TAIWANESE JOURNAL OF MATHEMATICS
Vol. 17, No. 4, pp. 1395-1412, August 2013
DOI: 10.11650/tjm.17.2013.2322
This paper is available online at http://journal.taiwanmathsoc.org.tw

SUBMAXIMAL INTEGRAL DOMAINS

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Abstract. It is proved that if D is a UFD and R is a D-algebra, such that $U(R) \cap D \neq U(D)$, then R has a maximal subring. In particular, if R is a ring which either contains a unit x which is not algebraic over the prime subring of R, or R has zero characteristic and there exists a natural number $n > 1$ such that $\frac{1}{n} \in R$, then R has a maximal subring. It is shown that if R is a reduced ring with $|R| > 2^{2^{\aleph_0}}$ or $J(R) \neq 0$, then any R-algebra has a maximal subring. Residually finite rings without maximal subrings are fully characterized. It is observed that every uncountable UFD has a maximal subring. The existence of maximal subrings in a noetherian integral domain R, in relation to either the cardinality of the set of divisors of some of its elements or the height of its maximal ideals, is also investigated.

1. Introduction

All rings in this article are commutative with $1 \neq 0$; all modules are unital. If S is a subring of a ring R, then $1_R \in S$. In this paper the characteristic of a ring R is denoted by $Char(R)$, and the set of all maximal ideals of a ring R is denoted by $Max(R)$. For any ring R, let $Z = \mathbb{Z} \cdot 1_R = \{n \cdot 1_R \mid n \in \mathbb{Z}\}$, be the prime subring of R. Rings with maximal subrings are called submaximal rings in [4] and [7]. Some important rings such as uncountable artinian rings, zero-dimensional rings which are either not integral over Z or with zero characteristic, noetherian rings R with $|R| > 2^{\aleph_0}$ and infinite direct product of rings are submaximal, see [4-7]. We should remind the reader that all finite rings except $\mathbb{Z}_n$ (up to isomorphism), where n is a natural number, are submaximal. It is also interesting to note that whenever S is a finite maximal subring of a ring R, then R must be finite, see [8, Theorem 8], [19], [17] and [20]. The latter interesting fact is also an easy consequence of [5, the proof of

Received August 27, 2012, accepted February 2, 2013.
Communicated by N. Mohan Kumar.
2010 *Mathematics Subject Classification*: 13B02, 13G05, 13E05.
Key words and phrases: Maximal subring, Integral domains, Noetherian, UFD.
The author is partially supported by Research Council of Chamran University of Ahvaz, Iran.

Theorem 2.9] or [6, Theorem 3.8]. Recently S.S. Korobkov determined which finite rings have exactly two maximal subrings, see [18].

We remind the reader that whenever S is a maximal subring of a ring R, then R is called minimal ring extension of S. Recently, D.E. Dobbs and J. Shapiro have extended the results in [15], to integral domains and certain commutative rings, see [13] and [14], respectively. Also see [23], [11] and [21]. T.G. Lucas, in [21], characterized minimal ring extensions of certain commutative rings especially in the case of minimal integral extension. It is interesting to know that every commutative ring R has a minimal ring extension, for if M is a simple R-module then the idealization $R(+)M$ is a minimal ring extension of R (note, for any R-module M, every R-subalgebra of $R(+)M$ has the form $R(+)N$, where N is a submodule of M, see [12]). For a generalization of minimal ring extensions, see also [10].

Unlike maximal ideals (resp. minimal ring extension) whose existence is guaranteed either by Zorn Lemma or noetherianity of rings (resp. by idealization or other techniques, see [12]), maximal subrings need not always exist, see [7] for such examples and in particular, for example of rings of any infinite cardinality, which are not submaximal. In fact by the above comment about the idealization, one can easily see that if K is any field with zero characteristic, then the ring $\mathbb{Z}(+)K$ is not submaximal, see [7, Example 3.19]. Also, in the latter reference and in [4] a good motivations for the study of maximal subrings related to algebraic geometry and elliptic curves are given.

In this paper, we are interested in finding submaximal integral domains, especially atomic and noetherian integral domains. A brief outline of this paper is as follow. Section 1, contains some preliminaries and also some generalizations of results which are to appeared in [7]. It is observed that whenever D is a UFD and R is a D-algebra in which a non-unit of D is invertible, then R is submaximal. In particular, if R is a ring with zero characteristic and there exists $n \in (Z \setminus \{1, -1\}) \cap U(R)$, then R is submaximal. Moreover, if D is a PID and $D \subseteq R$ is an integral domain such that D is integrally closed in R and $U(R) \neq U(D)$, then R is submaximal. Consequently it is proved that, if R is a $\mathbb{Z}$-algebra, then either R is submaximal or for any prime number p, there exists a maximal ideal M of R such that $Char(\frac{R}{M}) = p$. It is observed that every ring either is submaximal or is Hilbert. In particular, if R is a reduced ring with $|R| > 2^{2^{\aleph_0}}$ or $J(R) \neq 0$, then any R-algebra is submaximal. Consequently, it is shown that if R is a reduced non-submaximal ring with zero characteristic, then $\bigcap_{p\in\mathbb{P}} Rp = 0$, where $\mathbb{P}$ is the set of prime numbers. It is proved that if R is a residue finite non-submaximal ring, then R is a countable principal ideal ring which is either an integral domain with zero characteristic or it is an artinian ring with nonzero characteristic. Finally in Section 1, the existence of maximal subring in semi-local rings and localization of rings are investigated. In particular, it is proved that if R is a ring and S is a multiplicatively closed set in R such that R_S is semi-local, then either R_S is submaximal or every prime ideal of R_S has the form P_S, for some $P \in Max(R) \cap Min(R)$. Moreover, in the latter case, if R_S is submaximal, then R is submaximal too. Section 2, is devoted to the existence of maximal subrings in unique factorization domains, noetherian integral domains and certain atomic domains. It is observed that, every uncountable UFD is submaximal. We also generalized the latter result to certain uncountable atomic domains. In particular, it is proved that if R is an uncountable noetherian $\mathbb{Z}$-algebra, in which every natural number has at most countably many (irreducible) divisors, then R is submaximal. It is shown that, if R is a noetherian integral domain with zero characteristic and $tr.deg_Z R = n \geq 1$ (resp. with nonzero characteristic and $tr.deg_Z R = n \geq 2$) such that the height of every maximal ideal of R is greater or equal to $n+1$ (resp. greater or equal to n) and $Z[X] \subseteq R$ is a residually algebraic extension, where X is a transcendence basis for R over Z, then R is submaximal. Finally, we show that every uncountable Dedekind domain D with $|Max(D)| \leq \aleph_0$, is submaximal.

