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INEQUALITIES FOR MIXED COMPLEX PROJECTION BODIES

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Abstract. Complex projection bodies were introduced by Abardia and Bernig, recently. In this paper some geometric inequalities for mixed complex projection bodies which are analogs of inequalities for mixed real projection bodies are established.

1. Introduction

Let $\mathcal{K}^n$ denote the space of non-empty compact convex bodies in $\mathbb{R}^n$ with the Hausdorff topology. The projection body of $K \in \mathcal{K}^n$ is the convex body ΠK whose support function is defined by

$$h(\Pi K, u) = \frac{1}{2}\int_{S^{n-1}} |u \cdot v| dS(K, v), \tag{1.1}$$

where $S(K, \cdot)$ is the surface area measure of K.

Projection bodies have been widely studied since their introduction by Minkowski at the end of 19th century. They are objects of independent investigation in a number of mathematical disciplines such as geometric tomography, stereology, combinatorics, computational and stochastic geometry (see [3,5,7,9,18,20,21]). They have attracted increased interest in recent years (see [13,17,23]).

Mixed projection bodies were introduced in the classic volume of Bonnesen- Fenchel [4]. They are related to ordinary projection bodies in the same way that mixed volumes are related to ordinary volume. For $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$ and $u \in S^{n-1}$, the mixed projection body $\Pi(K_1, \ldots, K_{n-1})$ is defined by

$$h(\Pi(K_1, \ldots, K_{n-1}), u) = \frac{1}{2}\int_{S^{n-1}} |u \cdot v| dS(K_1, \ldots, K_{n-1}, v), \tag{1.2}$$

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where $S(K_1, \ldots, K_{n-1}, \cdot)$ is the mixed surface area measure of $K_1, \ldots, K_{n-1}$.

In [17] Lutwak considered the volume of mixed projection bodies and established analogs of the classical mixed volume inequalities, such as the Minkowski and Brunn-Minkowski inequalities.

Theorem A. [17]. *Let K be a convex body in $\mathbb{R}^n$. If $0 \leq i < j < n-1$, and $0 \leq k < n$, then*

$$W_k(\Pi_j K)^{n-i-1} \geq \omega_{n-1}^{(n-k)(j-i)} \omega_n^{j-i} W_k(\Pi_i K)^{n-j-1},$$

with equality if and only if K is a ball, where ω_n denotes the volume of the Euclidean unit ball B in $\mathbb{R}^n$ and $W_k(K)$ denotes the k-th Quermassintegral of K.

Theorem B. [17]. *Let K and L be convex bodies in $\mathbb{R}^n$. If $0 \leq i < n$, and $0 \leq j < n-2$, then*

$$W_i(\Pi_j(K+L))^{\overline{\frac{1}{(n-i)(n-j-1)}}} \geq W_i(\Pi_j K)^{\overline{\frac{1}{(n-i)(n-j-1)}}} + W_i(\Pi_j L)^{\overline{\frac{1}{(n-i)(n-j-1)}}},$$

with equality if and only if K and L are homothetic.

The theory of real convex bodies goes back to ancient times and continues to be a very active field now. Until recently the situation with complex convex bodies began to attract attention (see [1,2,8,10-12,19,24,25]). Some classical concepts of convex geometry in real vector space were extended to complex cases, such as complex intersection bodies [11], complex projection bodies [2] and complex difference bodies [1].

The real vector space $\mathbb{R}^n$ of real dimension n is replaced by a complex vector space $\mathbb{C}^n$ of dimension n. We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ using the standard mapping

$$(1.3) \qquad \xi = (\xi_1, \ldots, \xi_n) = (\xi_{11} + i\xi_{12}, \ldots, \xi_{n1} + i\xi_{n2}) \mapsto (\xi_{11}, \xi_{12}, \ldots, \xi_{n1}, \xi_{n2}).$$

The unit ball B in $\mathbb{C}^n$ is given by

$$B = \{\xi \in \mathbb{C}^n : \sum_{i=1}^{n} (\xi_{i1}^2 + \xi_{i2}^2) \leq 1\}.$$

The volume of the unit ball B in $\mathbb{C}^n$ is denoted by ω_{2n}.

