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Existence of Solutions to Quasilinear Schrödinger Equations Involving Critical Sobolev Exponent

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Abstract. By using variational approaches, we study a class of quasilinear Schrödinger equations involving critical Sobolev exponents

$$-\Delta u + V(x)u + \frac{1}{2}\kappa[\Delta(u^2)]u = |u|^{p-2}u + |u|^{2^*-2}u, \quad x \in \mathbb{R}^N,$$

where $V(x)$ is the potential function, $\kappa > 0$, $\max\{(N+3)/(N-2), 2\} < p < 2^* := 2N/(N-2)$, $N \geq 4$. If $\kappa \in [0, \overline{\kappa})$ for some $\overline{\kappa} > 0$, we prove the existence of a positive solution $u(x)$ satisfying $\max_{x \in \mathbb{R}^N} |u(x)| \leq \sqrt{1/(2\kappa)}$.

1. Introduction

This paper is motivated by the recent interests on the following type of quasilinear Schrödinger equations

$$i\psi_t + \Delta\psi - W(x)\psi + \rho(|\psi|^2)\psi + \frac{1}{2}k\Delta|\psi|^2\psi = 0, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $\psi \colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$, $W \colon \mathbb{R}^N \to \mathbb{R}$ is a given potential, k is the nonlocality or diffusion parameter, which can take any sign in plasma physics, ρ is a real function of essentially pure power form. Equation (1.1) has been discussed in the literature in the context of plasma physics [7], the continuum limit of discrete molecular structures and has been shown to posses bright and dark soliton solutions [18, 24].

Here our special interest is the standing wave solutions, i.e., solutions of type $\psi(x,t) = \exp(-iEt)u(x)$, where $E \in \mathbb{R}$ and $u > 0$ is a real function. Note that ψ satisfies (1.1) if and only if the function $u(x)$ solves the following equation of elliptic type with the formal variational structure

$$-\Delta u + V(x)u - \frac{1}{2}k[\Delta(u^2)]u = h(u), \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $V(x) = W(x) - E$ is the new potential function, h is the new nonlinearity.

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The existence of positive or sign-changing solutions for (1.2) with $k > 0$ has been studied extensively in recent years. In [5] Brüll and Lange studied the existence of ground states for (1.2) with $V(x) = \alpha$ and $h(u) = \beta u^p$ in one dimensional space, where α and β are positive constants. In [19] Poppenberg, Schmitt and Wang studied (1.2) with subcritical growth through a constrained minimization argument. A general existence result for (1.2) was derived in [15] by Liu, Wang and Wang. The idea in [15] is to make a change of variables $v = f^{-1}(u)$, where f is defined by ODE:

$$f'(t) = \frac{1}{\sqrt{1 + kf^2(t)}}, \quad t \in [0, +\infty), \tag{1.3}$$

and $f(t) = -f(-t)$, $t \in (-\infty, 0]$. By this way, they reduced (1.2) to a semilinear one and proved the existence of a positive solution in an Orlicz space via mountain pass theorem. The method of changing of variable was also used by Colin and Jeanjean in [6], but the difference lies in that they used the usual Sobolev space $H^1(\mathbb{R}^N)$ as the working space. Recently, in [21], Shen and Wang introduced the change of known variables $s = G^{-1}(t)$ for $t \in [0, +\infty)$ and $G^{-1}(t) = -G^{-1}(-t)$ for $t \in (-\infty, 0)$, where

$$G(s) = \int_0^s \sqrt{1 + kt^2}\, dt. \tag{1.4}$$

Then, using variational methods, they established the existence of nontrivial solutions for (1.1) with subcritical growth.

In [15], the authors pointed out that $2(2^*)$ behaves like a critical exponent for (1.2) and proposed an open problem whether (1.2) has nontrivial solutions for $h(u) = |u|^{2(2^*)-2}u$ when $k > 0$.

For such kind of problems with "critical growth", Silva and Vieira in [22] established the existence of solutions for asymptotically periodic quasilinear Schrödinger equations (1.2) with the nonlinearity $h(u) = |u|^{p-2}u$ replaced by a general nonlinearity $K(x)u^{2(2^*)-1} + g(x, u)$. In [14], the authors proved the existence of one positive and one sign-changing ground state solutions with $h(u) = |u|^{p-2}u + |u|^{2(2^*)-2}u$, $4 < p < 2(2^*)$ under some assumptions on the potential $V(x)$. In [10], He and Li study the existence, concentration and multiplicity of weak solutions to equation with $h(u) = W(x)u^{q-1} + u^{2(2^*)-1}$, $4 < q < 2(2^*)$ via minimax theorems and Ljusternik-Schnirelmann theory. For more results for problems with critical growth termp, we refer to [8, 12, 17, 23, 25–27].