Finally, let us recall some standard definitions and notations in commutative rings, see [16]. An integral domain D is called G-domain if the quotient field of D is finitely generated as a ring over D. A prime ideal P of a ring R is called G-ideal if $\frac{R}{P}$ is a G-domain. A ring R is called Hilbert if every G-ideal of R is maximal. We also call a ring R, not necessarily noetherian, semi-local (resp. local) if $Max(R)$ is finite (resp. $|Max(R)| = 1$). An integral domain D is called atomic, if every nonzero non-unit of D is a finite product of irreducible elements, not necessarily unique. An integral domain D is called idf-domain if every nonzero non-unit element of D has at most finitely many irreducible divisors, see [1]. In this paper the set of minimal prime ideals and prime ideals of a ring R are denoted by $Min(R)$ and $Spec(R)$, respectively. As usual, let $U(R)$ denote the set of all units of a ring R. The Jacobson and the nil radical ideals of a ring R are also denoted by $J(R)$ and $N(R)$, respectively. If P is a prime ideal of a ring R, then the height of P is denoted by $ht(P)$. If D is an integral domain, then from each set of associate irreducible elements of D, choose one to put into $Ir(D)$. We recall that if $D \subseteq R$ is an extension of integral domains, then as for the existence a transcendence basis for field extensions, one can easily see that there exists a subset X of R which is algebraically independent over D and R is algebraic over $D[X]$ (hence every integral domain is algebraic over a UFD). Moreover, in the latter case $|X| = tr.deg_F(E)$, where E and F are the quotient fields of R and D, respectively. Hence, similar to the field extensions, we can define the transcendence degree of R over D which is denoted by $tr.deg_D(R)$. Finally, we denote the set of all prime numbers by $\mathbb{P}$.

2. Preliminaries and Generalizations

We begin this section with the following useful fact about the existence of maximal subrings in subrings of a submaximal ring, which is the converse of [6, Proposition

2.1]. We remind the reader that a ring R is submaximal if and only if there exist a proper subring S of R and an element $x \in R \setminus S$ such that $S[x] = R$, see [3, Theorem 2.5]. Now the following is in order, and although its proof is in [7], we present it for the sake of the reader.

Proposition 2.1. [7, Theorem 2.19]. *Let $R \subseteq T$ be rings. If there exists a maximal subring V of T such that V is integrally closed in T and $U(R) \nsubseteq V$, then R is submaximal.*

Proof. First, we claim that whenever $x \in U(R) \setminus V$, then $x^{-1} \in V$. To see this, we observe that $x^{-1} \in R \subseteq T = V[x]$ (note, V is a maximal subring of T). Consequently, $x^{-1} = a_0 + a_1x + \cdots + a_nx^n$, where $a_0, a_1, \ldots, a_n \in V$. Now by multiplying the latter equality by x^{-n}, we infer that x^{-1} is integral over V, hence $x^{-1} \in V$. But $U(R) \nsubseteq V$ implies that $V \cap R$ is a proper subring of R and there exists $x \in U(R) \setminus V$ with $T = V[x]$. Finally, we claim that $R = (V \cap R)[x]$, which by the preceding comment, it implies that R is submaximal. To this end, let $y \in R$, hence $y \in V[x]$ and therefore $y = b_0 + b_1x + \cdots + b_mx^m$, where $b_0, b_1, \ldots, b_m \in V$, implies that $yx^{-m} \in V \cap R$ (note, $x^{-1} \in V$), i.e., $y \in (R \cap V)[x]$ and we are done. ∎

Next, we have the following fact which is needed in the sequel.

Theorem 2.2. *Let R be a ring and D be a subring of R which is a UFD. If there exists an irreducible element $p \in D$ such that $\frac{1}{p} \in R$, then R is submaximal. In particular, if $U(R) \cap D \neq U(D)$, then R is submaximal.*

Proof. We first assume that R is algebraic over D. We also may assume that R is an integral domain (note, if not, then there exists a prime ideal Q of R such that $D \cap Q = 0$ and therefore $\frac{R}{Q}$ contains a copy of D). Now suppose that K and E are the quotient fields of D and R, respectively. Thus E/K is an algebraic extension, since R is algebraic over D. Now, note that K has a maximal subring V such that $\frac{1}{p} \notin V$ (for example $V = D_{(p)}$). Hence E has a maximal subring W such that $W \cap K = V$, by [6, Proposition 2.1]. Therefore $\frac{1}{p} \notin W$. Thus we have $U(R) \nsubseteq W$ which implies that R is submaximal by the above proposition. Finally, assume that R is not algebraic over D, but by the preceding comment we may suppose that R is an integral domain too. Let X be a transcendence basis for R over D. Thus R is algebraic over $D[X]$. Now note that $D[X]$ is a UFD and p is an irreducible element in it. Hence we are done by the first part of the proof. The final part is evident. ∎

The following fact also serves to justify why in [4, Proposition 2.10] the proof is divided into two cases.

Remark 2.3. Let R be a ring satisfying the conditions of the above theorem, then there exists a maximal subring of R which dose not contain $\frac{1}{p}$. In particular, if K is a field with zero characteristic, then for any prime number p, there exists a maximal subring V_p of K such that $\frac{1}{p} \notin V_p$. Hence if M is the unique nonzero prime ideal of V_p, we infer that $Char(\frac{V_p}{M}) = p$.

The next three interesting facts are now immediate.

Corollary 2.4. *Let R be a UFD and S be a multiplicatively closed subset of R which contains a non-unit of R, then R_S is submaximal.*

Corollary 2.5. *Let R be a ring with zero characteristic. If there exists a natural number $n > 1$ such that $\frac{1}{n} \in R$, then R is submaximal.*

Corollary 2.6. *Let D be an integral domain with zero characteristic and X be a set of independent indeterminates over it. Then for any $x \in X$ and every natural number $n > 1$, the ring $\frac{D[X]}{(nx-1)D[X]}$ is submaximal.*

Corollary 2.7. *If R is a ring with $0 = Char(R) \neq Char(\frac{R}{J(R)})$, then any R-algebra T is submaximal.*

Proof. Assume that $Char(\frac{R}{J(R)}) = n$, thus $n \in J(R)$. Hence for any $k \in \mathbb{Z}$, we have $1 - kn \in U(R) \subseteq U(T)$ and therefore we are done by Corollary 2.5. ∎

Corollary 2.8. *Let R be a ring with zero characteristic which is not submaximal. Then $\{Char(\frac{R}{M}) \mid M \in Max(R)\} = \mathbb{P}$ and therefore $|Max(R)|$ is infinite.*

Proof. Since R is not submaximal, we infer that $Char(\frac{R}{M}) \neq 0$ for each maximal ideal M of R, by Corollary 2.5. Hence $\{Char(\frac{R}{M}) \mid M \in Max(R)\} \subseteq \mathbb{P}$. Now for each prime number q we claim that there exists a maximal ideal M of R with $Char(\frac{R}{M}) = q$, which proves the corollary. To see this, we note that $qR \neq R$, by Corollary 2.5. Consequently, there exists a maximal ideal M of R with $qR \subseteq M$, i.e., $Char(\frac{R}{M}) = q$. ∎

For more observations we need the following lemma.