Let $K_1, \ldots, K_{2n-1}$ be convex bodies in $\mathbb{C}^n$ and $C \subset \mathbb{C}$ be a convex subset. The mixed complex projection body $\Pi^C(K_1, \ldots, K_{2n-1})$ is the convex body whose support function is defined by[2]

$$(1.4) \quad h(\Pi^C(K_1, \ldots, K_{2n-1}), w) = \frac{1}{2n} \int_{S^{2n-1}} h(C \cdot w, \xi) dS(K_1, \ldots, K_{2n-1}, \xi),$$

where $C \cdot w := \{cw |\ c \in C\}, w \in \mathbb{C}^n$.

If $C = \{c\}(c \in \mathbb{C})$ is just a point, then $\Pi^C(K_1, \ldots, K_{2n-1}) = \{0\}$. Indeed, for every $w \in \mathbb{C}^n$,

$$\begin{aligned} h(\Pi^C(K_1, \ldots, K_{2n-1}), w) &= \frac{1}{2n} \int_{S^{2n-1}} h(cw, \xi) dS(K_1, \ldots, K_{2n-1}, \xi) \\ &= \frac{1}{2n} \int_{S^{2n-1}} cw \cdot \xi dS(K_1, \ldots, K_{2n-1}, \xi), \\ &= \frac{1}{2n} cw \cdot \int_{S^{2n-1}} \xi dS(K_1, \ldots, K_{2n-1}, \xi) \\ &= 0, \end{aligned}$$

since the centroid of the mixed surface area measure is the origin (see [6]). Thus, $\Pi^C(K_1, \ldots, K_{2n-1}) = \{0\}$.

If $K_1 = \cdots = K_{2n-i-1} = K$ and $M = (K_{2n-i}, \ldots, K_{2n-1})$, then the mixed projection body $\Pi^C(K, \ldots, K, K_{2n-i}, \ldots, K_{2n-1})$ is written as $\Pi_i^C(K, M)$. In particular, we write $\Pi_i^C(K, L)$ for the mixed complex projection body $\Pi^C(K, \ldots, K, L, \ldots, L)$ with i copies of L and $2n - i - 1$ copies of K. For the mixed complex projection body $\Pi_i^C(K, B)$ we simply write $\Pi_i^C K$.

Based on the standard proof of geometric inequalities which was mainly developed by Lutwak [14,15,17] and was successfully used by Schuster in [23], we establish analogs of the classical inequalities from the Brunn-Minkowski Theory (such as the Minkowski and Brunn-Minkowski inequalities) for mixed complex projection bodies.

Theorem 1.1. *Let K be a convex body in $\mathbb{C}^n$ and $C \subset \mathbb{C}$ be a convex subset which is not a point. If $0 \le i < j < 2n - 1$, while $0 \le k < 2n$, then*

$$W_k(\Pi_j^C K)^{2n-i-1} \ge r_C^{(2n-k)(j-i)} \omega_{2n}^{j-i} W_k(\Pi_i^C K)^{2n-j-1}, \tag{1.5}$$

with equality if and only if K is a ball, where r_C is given by $\Pi^C B = r_C B$.

Theorem 1.2. *Let K and L be convex bodies in $\mathbb{C}^n$ and $C \subset \mathbb{C}$ be a convex subset which is not a point. If $0 \le i < 2n$, while $0 \le j < 2n - 2$, then*

$$\begin{aligned} &W_i(\Pi_j^C (K + L))^{\frac{1}{(2n-i)(2n-j-1)}} \\ &\ge W_i(\Pi_j^C K)^{\frac{1}{(2n-i)(2n-j-1)}} + W_i(\Pi_j^C L)^{\frac{1}{(2n-i)(2n-j-1)}}, \end{aligned} \tag{1.6}$$

with equality if and only if K and L are homothetic.