Noting that most of the studies of recent papers on problem (1.2) mainly deal with $k > 0$, in this paper, we focus on the case $k < 0$. The difficulty for $k < 0$ is that neither the change of variable (1.3) nor (1.4) is suitable because $1 + kf^2(t)$ or $1 + kt^2$ may be negative. To overcome this difficulty, using variational methods combined with perturbation arguments, Alves, Wang and Shen in [3] proved the existence of positive solution of (1.2) with $h(u) = |u|^{p-2}u$, $2 < p < 2^*$. Then, another question arises: does

2^* is a critical exponent for (1.2) when $k < 0$? It seems that under their arguments, 2^* indeed behaves like a critical exponent for (1.2). Later, in [23], Wang considered the case $h(u) = \lambda|u|^{p-2}u + |u|^{q-2}u$ with $q \geq 2^*$ and $2 < p < 2^*$ and proved that there exist some $k_1 > 0$ and $\lambda_1 > 0$ such that for all $k \in (-k_1, 0)$ and $\lambda \in (0, \lambda_1)$, (1.2) has a positive solution.

For simplicity, we denote $\kappa = -k$ in the following and consider the problem

$$-\Delta u + V(x)u + \frac{1}{2}\kappa[\Delta(u^2)]u = |u|^{p-2}u + |u|^{2^*-2}u, \quad x \in \mathbb{R}^N, \tag{1.5}$$

where $\kappa > 0$, $2 < p < 2^*$, $N \geq 4$.

We assume the following conditions on $V(x)$:

(V1) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ and there exists some $V_0 > 0$ such that $0 < V_0 \leq V(x)$, $x \in \mathbb{R}^N$.

(V2) There is a constant V_∞ such that

$$\lim_{|x|\to\infty} V(x) = V_\infty, \quad V(x) \leq V_\infty, \quad V(x) \not\equiv V_\infty, \quad x \in \mathbb{R}^N.$$

We have the following main result.

Theorem 1.1. *Suppose that* (V1)–(V2)*,* $N \geq 4$ *and* $p \in (\max\{(N+3)/(N-2), 2\}, 2^*)$*. Then, there exists* $\overline{\kappa} > 0$ *such that for* $\kappa \in (0, \overline{\kappa}]$*,* (1.5) *has a positive solution* u *satisfying* $\max_{x\in\mathbb{R}^N} |u(x)| \leq \sqrt{1/(2\kappa)}$*.*

We point out that the conclusion of this paper is a supplement to the recent result in [23], where, using variational methods combined with perturbation arguments, the existence of nontrivial solutions of (1.5) with critical or supercritical exponent were established. For $\kappa = 0$, in [16], Miyagaki proved the existence of nontrivial solutions of the following equation:

$$-\Delta u + V(x)u = \lambda|u|^{p-2}u + |u|^{2^*-2}u, \quad x \in \mathbb{R}^N,$$

where $V(x)$ satisfies (V_0) and $V(x) \to +\infty$ as $|x| \to \infty$, $\lambda > 0$, either $4 < p < 2^*$ and $N = 3$ or $2 < p < 2^*$ and $N \geq 4$. For further related results we refer to the papers [1, 2].

2. Reformulation of the problem

Note that (1.2) is the Euler-Lagrange equation associated to the natural energy functional

$$\overline{I}_\kappa(u) = \frac{1}{2}\int_{\mathbb{R}^N}(1-\kappa u^2)|\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2\,dx - \frac{1}{p}\int_{\mathbb{R}^N}|u|^p\,dx - \frac{1}{2^*}\int_{\mathbb{R}^N} u^{2^*}\,dx. \tag{2.1}$$

From the variational point of view, the first difficulty that we have to deal with is to find some proper Sobolev space since (2.1) is not well defined in $H^1(\mathbb{R}^N)$ for $N \geq 3$ and $\kappa \neq 0$.

Besides, another difficulty is how to guarantee the positiveness of the principal part, i.e., $1-\kappa u^2>0$.