Lemma 2.9. *Let R be a ring and $x \in R$ be non-algebraic over the prime subring of R. Then at least one of the following conditions holds.*

(1) *If $Char(R) = 0$, then there exists a prime ideal Q of R such that R/Q contains a copy of $\mathbb{Z}[x]$.*

(2) *If $Char(R) = n > 0$, then for any prime divisor p of n, there exists a prime ideal Q of R such that R/Q contains a copy of $\mathbb{Z}_p[x]$.*

Proof. If R has zero (or prime) characteristic, we are done since $Z[x] \setminus \{0\}$ is a multiplicatively closed set in R. Now, suppose that R has nonzero characteristic, say n, which is also not a prime number. Assume that p is a prime divisor of n. Since

$\dim \mathbb{Z}_n[x] = 1$ and $P = \frac{p\mathbb{Z}}{n\mathbb{Z}}[x]$ is a non-maximal prime ideal of $\mathbb{Z}_n[x]$, hence we infer that P is a minimal prime ideal of $\mathbb{Z}_n[x]$. Thus, there exists a minimal prime ideal Q of R such that $Q \cap \mathbb{Z}_n[x] = P$. Now we have $\mathbb{Z}_p[x] \cong \frac{\mathbb{Z}_n[x]}{Q \cap \mathbb{Z}_n[x]} \subseteq \frac{R}{Q}$ and therefore we are done. ■

Remark 2.10. In fact in Corollary 2.8, we see that if R is not submaximal and $\mathbb{Z} \subseteq R$, then $|Max(R)| \geq |\mathbb{P}|$. We can generalize the previous fact to any non-submaximal ring which contains a UFD as follow. First, we recall that if R is a ring with $|Max(R)| > 2^{\aleph_0}$, then R is submaximal, see [4, Proposition 2.6]. Now assume that D is a UFD and let $Ir'(D)$ be a subset of $Ir(D)$ such that for any $p \neq q$ in $Ir'(D)$, we have $pD + qD = D$. Now, if R is a ring which contains D, then either R is submaximal or $|Ir'(D)| \leq |Max(R)| \leq 2^{\aleph_0}$. To see this assume that R is not submaximal, then for any $q \in Ir'(D)$ we have $qR \neq R$, by Theorem 2.2 and hence there exists a maximal ideal M_q of R, such that $qR \subseteq M_q$. It is clear that whenever $p \neq q$ in $Ir'(D)$, then we have $M_q \neq M_p$, and therefore we are done. In particular, if R is a non-submaximal ring with nonzero characteristic, say n, which is not algebraic over $\mathbb{Z}_n$, then $|Max(R)|$ is infinite. To see this note that by part (2) of the above lemma, for any prime divisor p of n there exists a prime ideal Q of R such that R/Q contains a copy of $\mathbb{Z}_p[x]$. Hence we are done by the first part of the proof.

The following proof greatly simplifies the proof of [7, Theorem 2.1 and Theorem 2.4].

Corollary 2.11. [7, Theorem 2.4]. *Let R be a ring with a unit element which is not algebraic over the prime subring of R. Then R is submaximal (in fact every R-algebra is submaximal).*

Proof. In view of Theorem 2.2 and Lemma 2.9 we are done. ■

Corollary 2.12. *Let R be a ring. Then either R is submaximal or every element of $J(R)$ is algebraic over the prime subring of R.*

We need the following immediate corollary in the next section.

Corollary 2.13. *Let R be a ring which is not algebraic over $\mathbb{Z}$. Then either R is submaximal or for any non-algebraic element $x \in R$ over $\mathbb{Z}$ and every natural number $n > 1$, we have $\mathbb{Z} \cap (nx - 1)R \neq 0$.*

Proof. If $(nx - 1)R = R$, then we are done by Corollary 2.11, and if not, then by using Corollary 2.5, we are done. ■

Corollary 2.14. *Let D be a PID and $R \supseteq D$ be an integral domain. If D is integrally closed in R and $U(R) \neq U(D)$, then R is submaximal. In particular, every proper overring of a PID is submaximal.*

Proof. Let $x \in U(R) \setminus U(D)$. If x is not algebraic over D, then we are done, by Corollary 2.11. Hence assume that x is algebraic over D, thus there exists $b \in D$ such that bx is integral over D and since D is integrally closed in R, we must have $bx = a \in D$. Therefore $x = \frac{a}{b}$. Now, since $x \notin U(D)$, we infer that either $x \notin D$ or $x^{-1} \notin D$. Therefore, in any case, there must exist $r, s \in D$ such that $(r, s) = 1$ and $z = \frac{r}{s} \in U(R) \setminus D$. Now since D is a PID, we infer that $\frac{1}{s} \in R$. Thus we are done, by Theorem 2.2. The final part is evident. ■

Lemma 2.15. *Let R be a ring with nonzero characteristic n which is square free (in particular, if R is reduced ring with nonzero characteristic). Then either R is submaximal or $U(R)$ is a torsion group.*

Proof. Without lose of generality we may assume that $Char(R) = p$, where p is a prime number. Now suppose that R is not submaximal, then $U(R)$ must be algebraic over $\mathbb{Z}_p$, by Corollary 2.11. Assume that $x \in U(R)$, thus we infer that $\mathbb{Z}_p[x] \cong \frac{\mathbb{Z}_p[t]}{I}$, where I is a nonzero ideal of the polynomial ring $\mathbb{Z}_p[t]$. Hence we infer that $\mathbb{Z}_p[x]$ is a finite ring, and therefore x is a torsion element. Thus $U(R)$ is a torsion group. ■

We recall that zero dimensional rings (in particular von Neumann regular rings) with zero characteristic are submaximal, see [6, Corollary 3.11]. We also have the following.

Proposition 2.16. *Let R be a von Neumann regular ring. Then either R is submaximal or R is a periodic ring.*

Proof. If R is not submaximal then by the above comment R has nonzero characteristic. Hence by the above lemma $U(R)$ is torsion. But it is well-known that von Neumann regular rings are unit regular, that is to say, for any $x \in R$, there exists $u \in U(R)$ such that $x = x^2u$. Hence by the above lemma, if $u^n = 1$, then we have $x^n = x^{2n}$ and thus we are done. ■

In fact the above result holds for any zero-dimensional ring R. For proof note that if R is not submaximal then R has nonzero characteristic, say n, and R is integral over $\mathbb{Z}_n$, by [6, Corollary 3.14]. Now note that for any $x \in R$, the ring $\mathbb{Z}_n[x]$ is finite and hence we are done. The next remark shows in some rings R, the group $U(R)$ may not be torsion.