2. Notation and Background Material

In this section some notation and basic facts about convex bodies are presented. For general reference the reader may wish to consult the books of Gardner [7] and Schneider [21].

A compact, convex set $K \in \mathcal{K}^n$ is uniquely determined by its support function $h(K, \cdot)$ on $\mathbb{R}^n$, defined by

$$h(K,u) = \max\{x \cdot u : \ x \in K\}, \ u \in \mathbb{R}^n. \tag{2.1}$$

Let $GL(n)$ denote the group of general linear transformations in $\mathbb{R}^n$. If $\phi \in GL(n)$ and $K \in \mathcal{K}^n$, then for every $u \in \mathbb{R}^n$,

$$h(\phi K, u) = h(K, \phi^t u), \tag{2.2}$$

where ϕ^t is the transpose of ϕ.

For $K_1, K_2 \in \mathcal{K}^n$ and $\lambda_1, \lambda_2 \geq 0$, the Minkowski sum $\lambda_1 K_1 + \lambda_2 K_2$ is the convex body defined by

$$h(\lambda_1 K_1 + \lambda_2 K_2, \cdot) = \lambda_1 h(K_1, \cdot) + \lambda_2 h(K_2, \cdot). \tag{2.3}$$

If $K_i \in \mathcal{K}^n (i = 1, 2, \ldots, m)$ and $\lambda_i (i = 1, 2, \ldots, m)$ are non-negative real numbers, then the volume of $\lambda_1 K_1 + \cdots + \lambda_m K_m$ is a homogeneous polynomial of degree n in λ_i given by

$$V(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1, \ldots, i_n} V(K_{i_1}, \ldots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n},$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ of positive integers not exceeding m. The coefficient $V(K_{i_1}, \ldots, K_{i_n})$ is called the mixed volume of $K_{i_1}, \ldots, K_{i_n}$. It is nonnegative, symmetric in its arguments and monotone (with respect to set inclusion in each component). In particular, $V(K, \ldots, K) = V(K)$. Let $K_1 = \cdots = K_{n-i} = K$ and $K_{n-i+1} = \cdots = K_n = L$, then the mixed volume $V(K_1, \ldots, K_n)$ is usually written as $V_i(K, L)$. We write B for the Euclidean unit ball in $\mathbb{R}^n$. If $L = B$, $V_i(K, B)$ is the i-th Quermassintegral of K and is written as $W_i(K)$. For $0 \leq i \leq n$, we write $W_i(K, L)$ for the mixed volume $V(\underbrace{K, \ldots, K}_{n-i-1}, \underbrace{B, \ldots, B}_{i}, L)$. If $M = (K_{n-i}, \ldots, K_{n-1})$, we write $V(K, n-i-1; M; L)$ for the mixed volume $V(\underbrace{K, \ldots, K}_{n-i-1}, K_{n-i}, \ldots, K_{n-1}, L)$.

The mixed volume $V(K_1, \ldots, K_n)$ has the following integral representation[17]:

$$V(K_1, \ldots, K_n) = \frac{1}{n} \int_{S^{n-1}} h(K_n, u) dS(K_1, \cdots, K_{n-1}, u). \tag{2.4}$$

One of the most general and fundamental inequalities for mixed volumes is the Aleksandrov-Fenchel inequality[17]: If $K_1, \ldots, K_n \in \mathcal{K}^n$ and $1 \leq m \leq n$, then

$$V(K_1, \ldots, K_n)^m \geq \prod_{j=1}^{m} V(\underbrace{K_j, \ldots, K_j}_{m}, K_{m+1}, \ldots, K_n), \tag{2.5}$$

Unfortunately, the equality conditions of this inequality are, in general, unknown.

An important special case of inequality (2.5), where the equality conditions are known, is the classical inequality between the Quermassintegrals: If $K \in \mathcal{K}^n$, and $0 \leq i < j < n$, then

$$\omega_n^{j-i} W_i(K)^{n-j} \leq W_j(K)^{n-i}, \tag{2.6}$$

with equality if and only if K is a ball.