In order to solve these difficulties, we first establish a nontrivial solution for a modified quasilinear Schrödinger equation. Precisely, we consider the existence of nontrivial solutions for the following quasilinear Schrödinger equation

$$-\operatorname{div}(g^2(u)\nabla u)+g(u)g'(u)|\nabla u|^2+V(x)u=|u|^{p-2}u+|u|^{2^*-2}u,\quad x\in\mathbb{R}^N, \tag{2.2}$$

where $g(t)\colon[0,+\infty)\to\mathbb{R}$ is given by

$$g(t)=\begin{cases}\sqrt{1-\kappa t^2} & \text{if } 0\le t<\sqrt{1/(2\kappa)},\\ \frac{1}{4\sqrt{2}\kappa t^2}+\frac{\sqrt{2}}{4} & \text{if } \sqrt{1/(2\kappa)}\le t.\end{cases}$$

Setting $g(t)=g(-t)$ for all $t\le 0$, it follows that $g\in C^1(\mathbb{R},(\sqrt{2}/4,1])$, g is an even function, increases in $(-\infty,0)$ and decreases in $[0,+\infty)$. Clearly, if $0\le u(x)<\sqrt{1/(2\kappa)}$, $x\in\mathbb{R}^N$, then equation (2.2) turns into (1.5). So, our goal is to prove the existence of a nontrivial solution u of (2.2) satisfying $\sup_{x\in\mathbb{R}^N}|u(x)|\le\sqrt{1/(2\kappa)}$.

Now, we note that (2.2) is the Euler-Lagrange equation associated to the natural energy functional

$$I_\kappa(u)=\frac{1}{2}\int_{\mathbb{R}^N}g^2(u)|\nabla u|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x)|u|^2\,dx-\frac{1}{p}\int_{\mathbb{R}^N}|u|^p\,dx-\frac{1}{2^*}\int_{\mathbb{R}^N}u^{2^*}\,dx.$$

In what follows, we set

$$G(t)=\int_0^t g(s)\,ds$$

and we observe that inverse function $G^{-1}(t)$ exists and it is an odd function. Moreover, it is very important to observe that $G,G^{-1}\in C^2(\mathbb{R})$.

By simple calculations, we get the following important properties involving functions g and G^{-1} which will be used later on.

Lemma 2.1. (1) $\lim_{t\to 0}G^{-1}(t)/t=1$*;*

(2) $\lim_{t\to\infty}G^{-1}(t)/t=2\sqrt{2}$*;*

(3) $1\le G^{-1}(t)/t\le 2\sqrt{2}$ *for all* $t\in\mathbb{R}$*;*

(4) $-1\le\frac{t}{g(t)}g'(t)\le 0$ *for all* $t\in\mathbb{R}$*;*

(5) $t/G^{-1}(t)\ge g(G^{-1}(t))$ *for all* $t\in\mathbb{R}$*.*

Proof. It follows from the definition of $g(t)$ that

$$\lim_{t\to 0}\frac{G^{-1}(t)}{t}=\lim_{t\to 0}\frac{1}{g(G^{-1}(t))}=\frac{1}{g(0)}=1$$

and

$$\lim_{t\to\infty}\frac{G^{-1}(t)}{t}=\lim_{t\to\infty}\frac{1}{g(G^{-1}(t))}=2\sqrt{2}.$$

Thus (1) and (2) is proved. Since $g(t)$ is decreasing in $|t|$, we have

$$G(t)\geq tg(t)\geq 0,\quad t\geq 0\quad\text{and}\quad G(t)<tg(t)<0,\quad t<0,$$

which implies (5). Moreover, we have

$$\frac{d}{dt}\left[\frac{G^{-1}(t)}{t}\right]=\frac{t-G^{-1}(t)g(G^{-1}(t))}{g(G^{-1}(t))t^2}\begin{cases}\geq 0 & \text{if } t\geq 0,\\ <0 & \text{if } t<0.\end{cases}$$

Combining (1) and (2), we get (3). Finally, we prove (4). We only consider $t\geq 0$ since the case $t<0$ can be proved in a similar way. The second inequality in (4) is clear. For $0\leq t<\sqrt{1/(2\kappa)}$, direct calculations show