Remark 2.17. Let R be a ring. If R is von Neumann regular with zero characteristic then clearly $U(R)$ is not torsion, by the proof of the above proposition, since R is not periodic. Also, if R is a ring with prime characteristic, say p, and there exists a non zero-divisor $x \in J(R)$, then $U(R)$ is not torsion. To see this note that if $U(R)$ is torsion, then there exists a natural number n such that $(1+x)^n = 1$. Hence we infer that there exists a natural number m such that $a_m x^m + \cdots + a_{n-1}x^{n-1} + x^n = 0$, where $a_i \in \mathbb{Z}_p$ and

$a_m \neq 0$ (note, x is not a zero-divisor). Since $a_m + \cdots + a_{n-1}x^{n-m-1} + x^{n-m} \in U(R)$, we infer that $x^m = 0$, which is a contradiction.

By Corollary 2.12, if R is a ring then either R is submaximal or every element of $J(R)$ is algebraic over Z. Now we also have the following result.

Proposition 2.18. *Let R be a ring with zero characteristic and $J(R) \neq 0$. Then either R is submaximal or for any $x \in J(R)$ and $f(t) \in \mathbb{Z}[t]$, where t is an indeterminate over $\mathbb{Z}$, if $f(x) = 0$, then $f(0) = 0$. In particular $J(R)$ consists of zero divisors.*

Proof. Assume that R is not submaximal and $x \in J(R)$, $f(t) \in \mathbb{Z}[t]$, and $f(x) = 0$. Now since $x \in J(R)$, we infer that if u is one of the elements $1 + f(0)$ or $1 - f(0)$, then $u \in U(R) \cap \mathbb{Z}$. Thus by Corollary 2.5, we have $u = 1$ or $u = -1$. This implies that either $f(0) = 0$, and therefore we are done, or $f(0) \in \{2, -2\}$. But in the latter case, we have $2 \in J(R)$ and therefore $1 - 2n \in U(R)$, for each $n \in \mathbb{Z}$, which is impossible by Corollary 2.5. ■

The following is a generalization of [7, Corollary 2.24].

Corollary 2.19. *Let R be an integral domain with $J(R) \neq 0$. Then any R-algebra T is submaximal. In particular, any algebra over a non-field G-domain is submaximal.*

Proof. If R has nonzero characteristic or if $Char(R) = Char(\frac{R}{J(R)}) = 0$, then one can easily see that $J(R)$ is not algebraic over the prime subring of R (note, if $0 \neq x \in J(R)$ and $a_n x^n + \cdots + a_1 x + a_0 = 0$, where $n \in \mathbb{N}$, a_i are in the prime subring of R and $a_0 \neq 0$, then we infer that $a_0 \in J(R)$ which is absurd). Therefore $U(R)$ is not algebraic over the prime subring of R. Thus $U(T)$ is not algebraic over the prime subring of T and therefore T is submaximal, by Corollary 2.11. Hence we may assume that $0 = Char(R) \neq Char(\frac{R}{J(R)})$ and hence T is submaximal by Corollary 2.7. The last part is now evident. ■

Remark 2.20. One can prove the above corollary by using the proof of Proposition 2.18, Lemma 2.15 and Remark 2.17.

Proposition 2.21. *Let $R \subseteq T$ be an extension of commutative rings with the lying-over property and R is not Hilbert. Then T is submaximal.*

Proof. Let P be a prime ideal in R such that P is not an intersection of a family of maximal ideals in R. Now assume Q is a prime ideal in T lying over P. Thus $R/P \subseteq T/Q$ and since $J(R/P) \neq 0$, we infer that T/Q is submaximal by Corollary 2.19. ■

We recall that if R is a ring with $|Max(R)| > 2^{\aleph_0}$, then R is submaximal, see [4, Proposition 2.6].

Corollary 2.22. *Let R be a ring. Then either R is submaximal or it is a Hilbert ring with $|Spec(R)| \leq 2^{2^{\aleph_0}}$.*

Proof. If R is not submaximal, then for any prime ideal P of R, the integral domain R/P is not submaximal too. Hence we infer that $J(R/P) = 0$, by Corollary 2.19, i.e., R is Hilbert and therefore P is an intersection of a set of maximal ideals of R. Thus by the above comment we infer that $|Spec(R)| \leq 2^{2^{\aleph_0}}$. ■

Remark 2.23. In fact if R is not submaximal, then for any prime ideal P and subring S of R, the prime ideal $P \cap S$ is an intersection of a family of maximal ideals of S. To see this note that R/P contains a copy of $S/(P \cap S)$, and since R is not submaximal, we infer that $J(S/(P \cap S)) = 0$, by Corollary 2.19. Hence we are done.

Lemma 2.24. *Let R be a ring. Then at least one of the following conditions holds,*

(1) *There exists a maximal ideal M of R, such that R/M is not an algebraic extension of a finite field (i.e., R/M is not absolutely algebraic field). In particular, R/M and therefore R are submaximal.*
(2) *For any subring S of R, we have $J(S) \subseteq J(R)$.*

Proof. If (1) does not hold, then for any maximal ideal M of R, the field R/M is algebraic over a finite field. Hence we infer that every subring of R/M is a field. Now note that if S is a subring of R, then $(S + M)/M$ is a subring of R/M and therefore $(S+M)/M$ is a field. Thus $S \cap M$ is a maximal ideal of S, for any maximal ideal M of R. This shows that $J(S) \subseteq J(R)$. For the final part in (1), note that by [4, Theorem 1.8], if R/M is not algebraic over a finite field, then R/M and therefore R are submaximal. ■

In [4, Proposition 2.9] it is proved that if R is a ring with $|R/J(R)| > 2^{2^{\aleph_0}}$, then R is submaximal.