The Minkowski inequality for mixed volumes states as follows[16]: If $K, L \in \mathcal{K}^n$ and $0 \leq i \leq n-2$, then

$$W_i(K, L)^{n-i} \geq W_i(K)^{n-i-1} W_i(L), \tag{2.7}$$

with equality if and only if K and L are homothetic.

A consequence of the Minkowski inequality is the Brunn-Minkowski inequality: If $K, L \in \mathcal{K}^n$ and $0 \leq i \leq n-2$, then

$$W_i(K+L)^{\frac{1}{n-i}} \geq W_i(K)^{\frac{1}{n-i}} + W_i(L)^{\frac{1}{n-i}}. \tag{2.8}$$

Equality holds if and only if K and L are homothetic.

A generalization of inequality (2.8) is also known (but without equality conditions): If $K, L, K_1, \ldots, K_i \in \mathcal{K}^n, 0 \leq i \leq n-2$, and $M = (K_1, \ldots, K_i)$, then

$$V_i(K+L, M)^{\frac{1}{n-i}} \geq V_i(K, M)^{\frac{1}{n-i}} + V_i(L, M)^{\frac{1}{n-i}}. \tag{2.9}$$

3. Main Results

Lemma 3.1. [2]. *If $K_1, \ldots, K_{2n-1}, L_1, \ldots, L_{2n-1}$ are convex bodies in $\mathbb{C}^n$ and $C \subset \mathbb{C}$ is a convex subset, then*

$$V(K_1, \ldots, K_{2n-1}, \Pi^C(L_1, \ldots, L_{2n-1})) = V(L_1, \ldots, L_{2n-1}, \Pi^{\overline{C}}(K_1, \ldots, K_{2n-1})),$$

where $\overline{C}$ is the complex conjugate of $C \subset \mathbb{C}$.

If $K_1 = \cdots = K_{2n-i-1} = K$, while $K_{2n-i} = \cdots = K_{2n-1} = B$, then Lemma 3.1 reduces to

Lemma 3.2. *If $K, L_1, \ldots, L_{2n-1}$ are convex bodies in $\mathbb{C}^n$ and $C \subset \mathbb{C}$ is a convex subset, then*

$$W_i(K, \Pi^C(L_1, \ldots, L_{2n-1})) = V(L_1, \ldots, L_{2n-1}, \Pi_i^{\overline{C}} K).$$

The special case of Lemma 3.2, where $L_1 = \cdots = L_{2n-j-1} = L$ and $L_{2n-j} = \cdots = L_{2n-1} = B$, states as follows:

Lemma 3.3. *Let K and L be convex bodies in $\mathbb{C}^n$ and $C \subset \mathbb{C}$ be a convex subset. If $0 \leq i \leq 2n-1$, while $0 \leq j \leq 2n-2$, then*

$$W_i(K, \Pi_j^C L) = W_j(L, \Pi_i^{\overline{C}} K).$$

Let $w \in S^{2n-1}$ and $SO(2n)$ be the rotation group in $\mathbb{R}^{2n}$. Then for every $\nu \in S^{2n-1}$, there exists a rotation transform $\phi \in SO(2n)$, such that $\nu = \phi w$. By (1.4) and (2.2), we have

$$\begin{aligned} h(\Pi^C B, \nu) = h(\Pi^C B, \phi w) &= \frac{1}{2n} \int_{S^{2n-1}} h(C \cdot \phi w, \xi) dS(B, \xi) \\ &= \frac{1}{2n} \int_{S^{2n-1}} h(C \cdot w, \phi^t \xi) dS(B, \xi), \\ &= \frac{1}{2n} \int_{S^{2n-1}} h(C \cdot w, u) dS(B, u) \\ &= h(\Pi^C B, w), \end{aligned}$$

equivalently, $\Pi^C B = r_C B$. Note that $h(\overline{C} \cdot w, \xi) = h(C \cdot \xi, w)$ and the surface area measure $S(B, \cdot)$ is constant in S^{2n-1}. From above argument, we obtain $\Pi^C B = \Pi^{\overline{C}} B = r_C B$.