$$\frac{t}{g(t)}g'(t)=-\frac{\kappa}{t^{-2}-\kappa}\geq -1, \tag{2.3}$$

while for $t\geq\sqrt{1/(2\kappa)}$, we have

$$\frac{t}{g(t)}g'(t)=-\frac{2}{1+2\kappa t^2}\geq -1. \tag{2.4}$$

Item (4) is an immediate consequence of (2.3) and (2.4). $\square$

Lemma 2.2. *For $t>0$, we have*

$$\begin{aligned}&t^{2^*}-\left(\frac{\sqrt{2}}{4}G^{-1}(t)\right)^{2^*}\\ &\leq(2\sqrt{2})^{2^*-1}\left\{\frac{(2+\pi)2^*}{8}\left(\frac{4+\pi}{4\sqrt{2}}\right)^{2^*-1}+\left[1-\left(\frac{\sqrt{2}}{4}\right)^{2^*}\right]\frac{1}{\sqrt{2}}\right\}\frac{1}{\sqrt{\kappa}}t^{2^*-1}\\ &:=\frac{C}{\sqrt{\kappa}}t^{2^*-1},\end{aligned} \tag{2.5}$$

where $C>0$ is independent of κ.

Proof. For $s=G^{-1}(t)>\sqrt{1/(2\kappa)}$, $t>0$, we have

$$\begin{aligned}G(s)&=\int_0^{\sqrt{1/(2\kappa)}}\sqrt{1-\kappa t^2}\,dt+\int_{\sqrt{1/(2\kappa)}}^{s}\left[\frac{1}{4\sqrt{2}\kappa t^2}+\frac{\sqrt{2}}{4}\right]dt\\ &=\frac{\sqrt{2}}{4}s-\frac{1}{4\sqrt{2}\kappa s}+\frac{2+\pi}{8\sqrt{\kappa}}.\end{aligned}$$

By mean value theorem, there exists $0<\theta<1$ such that

$$
\begin{aligned}
G(s)^{2^*}-\left(\frac{\sqrt{2}}{4}s\right)^{2^*} &= 2^*\left[\frac{\sqrt{2}}{4}s+\theta\left(\frac{2+\pi}{8\sqrt{\kappa}}-\frac{1}{4\sqrt{2}\kappa s}\right)\right]^{2^*-1}\left[\frac{2+\pi}{8\sqrt{\kappa}}-\frac{1}{4\sqrt{2}\kappa s}\right]\\
&\leq \frac{(2+\pi)2^*}{8\sqrt{\kappa}}\left[\frac{\sqrt{2}}{4}s+\frac{2+\pi}{8\sqrt{\kappa}}-\frac{1}{4\sqrt{2}\kappa s}\right]^{2^*-1}\\
&\leq \frac{(2+\pi)2^*}{8\sqrt{\kappa}}\left(\frac{4+\pi}{4\sqrt{2}}\right)^{2^*-1}s^{2^*-1}.
\end{aligned}
$$

Thus, from Lemma 2.1(3), we have

$$
t^{2^*}-\left(\frac{\sqrt{2}}{4}G^{-1}(t)\right)^{2^*}\leq \frac{(2+\pi)2^*(2\sqrt{2})^{2^*-1}}{8\sqrt{\kappa}}\left(\frac{4+\pi}{4\sqrt{2}}\right)^{2^*-1}t^{2^*-1}. \tag{2.6}
$$

On the other hand, for $G^{-1}(t)\leq\sqrt{1/(2\kappa)}$, $t>0$, from Lemma 2.1(3), we have

$$
\begin{aligned}
t^{2^*}-\left(\frac{\sqrt{2}}{4}G^{-1}(t)\right)^{2^*} &\leq \left[1-\left(\frac{\sqrt{2}}{4}\right)^{2^*}\right]\frac{1}{\sqrt{2\kappa}}G^{-1}(t)^{2^*-1}\\
&\leq \left[1-\left(\frac{\sqrt{2}}{4}\right)^{2^*}\right](2\sqrt{2})^{2^*-1}\frac{1}{\sqrt{2\kappa}}t^{2^*-1}.
\end{aligned} \tag{2.7}
$$

Combining (2.6) and (2.7), we get (2.5). □

Now, we introduce the following change variable

$$
v=G(u)=\int_0^u g(s)\,ds.
$$

We observe that functional I_κ can be written in the following way

$$
\begin{aligned}
J_\kappa(v) = {} & \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x)|G^{-1}(v)|^2\,dx\\
&-\frac{1}{p}\int_{\mathbb{R}^N}|G^{-1}(v)|^p\,dx-\frac{1}{2^*}\int_{\mathbb{R}^N}|G^{-1}(v)|^{2^*}\,dx.
\end{aligned}
$$

From Lemma 2.1, J_κ is well defined in $H^1(\mathbb{R}^N)$, $J_\kappa\in C^1(H^1(\mathbb{R}^N),\mathbb{R})$ and

$$
\begin{aligned}
&\langle J_\kappa'(v),\psi\rangle\\
&=\int_{\mathbb{R}^N}\left[\nabla v\nabla\psi+V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\psi-\frac{|G^{-1}(v)|^{p-2}G^{-1}(v)}{g(G^{-1}(v))}\psi-\frac{|G^{-1}(v)|^{2^*-2}G^{-1}(v)}{g(G^{-1}(v))}\psi\right]dx
\end{aligned} \tag{2.8}
$$

for all $v,\psi\in H^1(\mathbb{R}^N)$.