Corollary 2.25. *Let R be a reduced ring. If either $J(R) \neq 0$ or $|R| > 2^{2^{\aleph_0}}$, then R is submaximal. Moreover, every R-algebra T, is submaximal too.*

Proof. If R is not submaximal, then by Corollary 2.22, R is Hilbert ring and therefore $J(R) = N(R)$. Hence we infer that $J(R) = 0$ and by the above comment also we have $|R| \leq 2^{2^{\aleph_0}}$ which contradicts our assumptions. For the final part note that $T/N(T)$ contains a copy of R, hence by our assumptions, either by the above lemma $J(T/N(T)) \neq 0$, or $|T/N(T)| > 2^{2^{\aleph_0}}$. Thus by the first part, $T/N(T)$ and therefore T are submaximal. ■

Hence by the above corollary if T is a non-submaximal ring, then for any reduced subring R of T we have $J(R) = 0$. More generally, for any subring R of T we have

$N(R) = J(R)$. To see this, note that $R + N(T)$ is a subring of T. Now since $T/N(T)$ contains a copy of $R/N(R)$, we infer that $J(R/N(R)) = 0$, hence we are done. The following is also interesting.

Corollary 2.26. *Let R be a reduced ring with zero characteristic, then either R is submaximal or $\bigcap_{p\in\mathbb{P}} Rp = 0$.*

Proof. If R is not submaximal then by Corollary 2.8, we infer that $\bigcap_{p\in\mathbb{P}} Rp \subseteq J(R)$. But by the above corollary we also have $J(R) = 0$. Hence we are done. ■

We recall that each zero dimensional ring with nonzero characteristic which is not integral over its prime subring, is submaximal, see [6, Corollary 3.14]. The following is a generalization of the existence of maximal subrings in artinian rings, see [5].

Corollary 2.27. *Let R be a semi-local ring. Then either R is submaximal or R has nonzero characteristic, say n, which is integral over $\mathbb{Z}_n$ (thus R is zero-dimensional). In particular, every semi-local ring with zero characteristic is submaximal. Consequently,*

(1) *Non-submaximal semi-local integral domains are exactly non-submaximal fields.*
(2) *Every non-submaximal noetherian semi-local ring, is countable artinian.*

Proof. If $Char(R) = 0$, then we are done by Corollary 2.8. Hence assume that R has nonzero characteristic. If R is not submaximal, then R is a Hilbert ring by Corollary 2.22. Therefore every non-maximal prime ideal of R is an intersection of infinitely many maximal ideals. Hence we infer that R is zero dimensional, since $|Max(R)| < \aleph_0$. Now by the above comment we infer that R must be integral over its prime subring. For part (1), we note that the prime subring of an integral domain with nonzero characteristic is a field; and for (2) note that R is a zero-dimensional ring. Hence R is artinian. Thus by [5, Proposition 2.4], R must be countable too. ■

Proposition 2.28. *Let $R_1 \subseteq R_2$ be extension of rings. Assume that R_1 is semi-local. Then either R_2 is submaximal or R_1 is zero-dimensional. In other words, every algebra over a semi-local ring which is not zero dimensional, is submaximal.*

Proof. First note that, if P is a prime ideal of R_2, then the ring R_2/P contains a copy of $S = R_1/(R_1 \cap P)$. Hence if $J(S) \neq 0$, then R_2/P and therefore R_2 are submaximal by Corollary 2.19. If not, then we infer that $P \cap R_1$ is a maximal ideal of R_1, since R_1 is semi-local. Hence, we may assume that for any prime ideal P of R_2, $R_1 \cap P$ is a maximal ideal in R_1. Now, if Q is a prime ideal in R_1, then there exists a prime ideal P of R_2 such that $P \cap R_1 \subseteq Q$ (note, by Zorn's Lemma there exists a prime ideal P of R_2 with $P \cap (R_1 \setminus Q) = \emptyset$). Hence we infer that $Q = P \cap R_1$ and therefore Q is maximal in R_1. Hence R_1 is a zero dimensional ring. ■

We recall the reader that a ring R is called residue finite if R/I is a finite ring for every nonzero ideal I of R. It is clear that if R is a residue finite ring, then $dim(R) \leq 1$ and in fact $dim(R) = 1$ if and only if R is a non-field integral domain. In the next theorem we give the structure of non-submaximal residue finite rings.

Theorem 2.29. *Let R be a residue finite ring which is not submaximal. Then R is a countable principal ideal ring. Moreover, exactly one of the following holds:*

(1) *If $dim(R) = 1$, then $R = U(R)\mathbb{Z}$ and R is algebraic over $\mathbb{Z}$.*

(2) *If $dim(R) = 0$, then R is an artinian ring with nonzero characteristic, say n, which is also integral over $\mathbb{Z}_n$. Moreover, R has only finitely many ideals. In particular, if R is reduced then R is finite.*

Proof. First note that since R is not submaximal then for any nonzero ideal I of R we infer that $R/I \cong \mathbb{Z}_m$ for some natural number m (note, it is clear that all finite rings except $\mathbb{Z}_n$, up to isomorphism, where n is a natural number, are submaximal). This shows that I is principal and therefore R is a principal ideal ring. Now by the above comment we have two cases, either $dim(R) = 1$ or $dim(R) = 0$. First assume that $dim(R) = 1$ and therefore R is a non-field integral domain. Hence we have two cases.

(1) If R has nonzero characteristic, say p (where $p \in \mathbb{P}$), then we infer that for any nonzero ideal I of R we have $R/I \cong \mathbb{Z}_p$, which is absurd, by the first part of the proof (note, in this case $R \cong \mathbb{Z}_p$ which is impossible).

(2) If R has zero characteristic. Then R is a PID with $Ir(R) = \mathbb{P}$, by the first part of the proof. Hence we infer that $R = U(R)\mathbb{Z}$. Also note that by Corollary 2.11, $U(R)$ is algebraic over $\mathbb{Z}$. Therefore $U(R)$ is countable and hence R is countable too. Thus we are done.

Now assume that R is zero-dimensional ring. Thus R is artinian, since R is noetherian (note, every ideal of R is principal). Therefore by [5, Proposition 2.4], R is countable and has a nonzero characteristic, say n, which is also integral over $\mathbb{Z}_n$, by [5, Corollary 2.5]. Moreover, by the first part of the proof either R is a finite ring or every nonzero ideal of R has the form $I = Rm$, where $m|n$. Thus R has only finitely many ideals. Also note that if R is reduced, then by Corollary 2.25, we infer that $J(R) = 0$ and therefore $R \cong \mathbb{Z}_n$ (where n is square free) and hence we are done. ∎

Proposition 2.30. *Let D be an integral domain and S be a multiplicatively closed set in it such that $S \nsubseteq U(D)$. If D_S is not submaximal then the following conditions hold.*

(1) *D has zero characteristic. S is algebraic over $\mathbb{Z}$ and therefore $|S| \leq \aleph_0$. In particular, $\mathbb{Z}$ is not integrally closed in D_S.*

(2) *There exists an infinite subset $\mathcal{M}$ of $Max(D)$ such that $Max(D_S) = \{Q_S \mid Q \in \mathcal{M}\}$. In particular, $\bigcap \mathcal{M} = 0$.*

(3) *For any non-maximal prime ideal* P_S *of* D_S*, either* $Char(\frac{D}{P}) = 0$ *or* $\frac{D_S}{P_S} \cong \frac{D}{P}$.