Take $K_1 = \cdots = K_{2n-1} = B$ in Lemma 3.1 and use $\Pi^C B = \Pi^{\overline{C}} B = r_C B$ to get

Lemma 3.4. *Let $L_1, \ldots, L_{2n-1}$ be convex bodies in $\mathbb{C}^n$ and $C \subset \mathbb{C}$ be a convex subset. Then*

$$W_{2n-1}(\Pi^C(L_1, \ldots, L_{2n-1})) = r_C V(L_1, \ldots, L_{2n-1}, B). \tag{3.1}$$

For $L_1 = \cdots = L_{2n-2} = K$ and $L_{2n-1} = L$, identity (3.1) becomes

$$W_{2n-1}(\Pi_1^C(K, L)) = r_C W_1(K, L), \tag{3.2}$$

for $L_1 = \cdots = L_{2n-i-1} = K$ and $L_{2n-i} = \cdots = L_{2n-1} = B$, identity (3.1) becomes,

$$W_{2n-1}(\Pi_i^C K) = r_C W_{i+1}(K). \tag{3.3}$$

In [2], Abardia and Bernig established the following Minkowski type inequality for mixed complex projection bodies.

Theorem 3.5. [2]. *Let K and L be convex bodies in $\mathbb{C}^n$ and $C \subset \mathbb{C}$ be a convex subset which is not a point. If $0 \leq i \leq 2n-1$, then*

$$W_i(\Pi_1^C(K, L))^{2n-1} \geq W_i(\Pi^C K)^{2n-2} W_i(\Pi^C L), \tag{3.4}$$

with equality if and only if K and L are homothetic.

Much more general than the Minkowski inequality is the Aleksandrov-Fenchel inequality for mixed complex projection bodies which was obtained by Abardia and Bernig [2].

Theorem 3.6. [2]. *Let $K_1, \ldots, K_{2n-1}$ be convex bodies in $\mathbb{C}^n$ and $C \subset \mathbb{C}$ be a convex subset. If $0 \leq i \leq 2n-1$, while $0 \leq k \leq 2n-2$, then*

$$W_i(\Pi^C(K_1, \ldots, K_{2n-1}))^k \geq \prod_{j=1}^{k} W_i(\Pi^C(\underbrace{K_j, \ldots, K_j}_{k}, K_{k+1}, \ldots, K_{2n-1})). \tag{3.5}$$

From the case of $k = 2n-2$ of inequality (3.5), it follows that

$$\begin{aligned} &W_i(\Pi^C(K_1, \ldots, K_{2n-1}))^{2n-2} \\ &\geq W_i(\Pi_1^C(K_1, K_{2n-1})) \cdots W_i(\Pi_1^C(K_{2n-2}, K_{2n-1})). \end{aligned} \tag{3.6}$$

Combine inequality (3.6) and Theorem 3.5, and the result is

Corollary 3.7. *Let $K_1, \ldots, K_{2n-1}$ be convex bodies in $\mathbb{C}^n$ and $C \subset \mathbb{C}$ be a convex subset which is not a point. If $0 \leq i \leq 2n-1$, then*

$$W_i(\Pi^C(K_1, \ldots, K_{2n-1}))^{2n-1} \geq W_i(\Pi^C K_1) \cdots W_i(\Pi^C K_{2n-1}),$$

with equality if and only if the K_i are homothetic.

The special case of Corollary 3.7, where we have $K_1 = \cdots = K_{2n-j-1} = K$, and $K_{2n-j} = \cdots = K_{2n-1} = L$, states as follow:

Corollary 3.8. *Let K, L be convex bodies in $\mathbb{C}^n$ and $C \subset \mathbb{C}$ be a convex subset which is not a point. If $0 \leq i \leq 2n-1$ and $1 \leq j \leq 2n-2$, then*

$$W_i(\Pi_j^C(K, L))^{2n-1} \geq W_i(\Pi^C K)^{2n-j-1} W_i(\Pi^C L)^j,$$

with equality if and only if K and L are homothetic.