Therefore, in order to find a nontrivial solution of (2.2), it suffices to study the existence of nontrivial solutions of the following equation

$$
-\Delta v+V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}-\frac{|G^{-1}(v)|^{p-2}G^{-1}(v)}{g(G^{-1}(v))}-\frac{|G^{-1}(v)|^{2^*-2}G^{-1}(v)}{g(G^{-1}(v))}=0,\quad x\in\mathbb{R}^N. \tag{2.9}
$$

3. Existence result of modified problem

In this section, we consider the existence of positive solutions of (2.9). From variational methods, we will study the positive critical points of the following functional

$$\begin{aligned} J_\kappa^+(v) = &\frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2\,dx \\ &- \frac{1}{p}\int_{\mathbb{R}^N} |G^{-1}(v)^+|^p\,dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |G^{-1}(v)^+|^{2^*}\,dx. \end{aligned}$$

In order to avoid cumbersome notation, in the rest of this paper, we still denote $J_\kappa(v)$, $|G^{-1}(v)|^p$ and $|G^{-1}(v)|^{2^*}$ by $J_\kappa^+(v)$, $|G^{-1}(v)^+|^p$ and $|G^{-1}(v)^+|^{2^*}$, respectively. Therefore, if v is a nontrivial solution of (2.9), by Strong maximum principle [9], v is positive.

By (V1) and (V2), the norm

$$\|v\| = \left[\int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)v^2)\,dx\right]^{1/2}$$

is equivalent to the usual norm in $H^1(\mathbb{R}^N)$.

Now, we establish the geometric hypotheses of the Mountain Pass Theorem for J_κ.

Lemma 3.1. *For $2 < p < 2^*$, there exist $\rho_0, a_0 > 0$, such that $J_\kappa(v) \geq a_0$ for $\|v\| = \rho_0$. Moreover, there exists $e \in H^1(\mathbb{R}^N)$ such that $J_\kappa(e) < 0$.*

Proof. By Lemma 2.1(3) and Sobolev embedding,

$$\begin{aligned} J_\kappa(v) &\geq \frac{1}{2}\int_{\mathbb{R}^N} \left[|\nabla v|^2 + V(x)|G^{-1}(v)|^2\right]\,dx - \frac{1}{p}\int_{\mathbb{R}^N} |G^{-1}(v)|^p\,dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |G^{-1}(v)|^{2^*}\,dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|v|^2\,dx - C\int_{\mathbb{R}^N} |v|^p\,dx - C\int_{\mathbb{R}^N} |v|^{2^*}\,dx \\ &= \frac{1}{2}\|v\|^2 - C\|v\|^p - C\|v\|^{2^*}. \end{aligned}$$

Thereby, by choosing ρ_0 small, we get

$$a_0 = \frac{1}{2}\rho_0^2 - C\rho_0^p - C\rho_0^{2^*} > 0,$$

and so,

$$J_\kappa(v) \geq a_0 \quad \text{for } \|v\| = \rho_0.$$

In order to prove the existence of $e \in H^1(\mathbb{R}^N)$ such that $J_\kappa(e) < 0$, we fix $\varphi \in C_0^\infty(\mathbb{R}^N, [0,1])$ with $\operatorname{supp}\varphi \subset B_1(0)$ and show that $J_\kappa(t\varphi) \to -\infty$ as $t \to \infty$, because the result follows by taking $e = t\varphi$ with t large enough. By Lemma 2.1(3),

$$J_\kappa(t\varphi) \leq Ct^2\int_{\mathbb{R}^N} (|\nabla\varphi|^2 + V(x)\varphi^2)\,dx - Ct^p\int_{\mathbb{R}^N} \varphi^p\,dx.$$

Since $p > 2$, it follows that $J_\kappa(t\varphi) \to -\infty$ as $t \to \infty$. $\square$

Now, in view of Lemma 3.1, applying a version of Mountain Pass Theorem without $(\mathrm{PS})_c$ condition due to Ambrosetti-Rabinowitz [20], it follows that there exists a $(\mathrm{PS})_c$ sequence $\{v_n\} \subset H^1(\mathbb{R}^N)$, i.e., a sequence such that $J_\kappa(v_n) \to c_\kappa$ and $J'_\kappa(v_n) \to 0$, where c_κ is the Mountain Pass level of J characterized by

$$c_\kappa = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J_\kappa(\gamma(t)), \tag{3.1}$$

where $\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, J_\kappa(\gamma(1)) < 0, \gamma(1) \neq 0\}$.