Proof. Since D_S is not submaximal then by Corollary 2.11, we infer that S is algebraic over Z. Hence if $Char(D) \neq 0$, then $S \subseteq U(D)$ which is absurd. Thus D has zero characteristic. Hence by Corollary 2.14, $\mathbb{Z}$ is not integrally closed in D_S. Now assume that P_S is a maximal ideal in D_S and $P \in Spec(D) \setminus Max(D)$. Thus we have $\frac{D_S}{P_S} \cong (\frac{D}{P})_{\bar{S}} = Frac(\frac{D}{P})$, where $\bar{S} = \{s + P \mid s \in S\}$ and $Frac(\frac{D}{P})$ is the quotient field of $\frac{D}{P}$. Since P is not maximal we infer that $Frac(\frac{D}{P})$ is submaximal by Corollary 2.11, and therefore D_S is submaximal which is absurd. Hence $P \in Max(D)$. Also, note that by Corollary 2.27, $Max(D_S)$ is infinite since D_S is not submaximal; and by Corollary 2.19, we have $J(D_S) = 0$ and therefore $\bigcap \mathcal{M} = 0$. Finally, for part (3), assume that $Char(\frac{D}{P}) = q > 0$, then by part (1), either $\bar{S} \subseteq U(\frac{D}{P})$ and therefore $\frac{D_S}{P_S} \cong (\frac{D}{P})_{\bar{S}} = \frac{D}{P}$ and we are done; or $\bar{S} \nsubseteq U(\frac{D}{P})$ and therefore $(\frac{D}{P})_{\bar{S}}$ is submaximal. Thus D_S is submaximal which is absurd. ∎

Note that in the above proposition clearly for any maximal ideal Q_S of D_S we also have $\frac{D_S}{Q_S} \cong \frac{D}{Q}$. More generally, if R is a ring and S be a multiplicatively closed set in R, then the non-submaximality of R_S implies that every maximal ideal of R_S has the form P_S for some maximal ideal P of R, by the preceding proof. In particular, if R has nonzero characteristic then one can easily see that, by a similar proof, for every prime ideal P_S of R_S we have $\frac{R_S}{P_S} \cong \frac{R}{P}$. The following is a generalization of [7, Theorem 3.2].

Theorem 2.31. *Let* R *be a ring and* S *be a multiplicatively closed set in* R*, such that* R_S *is semi-local. Then at least one of the following holds.*

(1) R_S *is submaximal.*

(2) $Spec(R_S) = Max(R_S) = \{P_S \mid P \in \mathcal{M}\}$*, where* $\mathcal{M}$ *is a finite subset of* $Min(R) \cap Max(R)$.

In particular if (2) *holds and* R_S *is submaximal, then* R *is submaximal too.*

Proof. Assume that R_S is not submaximal, then by the above comment $Max(R_S) = \{P_S \mid P \in \mathcal{M}\}$, where $\mathcal{M}$ is a finite subset of $Max(R)$. But since R_S is a semi-local non-submaximal ring, then we infer that R_S is zero-dimensional, by Corollary 2.27. Hence $\mathcal{M} \subseteq Min(R)$. Now assume that (2) holds and R_S is submaximal. Thus by [7, Theorem 2.26], at least one of the following holds (note R_S is zero-dimensional).

(1) There exists a maximal ideal P_S of R_S, such that R_S/P_S is submaximal. Since $R_S/P_S \cong R/P$, we infer that R/P and therefore R are submaximal.

(2) There exist distinct maximal ideals P_S and Q_S of R_S such that $R_S/P_S \cong R_S/Q_S$. Hence similar to (1), we infer that $R/P \cong R/Q$ and therefore R is submaximal, by [3, Theorem 2.2].

(3) There exist an ideal I_S and a maximal ideal P_S of R_S, such that $(P_S)^2 \subseteq I_S \subseteq P_S$ and $R_S/I_S \cong K[x]/(x^2)$, for some field K. Hence we infer that I is a P-primary ideal in R. Therefore R/I is a local ring with unique prime ideal P/I. Thus $R_S/I_S \cong (R/I)_{\bar{S}} \cong (R/I)_{P/I} = R/I$, where $\bar{S} = \{s + I \mid s \in S\}$, see [16, P. 24, Ex. 7]. Hence R/I and therefore R are submaximal. ∎

The following remark which is a generalization of [4, Corollary 1.15] is interesting.

Remark 2.32. Let $\mathcal{F}$ be the set of all fields, up to isomorphism, which are not submaximal (note, by [4, Corollary 1.15], $\mathcal{F}$ is a set with $|\mathcal{F}| = 2^{\aleph_0}$) and let $\mathcal{D}$ be the class of all integral domains (or reduced rings), up to isomorphism, which are not submaximal. Now for any $D \in \mathcal{D}$ we have the following facts:

(1) For any $M \in Max(D)$, we have $D/M \in \mathcal{F}$.

(2) For any $M, N \in Max(D)$, with $M \neq N$ we have $D/M \ncong D/N$ (For otherwise, by [3, Theorem 2.2], D is submaximal). In other words there exists an injection Φ_D from $Max(D)$ into $\mathcal{F}$, sending M into D/M.

(3) $|Max(D)| \leq |\mathcal{F}|$, by [4, Proposition 2.6] or (2).

(4) $J(D) = 0$, by Corollary 2.19 or 2.25.

(5) Hence we have the natural rings embedding $D \hookrightarrow \prod_{M \in Max(D)} D/M \hookrightarrow \prod_{E \in \mathcal{F}} E$ (i.e., every non-submaximal integral domain (or reduced ring) can be embedded in $\prod_{E \in \mathcal{F}} E$).

Now, for any $D \in \mathcal{D}$, let $RdMax(D) = Im(\Phi_D)$. Two non submaximal integral domains D_1 and D_2 are called $RdMax$-equivalent, if $RdMax(D_1) = RdMax(D_2)$. Now, let $\mathcal{D}'$ be the set of equivalent classes of this relation. We claim that $|\mathcal{D}'| \leq 2^{|\mathcal{F}|} = 2^{2^{\aleph_0}}$ and $\mathcal{F} \subseteq \mathcal{D}'$. To show this, it is clear that $\mathcal{F} \subseteq \mathcal{D}'$. Also note that for any $[D] \in \mathcal{D}'$, the function that send $[D]$ into $RdMax(D)$ is well-defined and one-one from $\mathcal{D}'$ into $P(\mathcal{F})$, the set of all subsets of $\mathcal{F}$. Hence we are done, since $|\mathcal{F}| = 2^{\aleph_0}$, by [4, Corollary 1.15].