An immediate consequence of Corollary 3.8 states as follows:

Theorem 3.9. *Let K, L be convex bodies in $\mathbb{C}^n$ and $\mathcal{M} \subset \mathbb{C}^n$ be a subset which contains K and L. Suppose $C \subset \mathbb{C}$ is a convex subset which is not a point, $0 \leq i \leq 2n-1$ and $1 \leq j \leq 2n-2$. If either*

$$W_i(\Pi_j^C(K, Q)) = W_i(\Pi_j^C(L, Q)), \ for\ all\ Q \in \mathcal{M}, \tag{3.7}$$

or

$$W_i(\Pi_j^C(Q,K)) = W_i(\Pi_j^C(Q,L)),\ for\ all\ Q \in \mathcal{M}, \tag{3.8}$$

hold, then it follows that $K = L$, up to translation.

Proof. Suppose that (3.7) holds. Take K for Q in (3.7), use Corollary 3.8 to get

$$W_i(\Pi^C K) \geq W_i(\Pi^C L), \tag{3.9}$$

with equality if and only if K and L are homothetic.

Take L for Q in (3.7), use Corollary 3.8 to get

$$W_i(\Pi^C L) \geq W_i(\Pi^C K).$$

Hence, there is equality in (3.9) and thus, there is a $\lambda > 0$ for which K and λL are translates. But equality in (3.9) implies that $\lambda = 1$.

Exactly the same sort of argument shows that condition (3.8) implies that K and L must be translates. ■

Proof of Theorem 1.1. From (3.3), it follows that the case $k = 2n-1$ of inequality (1.5) reduces to (2.6), and hence, it may be assumed that $k < 2n-1$.

Suppose Q is a convex body in $\mathbb{C}^n$. From Lemma 3.3,

$$W_k(Q, \Pi_j^C K) = W_j(K, \Pi_k^{\overline{C}} Q). \tag{3.10}$$

From inequality (2.5), it follows that

$$W_j(K, \Pi_k^{\overline{C}} Q)^{2n-i-1} \geq W_{2n-1}(\Pi_k^{\overline{C}} Q)^{j-i} W_i(K, \Pi_k^{\overline{C}} Q)^{2n-j-1}. \tag{3.11}$$

From (3.3) and inequality (2.6), it follows that

$$W_{2n-1}(\Pi_k^{\overline{C}} Q) = r_C W_{k+1}(Q) \geq r_C \omega_{2n}^{\frac{1}{2n-k}} W_k(Q)^{\frac{2n-k-1}{2n-k}}, \tag{3.12}$$

with equality if and only if Q is a ball.

For the second term on the right of (3.11), note that by Lemma 3.3,

$$W_i(K, \Pi_k^{\overline{C}} Q) = W_k(Q, \Pi_i^C K).$$

Apply inequality (2.7) to the quantity on the right and get:

$$W_i(K, \Pi_k^{\overline{C}} Q) \geq W_k(Q)^{\frac{2n-k-1}{2n-k}} W_k(\Pi_i^C K)^{\frac{1}{2n-k}}, \tag{3.13}$$

with equality if and only if Q and $\Pi_i K$ are homothetic.

Now take $Q = \Pi_j^C K$, note that $W_k(Q,Q) = W_k(Q)$, and combine (3.10) with (3.11), (3.12) and (3.13) to obtain the desired inequality of Theorem 1.1.