Lemma 3.2. *There exists $\widetilde{\kappa} > 0$ such that for $\kappa \in (0, \widetilde{\kappa}]$, the minimax level c_κ in* (3.1) *satisfies*

$$c_\kappa < \frac{1}{N} \left(\frac{\sqrt{2}}{4} \right)^N S^{N/2},$$

where S is the best constant for the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$.

Proof. It suffices to show that there exists $v_0 \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that

$$\max_{t \geq 0} J_\kappa(tv_0) < \frac{1}{N} \left(\frac{\sqrt{2}}{4} \right)^N S^{N/2}.$$

We follow the strategy used in [4]. First, we choose a cut-off function $\varphi \in C_0^\infty(\mathbb{R}^N, [0,1])$ such that $\varphi \equiv 1$ on $B_1(0)$ and $\varphi \equiv 0$ on $\mathbb{R}^N \setminus B_2(0)$ and $0 \leq \varphi(x) \leq 1$ on $B_2(0)$. Let $\psi_\kappa(x) = \varphi(x) w_\kappa(x)$, where

$$w_\kappa = \frac{[N(N-2)\kappa^2]^{(N-2)/4}}{(\kappa^2 + |x|^2)^{(N-2)/2}}.$$

It is known that w_κ satisfies the equation $-\Delta u = u^{2^*-1}$ in $\mathbb{R}^N$ and

$$\int_{\mathbb{R}^N} |\nabla w_\kappa|^2 \, dx = \int_{\mathbb{R}^N} w_\kappa^{2^*} = S^{N/2}, \quad \int_{B_1(0)} |\nabla w_\kappa|^2 \, dx \leq \int_{B_1(0)} w_\kappa^{2^*}, \tag{3.2}$$

$$\int_{\mathbb{R}^N \setminus B_1(0)} |\nabla w_\kappa|^2 = O(\kappa^{N-2}) \quad \text{as } \kappa \to 0. \tag{3.3}$$

Thus, if we define the function $v_\kappa(x) = \psi_\kappa / |\psi_\kappa|_{2^*}$, then, by (3.2) and (3.3), as $\kappa \to 0$, we have

$$\int_{\mathbb{R}^N} |\nabla v_\kappa|^2 \, dx = S + O(\kappa^{N-2}), \quad |v_\kappa|_{2^*-1}^{2^*-1} = O(\kappa^{(N-2)/2}), \quad \text{if } N \geq 3$$

and

$$|v_\kappa|_2^2 = \begin{cases} O(\kappa) & \text{if } N = 3, \\ O(\kappa^2 |\log \kappa|) & \text{if } N = 4, \\ O(\kappa^2) & \text{if } N \geq 5. \end{cases} \tag{3.4}$$

In view of Lemma 3.1, we also have $\lim_{t\to+\infty} J_\kappa(tv_\kappa) = -\infty$ and there exists $t_\kappa > 0$ such that $J_\kappa(t_\kappa v_\kappa) = \max_{t>0} J_\kappa(tv_\kappa)$. We claim that there exist positive constants $t_0, t_1 > 0$ such that $t_0 \le t_\kappa \le t_1$ for some $\kappa_0 > 0$ with $0 < \kappa < \kappa_0$. First, we prove that t_κ is bounded from below by a positive constant. Otherwise, we could find a sequence $\kappa_n \to 0$ such that $t_{\kappa_n} \to 0$. Up to a subsequence (still denote by κ_n), we have $t_{\kappa_n} v_{\kappa_n} \to 0$. Therefore, $0 < c_\kappa \le \sup_{t\ge 0} J_\kappa(t_{\kappa_n} v_{\kappa_n}) \to 0$, which is a contradiction. On the other hand, from Lemma 2.1(3), we have

