\right)^{\frac{1}{m+1}},$$

we obtain

$$\mathcal{S}(x) \ll \frac{x}{\log\log x}.$$

This completes the proof of Theorem 2. ■

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REFERENCES

1. Y.-G. Chen, On integers of the forms $k^r - 2^n$ and $k^r 2^n + 1$, *J. Number Theory*, **98** (2003), 310-319.
2. Y.-G. Chen, Five consecutive positive odd numbers, none of which can be expressed as a sum of two prime powers, *Math. Comp.*, **74** (2005), 1025-1031.
3. Y.-G. Chen, On integers of the forms $k \pm 2^n$ and $k2^n \pm 1$, *J. Number Theory*, **125** (2007), 14-25.
4. Y.-G. Chen, R. Feng and N. Templier, Fermat numbers and integers of the form $a^k + a^l + p^\alpha$, *Acta Arith.*, **135** (2008), 51-61.
5. Y.-G. Chen, Romanoff theorem in a sparse set, *Sci. China Math.*, **53** (2010), 2195-2202.
6. F. Cohen and J. L. Selfridge, Not every number is the sum or difference of two prime powers, *Math. Comp.*, **29** (1975), 79-81.
7. J. G. van der Corput, On de Polignac's conjecture, *Simon Stevin*, **27** (1950), 99-105.
8. R. Crocker, On the sum of a prime and two powers of two, *Pacific J. Math.*, **36** (1971), 103-107.
9. R. Crocker, On the sum of two squares and two powers of k, *Colloq. Math.*, **112** (2008), 235-267.
10. P. Erdős, On integers of the form $2^k + p$ and some related problems, *Summa Brasil. Math.*, **2** (1950), 113-123.
11. M. Filaseta, C. Finch and M. Kozek, On powers associated with Sierpiński numbers, Riesel numbers and Polignac's conjecture, *J. Number Theory*, **128** (2008), 1916-1940.
12. H. Halberstam and H.-E. Richert, *Sieve Methods*, Academic Press, New York, 1974.
13. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 5th ed., Oxford Univ. Press, 1979.
14. F. Luca and P. Stănică, Fibonacci numbers that are not sums of two prime powers, *Proc. Amer. Math. Soc.*, **133** (2005), 1887-1890.
15. M. B. Nathanson, *Additive Number Theory: The Classical Bases, Grad. Texts Math.*, 164, Springer-Verlag, New York, 1996.

16. H. Pan, On the integers not of the form $p + 2^a + 2^b$, *Acta Arith.*, **148** (2011), 55-61.

17. H. Pan and W. Zhang, On the integers of the form $p^2 + b^2 + 2^n$ and $b_1^2 + b_2^2 + 2^{n^2}$, *Math. Comp.*, **80** (2011), 1849-1864.

18. A. de Polignac, Six propositions arithmologiques déduites du crible d'Eratosthéne, *Nouv. Ann. Math.*, **8** (1849), 423-429.

19. N. P. Romanoff, Über einige Sätze der additiven Zahlentheorie, *Math. Ann.*, **109** (1934), 668-678.

20. X.-G. Sun and J.-H. Fang, On the density of integers of the form $(p - 1)2^{-n}$ in arithmetic progressions, *Bull. Aust. Math. Soc.*, **78** (2008), 431-436.

21. Z.-W. Sun, On integers not of the form $\pm p^a \pm q^b$, *Proc. Amer. Math. Soc.*, **128** (2000), 997-1002.

22. Z.-W. Sun and M.-H. Le, Integers not of the form $c(2^a + 2^b) + p^\alpha$, *Acta Arith.*, **99** (2001), 183-190.

23. Z.-W. Sun and S.-M. Yang, A note on integers of the form $2^n + cp$, *Proc. Edinb. Math. Soc.*, **45** (2002), 155-160.

24. K.-J. Wu and Z.-W. Sun, Covers of the integers with odd moduli and their applications to the forms $x^m - 2^n$ and $x^2 - F_{3n}/2$, *Math. Comp.*, **78** (2009), 1853-1866.

25. T. Tao, A remark on primality testing and decimal expansions, *J. Aust. Math. Soc.*, **91** (2011), 405-413.

26. P.-Z. Yuan, Integers not of the form $c(2^a + 2^b) + p^\alpha$, *Acta Arith.*, **115** (2004), 23-28.

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