Suppose there is equality in inequality (1.5):

$$W_k(\Pi_j^C K)^{2n-i-1} = r_C^{(j-i)(2n-k)} \omega_{2n}^{j-i} W_k(\Pi_i^C K)^{2n-j-1}. \tag{3.14}$$

From the equality conditions of inequalities (3.12) and (3.13), this implies that $\Pi_i^C K$ and $\Pi_j^C K$ must be centered balls. Thus there exist $\lambda, \mu > 0$ and $x_1, x_2 \in \mathbb{C}^n$, such that

$$\Pi_i^C K = \lambda B + x_1, \ \textit{and}\ \Pi_j^C K = \mu B + x_2. \tag{3.15}$$

Since Quermassintegrals are translation invariant, from (3.14), it follows that

$$\mu^{2n-i-1} = r_C^{j-i} \lambda^{2n-j-1},$$

equivalently,

$$\omega_{2n}^{j-i} [\frac{\lambda \omega_{2n}}{r_C}]^{2n-j-1} = [\frac{\mu \omega_{2n}}{r_C}]^{2n-i-1}.$$

Moreover, (3.3) and (3.15) imply

$$W_{i+1}(K) = \frac{\lambda \omega_{2n}}{r_C} \ \textit{and}\ W_{j+1}(K) = \frac{\mu \omega_{2n}}{r_C}.$$

Hence, we have

$$\omega_{2n}^{j-i} W_{i+1}(K)^{2n-j-1} = W_{j+1}(K)^{2n-i-1},$$

which implies, by (2.6), that K is a ball. ■

Theorem 3.10. *Let $K, L, M_1, \ldots, M_i, N_1, \ldots, N_j$ be convex bodies in $\mathbb{C}^n$ and $C \subset \mathbb{C}$ be a convex subset. Let $M = (M_1, \ldots, M_i), N = (N_1, \ldots, N_j)$. If $0 \le i \le 2n-1$, while $0 \le j \le 2n-2$, then*

$$\begin{aligned} &V_i(\Pi_j^C(K+L, N), M)^{\frac{1}{(2n-i)(2n-j-1)}} \\ &\ge V_i(\Pi_j^C(K, N), M)^{\frac{1}{(2n-i)(2n-j-1)}} + V_i(\Pi_j^C(L, N), M)^{\frac{1}{(2n-i)(2n-j-1)}}. \end{aligned}$$

Proof. If $j = 2n-2$, by (1.4), we have that, for every $w \in S^{2n-1}$,

$$\begin{aligned} h(\Pi_{2n-2}^C(K+L, N), w) &= \frac{1}{2n} \int_{S^{2n-1}} h(C \cdot w, \xi) dS(K+L, N, \ldots, N, \xi) \\ &= V(K+L, N, \ldots, N, C \cdot w) \\ &= V(K, N, \ldots, N, C \cdot w) + V(L, N, \ldots, N, C \cdot w) \\ &= h(\Pi_{2n-2}^C(K, N), w) + h(\Pi_{2n-2}^C(L, N), w). \end{aligned}$$

From (2.3), it follows that

$$\Pi^C_{2n-2}(K+L,N)=\Pi^C_{2n-2}(K,N)+\Pi^C_{2n-2}(L,N).$$

Hence, for $j=2n-2$, the inequality of Theorem 3.10 reduces to inequality (2.9). If $i=2n-1$, then from Lemma 3.1, it follows that the inequality of Theorem 3.10 reduces to (2.9). Thus, only the cases where $j<2n-2$ and $i<2n-1$ need be treated.

Let $Q\in\mathbb{C}^n$ be a convex body. From Lemma 3.1, (2.9) and (2.5), it follows that

$$\begin{aligned}
&V(Q,2n-i-1;M;\Pi^C_j(K+L,N)^{\frac{1}{2n-j-1}}\\
&=V(K+L,2n-j-1;N;\Pi^{\overline{C}}_i(Q,M))^{\frac{1}{2n-j-1}}\\
&\geq V(K,2n-j-1;N;\Pi^{\overline{C}}_i(Q,M))^{\frac{1}{2n-j-1}}+V(L,2n-j-1;N;\Pi^{\overline{C}}_i(Q,M))^{\frac{1}{2n-j-1}}\\
&=V(Q,2n-i-1;M;\Pi^C_j(K,N))^{\frac{1}{2n-j-1}}+V(Q,2n-i-1;M;\Pi^C_j(L,N))^{\frac{1}{2n-j-1}}\\
&\geq V_i(Q,M)^{\frac{2n-i-1}{(2n-i)(2n-j-1)}}[V_i(\Pi^C_j(K,N),M)^{\frac{1}{(2n-i)(2n-j-1)}}\\
&\quad+V_i(\Pi^C_j(L,N),M)^{\frac{1}{(2n-i)(2n-j-1)}}].
\end{aligned}$$