$$\begin{aligned}
c_\kappa &\le J_\kappa(t_\kappa v_\kappa)\\
&\le \frac{1}{2}t_\kappa^2 \int_{B_2(0)} |\nabla v_\kappa|^2\,dx + \frac{1}{2}\int_{B_2(0)} V(x)|G^{-1}(t_\kappa v_\kappa)|^2\,dx - \frac{1}{2^*}\int_{B_2(0)} |G^{-1}(t_\kappa v_\kappa)|^{2^*}\,dx\\
&\le \frac{1}{2}t_\kappa^2 \int_{B_2(0)} |\nabla v_\kappa|^2\,dx + 4V_\infty t_\kappa^2 \int_{B_2(0)} |v_\kappa|^2\,dx - \frac{1}{2^*}t_\kappa^{2^*}\int_{B_2(0)} |v_\kappa|^{2^*}\,dx\\
&\le Ct_\kappa^2\|v_\kappa\|^2 - \frac{1}{2^*}t_\kappa^{2^*}\\
&\le Ct_\kappa^2\left[S^2 + O(\kappa)\right] - \frac{1}{2^*}t_\kappa^{2^*}
\end{aligned}$$

which implies the claim for $0 < \kappa < \kappa_0$.

Now, we have

$$\begin{aligned}
J_\kappa(t_\kappa v_\kappa) \le{}& \frac{1}{2}t_\kappa^2 \int_{B_2(0)} |\nabla v_\kappa|^2\,dx + 4V_\infty t_\kappa^2 \int_{B_2(0)} |v_\kappa|^2\,dx\\
&+ C\frac{1}{\sqrt{\kappa}}t_\kappa^{2^*-1}\int_{B_2(0)} |v_\kappa|^{2^*-1}\,dx - \frac{1}{p}t_\kappa^p \int_{B_2(0)} |v_\kappa|^p\,dx - \frac{1}{2^*}(2\sqrt{2})^{2^*}t_\kappa^{2^*}.
\end{aligned}\tag{3.5}$$

Let $A = \int_{\mathbb{R}^N} |\nabla v_\kappa|^2\,dx$ and $B = (2\sqrt{2})^{2^*}$, considering the function $\xi\colon [0,+\infty) \to \mathbb{R}$ given by

$$\xi(t) = \frac{1}{2}t^2 A - \frac{1}{2^*}Bt^{2^*},$$

we have $t_0 = (AB^{-1})^{1/(2^*-2)}$ is the maximum point of ξ and

$$\xi(t_0) = \frac{1}{N}\left(\frac{\sqrt{2}}{4}\right)^N A^{N/2}.$$

Thus, from (3.5) and recalling $t_0 \le t_\kappa \le t_1$, we deduce that

$$J_\kappa(t_\kappa v_\kappa) \le \frac{1}{N}\left(\frac{\sqrt{2}}{4}\right)^N \left[S + O(\kappa^{N-2})\right]^{N/2} + C|v_\kappa|_2^2 + C\frac{1}{\sqrt{\kappa}}|v_\kappa|_{2^*-1}^{2^*-1} - C\int_{B_2(0)} |v_\kappa|^p\,dx.$$

Therefore, by using the following inequality:

$$(a+b)^r \le a^r + r(a+b)^{r-1}b \quad \text{for any } a, b > 0,\ r \ge 1,$$

we have

$$(3.6)\quad J_\kappa(t_\kappa v_\kappa) \le \frac{1}{N}\left(\frac{\sqrt{2}}{4}\right)^N S^{N/2} + C|v_\kappa|_2^2 + C\frac{1}{\sqrt{\kappa}}|v_\kappa|_{2^*-1}^{2^*-1} - C\int_{B_2(0)} |v_\kappa|^p\,dx + O(\kappa^{N-2}).$$

For $|x| \le \kappa$ and $0 < \kappa \le \kappa_0 < 2$, we have

$$\int_{B_2(0)} |v_\kappa|^p\,dx \ge \int_{B_\kappa(0)} |v_\kappa|^p\,dx \ge C\kappa^{(2-N)p/2+N}.$$

Therefore, from (3.6), we get

$$J_\kappa(t_\kappa v_\kappa) \le \frac{1}{N}\left(\frac{\sqrt{2}}{4}\right)^N S^{N/2} + C|v_\kappa|_2^2 + C\frac{1}{\sqrt{\kappa}}|v_\kappa|_{2^*-1}^{2^*-1} - C\kappa^{(2-N)p/2+N} + O(\kappa^{N-2}).$$