3. Submaximal Integral Domains

In [5, Corollary 1.3], it is proved that uncountable fields, are submaximal. The following interesting result is a generalization of this fact.

Theorem 3.1. *Let R be an uncountable UFD, then R is submaximal.*

Proof. If $U(R)$ is uncountable, then we are done by Corollary 2.11. Hence we may assume that $U(R)$ is countable. Thus $|Ir(R)| = |R|$. Now, note that there exists a $p \in Ir(R)$ such that $1 - p \notin U(R)$. Hence let p be an element in $Ir(R)$, such that $1 - p \notin U(R)$ and $q_0 \in Ir(R)$ such that $q_0 | 1 - p$. Thus $q_0 \in A = \{q \in$

$Ir(R) \mid pR + qR = R$ }. Now we show that A must be an uncountable set. Let us assume that A is countable and put $B = \{pq + 1 \mid q \in Ir(R) \setminus A\}$. It is clear that B is an uncountable set and therefore there exists a non-unit element $x \in B$ such that x has an irreducible divisor $q' \in Ir(R) \setminus A$ (note, $U(R)$ and A are countable, thus the set of all elements which are of the form $uq_1 \cdots q_n$, where $u \in U(R)$ and $q_i \in A$, $n \in \mathbb{N} \cup \{0\}$ must be a countable set). Hence $q'R + pR = R$ and $q' \notin A$, which is a contradiction. Thus A must be uncountable. Now for any $q \in A$, $p + (q)$ is a unit in the ring $R/(q)$, hence if there exists $q \in A$ such that $p + (q)$ is not algebraic over the prime subring of $R/(q)$, then by Corollary 2.11, $R/(q)$ and therefore R are submaximal. Consequently, we may assume that for any $q \in A$, $p + (q)$ is algebraic over the prime subring of $R/(q)$. Thus for any $q \in A$, $Z[p] \cap (q) \neq 0$, where Z is the prime subring of R. But $Z[p]$ is a countable set and the set $\{(q)\}_{q \in A}$ is uncountable, thus there exists a nonzero element $f \in Z[p]$ which belongs to an infinite (in fact uncountable) number of (q), where $q \in A$, which is a contradiction. This proves the theorem. ∎

Corollary 3.2. *Let R be a non-submaximal non-field PID, then R is countable and $|Ir(R)| = |R|$.*

Proof. By the above theorem, R and $Ir(R)$ are countable. Now note that if $Ir(R)$ is finite then R is a G-domain and therefore R is submaximal by Corollary 2.19, hence we are done. ∎

Corollary 3.3. *Every localization of an uncountable UFD is submaximal.*

Proposition 3.4. *Let D be an uncountable atomic (or noetherian) domain. Assume that there exists an irreducible element p of D such that $1 - p \notin U(D)$ and every element of $Z[p]$ has at most countably many (irreducible) divisors. Then D is submaximal. In particular, if D is an uncountable atomic (or noetherian) domain such that every element of it has at most countably many (irreducible) divisors, then D is submaximal. Consequently, every uncountable noetherian idf-domain is submaximal.*

Proof. Note that any noetherian integral domain is an atomic domain, and by using the proof of the previous theorem word-for-word, one can easily complete the proof. ∎

Theorem 3.5. *Let R be an uncountable atomic (or noetherian) integral domain with zero characteristic. If every $n \in \mathbb{N}$ has at most countably many (irreducible) divisors, then R is submaximal.*

Proof. We may assume that $U(R)$ is algebraic over $\mathbb{Z}$ and therefore it is countable, by Corollary 2.11. Hence we infer that $|Ir(R)| = |R|$ and therefore $Ir(R)$ is uncountable. Let X be a transcendence basis for R over $\mathbb{Z}$. Now, if there exist a natural number $n > 1$ and $x \in X$ such that $\mathbb{Z} \cap (nx - 1)R = 0$, then R is submaximal by

Corollary 2.13. Hence we may assume that $\mathbb{Z} \cap (nx-1)R \neq 0$ for any natural number $n > 1$ and $x \in X$. Since X is uncountable (note, since R is uncountable we infer that its quotient field, say E, is uncountable too. Hence we have $|X| = tr.deg_{\mathbb{Q}}(E)$, which clearly is uncountable) and the number of ideals of $\mathbb{Z}$ is countable, we infer that there exists an uncountable subset Y of X such that for any $y \in Y$ we have $\mathbb{Z} \cap (ny-1)R = m\mathbb{Z}$ for some fixed natural numbers $n > 1$ and m. Hence for any $y \in Y$ we have $ny - 1|m$. Now we show that m has uncountably many irreducible divisors. Assume that $P = \{q \in Ir(R) \ : q|ny-1, \text{for some } y \in Y\}$. If P is countable, then we infer that $\{ny-1 \ : \ y \in Y\}$ is countable too (note $U(R)$ is countable) which is a contradiction. Hence P is uncountable. Now note that any $q \in P$ is an irreducible divisor of m, i.e., m has uncountable many irreducible divisors, which is a contradiction. Thus for any natural number $n > 1$, the set $\{x \in X \mid \mathbb{Z} \cap (nx-1)R \neq 0\}$ is countable, and therefore R is submaximal, by Corollary 2.13. ∎

For more observations we need the following definition, see [2].

Definition 3.6. An extension $R \subseteq T$ of rings is called residually algebraic extension, if for any prime ideal Q of T, the ring T/Q is algebraic over $R/(Q \cap R)$.

One can easily see that if $R \subseteq T$ is a residually algebraic extension then T must be algebraic over R. Also see [2] for more interesting results about residually algebraic extensions. In particular, see [2, Section 4, b-Maximal subrings] which contains interesting results related to the subject of this paper. The following lemma is needed for the next theorem.

Lemma 3.7. *Let $R \subseteq T$ be a residually algebraic extension of rings where $dim(R) < \infty$. Then T has finite dimension too and we have $dim(T) \leq dim(R)$.*

Proof. Assume that $n = dim(R)$. First suppose that T is an integral domain, and we prove the lemma by induction on n. If $n = 0$, then R is a field and therefore T is a field too, hence we are done. Thus assume that $n \geq 1$ and the lemma holds for any residually algebraic extension (of integral domains) $R \subseteq T$ with $dim(R) < n$. Now assume that $R \subseteq T$ is a residually algebraic extension of integral domains with $dim(R) = n$. Hence for any nonzero prime ideal Q of T, the extension $R/(Q \cap R) \subseteq T/Q$ is also a residually algebraic extension of integral domains and $dim(R/(Q \cap R)) < n$ (note that T is algebraic over R and $Q \neq 0$, hence $Q \cap R \neq 0$). Hence we infer that $dim(T/Q) < n$. This immediately implies that $dim(T) \leq n$ and therefore we are done. Now assume that $R \subseteq T$ be any residually algebraic extension, $dim(R) = n$ and Q be a prime ideal of T. Hence $R/(Q \cap R) \subseteq T/Q$ is a residually algebraic extension of integral domains and $dim(R/(Q \cap R)) \leq n$. Thus by the first part of the proof we infer that $dim(T/Q) \leq n$ and since the latter inequality holds for any prime ideal Q of T, we must have $dim(T) \leq n$. Therefore we are done. ∎

The following is now in order.