Take $\Pi^C_j(K+L,N)$ for Q, and recall that $V(Q,2n-i-1;M;Q)=V_i(Q,M)$ to obtain that

$$\begin{aligned}
&V_i(\Pi^C_j(K+L,N),M)^{\frac{1}{(2n-i)(2n-j-1)}}\\
&\geq V_i(\Pi^C_j(K,N),M)^{\frac{1}{(2n-i)(2n-j-1)}}+V_i(\Pi^C_j(L,N),M)^{\frac{1}{(2n-i)(2n-j-1)}}. \qquad\blacksquare
\end{aligned}$$

The most interesting case of the inequality of Theorem 3.10 is the special case where $N=(B,\dots,B)$. In this case the inequality of Theorem 3.10 reads

$$V_i(\Pi^C_j(K+L),M)^{\frac{1}{(2n-i)(2n-j-1)}}\geq V_i(\Pi^C_jK,M)^{\frac{1}{(2n-i)(2n-j-1)}}+V_i(\Pi^C_jL,M)^{\frac{1}{(2n-i)(2n-j-1)}}.$$

For the special case where $M=(B,\dots,B)$, the equality conditions of the above inequality will be established.

Proof of Theorem 1.2. Taking $N=(B,\dots,B)$ and $M=(B,\dots,B)$ in Theorem 3.10, we obtain

$$\begin{aligned}
&W_i(\Pi^C_j(K+L))^{\frac{1}{(2n-i)(2n-j-1)}}\\
&\geq W_i(\Pi^C_jK)^{\frac{1}{(2n-i)(2n-j-1)}}+W_i(\Pi^C_jL)^{\frac{1}{(2n-i)(2n-j-1)}}.
\end{aligned}\tag{3.16}$$

By the equality condition of (2.7), equality in (3.16) holds if and only if $\Pi^C_j(K+L),\Pi^C_jK$ and Π^C_jL are homothetic. If there is equality in (3.16), then there exist $\lambda_1,\lambda_2>0$ and $x_1,x_2\in\mathbb{C}^n$, such that

$$\Pi^C_jK=\lambda_1\Pi^C_j(K+L)+x_1 \textit{ and } \Pi^C_jL=\lambda_2\Pi^C_j(K+L)+x_2.\tag{3.17}$$

From equality in (3.16), it follows that

$$\lambda_1^{\frac{1}{2n-j-1}} + \lambda_2^{\frac{1}{2n-j-1}} = 1.$$

Moreover, (3.3) and (3.17) imply

$$\lambda_1 = \frac{W_{j+1}(K)}{W_{j+1}(K+L)} \; and \; \lambda_2 = \frac{W_{j+1}(L)}{W_{j+1}(K+L)}. \tag{3.18}$$

Hence, we have

$$W_{j+1}(K+L)^{\frac{1}{2n-j-1}} = W_{j+1}(K)^{\frac{1}{2n-j-1}} + W_{j+1}(L)^{\frac{1}{2n-j-1}},$$

which implies, by the equality condition of (2.8), that K and L are homothetic. ■

Corollary 3.11. Let K, L be convex bodies in $\mathbb{C}^n$ and $C \subset \mathbb{C}$ be a convex subset which is not a point. If $0 \leq i \leq 2n-1$, then

$$W_i(\Pi^C(K+L))^{\frac{1}{(2n-i)(2n-1)}} \geq W_i(\Pi^C K)^{\frac{1}{(2n-i)(2n-1)}} + W_i(\Pi^C L)^{\frac{1}{(2n-i)(2n-1)}},$$

with equality if and only if K and L are homothetic.

Remark. The case $i = j = 0$ of Theorem 3.10 was first established by Abardia and Bernig [2].

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