Theorem 3.8. *Let R be a noetherian integral domain with $tr.deg_Z(R) = n < \aleph_0$ and assume that X is a transcendence basis for R over Z. Moreover let $Z[X] \subseteq R$ be a residually algebraic extension and at least one of the following holds.*

(1) *If $Char(R) = 0$ and $n \geq 1$, then for any maximal ideal M of R we have $ht(M) \geq n + 1$.*
(2) *If $Char(R) = p > 0$ and $n \geq 2$, then for any maximal ideal M of R we have $ht(M) \geq n$.*

Then R is submaximal.

Proof. (1) Let $x \in X$, if $R(2x - 1) = R$, then we are done, by Corollary 2.11. Hence assume that $R(2x - 1) \neq R$, and let P be a prime ideal of R which is minimal over $R(2x - 1)$. Thus by the Krull's principal ideal theorem we infer that $ht(P) = 1$ and therefore by our assumption P is not a maximal ideal in R. Hence $(0) \subsetneq (2x - 1)\mathbb{Z}[X] \subseteq P \cap \mathbb{Z}[X]$. Thus we have two cases. First, if $(2x - 1)\mathbb{Z}[X] = P \cap \mathbb{Z}[X]$, then $\frac{\mathbb{Z}[X]}{(2x-1)\mathbb{Z}[X]} \subseteq \frac{R}{P}$ and therefore $\frac{1}{2} \in U(\frac{R}{P})$, hence we are done, by Corollary 2.5. Thus we may assume that $Q = P \cap \mathbb{Z}[X] \neq (2x - 1)\mathbb{Z}[X]$. Therefore $ht(Q) \geq 2$ and since $dim(\mathbb{Z}[X]) = n + 1$, we infer that $dim(\frac{\mathbb{Z}[X]}{Q}) \leq n - 1$. But $\frac{\mathbb{Z}[X]}{Q} \subseteq \frac{R}{P}$ is a residually algebraic extension, hence by the above lemma, we conclude that $dim(\frac{R}{P}) \leq n - 1$. Now since $ht(P) = 1$, the latter inequality immediately implies that $ht(M) \leq n$, for any maximal ideal $M \supseteq P$, which is absurd. Thus we are done.

(2) Let $x, y \in X$ and $x \neq y$. If $R(1 - xy) = R$, then we are done by Corollary 2.11. Hence assume that $R(1 - xy) \neq R$ and let P be a prime ideal of R which is minimal over $R(xy - 1)$. Thus by the Krull's principal ideal theorem we infer that $ht(P) = 1$ and therefore by our assumption P is not a maximal ideal in R. Hence $(0) \subsetneq (xy-1)\mathbb{Z}_p[X] \subseteq P \cap \mathbb{Z}_p[X]$. Thus we have two cases. First, if $(xy-1)\mathbb{Z}_p[X] = P \cap \mathbb{Z}_p[X]$, then $\frac{\mathbb{Z}_p[X]}{(xy-1)\mathbb{Z}_p[X]} \subseteq \frac{R}{P}$ and therefore $x + (xy - 1)\mathbb{Z}_p[X] \in U(\frac{R}{P})$, hence we are done by Corollary 2.11 (note, $x + (xy - 1)\mathbb{Z}_p[X]$ is not algebraic over $\mathbb{Z}_p$, since $\mathbb{Z}_p[X]$ is a UFD). Thus assume that $Q = P \cap \mathbb{Z}_p[X] \neq (xy - 1)\mathbb{Z}_p[X]$. Therefore $ht(Q) \geq 2$ and since $dim(\mathbb{Z}_p[X]) = n$ we infer that $dim(\frac{\mathbb{Z}_p[X]}{Q}) \leq n - 2$. But $\frac{\mathbb{Z}_p[X]}{Q} \subseteq \frac{R}{P}$ is a residually algebraic extension, hence by the above lemma we infer that $dim(\frac{R}{P}) \leq n - 2$. Now since $ht(P) = 1$, the latter inequality immediately implies that $ht(M) \leq n - 1$, for any maximal ideal $M \supseteq P$, which is absurd. Thus we are done. ■

Proposition 3.9. *Let a non-singleton $X \neq \emptyset$ be a set of algebraically independent indeterminates in a noetherian ring R over Z, where Z is the prime subring of R. If $Char(R) \in \mathbb{P} \cup \{0\}$ and R is integral over $Z[X]$, then R is submaximal.*

Proof. Let $x, y \in X$ and $x \neq y$. If $R(1-xy) = R$, then we are done. Hence assume that P is a minimal prime ideal of $R(1-xy)$. Hence $ht(P) \leq 1$, by the Krull's principal ideal theorem. Thus $ht(P \cap Z[X]) \leq 1$. But $(1-xy)Z[X]$ is a prime ideal in $Z[X]$, which is contained in $P \cap Z[X]$. So we infer that $(1-xy)Z[X] = P \cap Z[X]$ and therefore $T = Z[X]/(1-xy)Z[X] \subseteq R/P$. Now $\bar{x}$ and $\bar{y}$ are units in T, which are not algebraic over the prime subring of T (note, $Z[X]$ is a UFD). Hence R/P has unit elements which are not algebraic over its prime subring and therefore we are done, by Corollary 2.11. ■

We conclude this article with the following fact about Dedekind domains.

Proposition 3.10. *Let D be an uncountable Dedekind domain with countable set of maximal ideals. Then $U(D)$ is uncountable. In particular, D is submaximal.*

Proof. Let $U(D)$ be countable and seek a contradiction. It is now clear that $Ir(D)$ is uncountable. Hence we infer that the set of principal ideals of D is uncountable. But since D is a Dedekind domain, every nonzero ideal of D is a finite product of prime ideals. Since the set of prime ideals of D is countable, we infer that the set of ideals D is countable too, which is a contradiction. Thus $U(D)$ is uncountable and therefore we are done, by Corollary 2.11. ■

Acknowledgment

The author would like to thank the referee for giving useful comments and corrections.

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