Let

$$B(\kappa) = C|v_\kappa|_2^2 + C\kappa^{(N-3)/2} - C\kappa^{(2-N)p/2+N} + O(\kappa^{N-2}).$$

We will prove our result if we show that $B(\kappa) < 0$ for small κ. In fact, by (3.4), if $N \ge 4$ and $p > \max\{(N+3)/(N-2), 2\}$, the result follows. □

Lemma 3.3. *The* $(\mathrm{PS})_{c_\kappa}$ *sequence is bounded in* E*.*

Proof. Let $\{v_n\}$ be a $(\mathrm{PS})_{c_\kappa}$ sequence, that is,

$$(3.7)\quad \begin{aligned} J_\kappa(v_n) &= \frac{1}{2}\int_{\mathbb{R}^N} \left[|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2\right]\,dx - \frac{1}{p}\int_{\mathbb{R}^N} |G^{-1}(v_n)|^p\,dx \\ &\quad - \frac{1}{2^*}\int_{\mathbb{R}^N} |G^{-1}(v_n)|^{2^*}\,dx \\ &= c_\kappa + o(1) \end{aligned}$$

and

$$(3.8)\quad \begin{aligned} \langle J_\kappa'(v_n), \psi\rangle &= \int_{\mathbb{R}^N} \left[\nabla v_n \nabla\psi + V(x)\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}\psi\right]\,dx \\ &\quad - \int_{\mathbb{R}^N} \frac{|G^{-1}(v_n)|^{p-2}G^{-1}(v_n) + |G^{-1}(v_n)|^{2^*-2}G^{-1}(v_n)}{g(G^{-1}(v_n))}\psi\,dx \\ &= o(1)\|\psi\|. \end{aligned}$$

By Lemma 2.1(4), we have

$$\begin{aligned} \int_{\mathbb{R}^N} \left|\nabla[G^{-1}(v_n)g(G^{-1}(v_n))]\right|^2 &= \int_{\mathbb{R}^N} \left|1 + \frac{G^{-1}(v_n)g'(G^{-1}(v_n))}{g(G^{-1}(v_n))}\right|^2 |\nabla v_n|^2\,dx \\ &\le \int_{\mathbb{R}^N} |\nabla v_n|^2\,dx. \end{aligned}$$

Recalling $|G^{-1}(v_n)g(G^{-1}(v_n))| \leq |v_n|$, it follows that $G^{-1}(v_n)g(G^{-1}(v_n)) \in H^1(\mathbb{R}^N)$. By choosing $\psi = G^{-1}(v_n)g(G^{-1}(v_n))$ as a test function in (3.8) and combining (3.7), we get

$$\begin{aligned} pc_\kappa + o(1) + o(1)\|v_n\| &= pJ_\kappa(v_n) - \langle J'_\kappa(v_n), G^{-1}(v_n)g(G^{-1}(v_n))\rangle \\ &\geq \int_{\mathbb{R}^N} \left[\frac{p-2}{2} - \frac{G^{-1}(v_n)g'(G^{-1}(v_n))}{g(G^{-1}(v_n))}\right] |\nabla v_n|^2\, dx \\ &\geq \frac{p-2}{2}\|v_n\|^2. \end{aligned}$$

The proof of the lemma is complete. □

Lemma 3.4. *Let $\{v_n\}$ be a $(\mathrm{PS})_{c_\kappa}$ sequence with $c_\kappa < \frac{1}{N}\big(\frac{\sqrt{2}}{4}\big)^N S^{N/2}$, then there is a sequence $\{z_n\} \subset \mathbb{R}^N$ and $R > 0$, $\beta > 0$ such that*

$$\int_{B_R(z_n)} v_n^2\, dx \geq \beta. \tag{3.9}$$

Proof. Suppose by contradiction (3.9) does not hold. Then by Lions compactness lemma [13] it follows that

$$\int_{\mathbb{R}^N} |v_n|^p\, dx = o(1), \quad \forall\, p \in (2, 2^*). \tag{3.10}$$

Thus, by Lemma 2.1(3), we have

$$\int_{\mathbb{R}^N} |G^{-1}(v_n)|^p\, dx = o(1),\ \int_{\mathbb{R}^N} \frac{|G^{-1}(v_n)|^{p-2}G^{-1}(v_n)v_n}{g(G^{-1}(v_n))}\, dx = o(1),\ \forall\, p \in (2, 2^*). \tag{3.11}$$

Therefore, in view of $J_\kappa(v_n) = c_\kappa + o(1)$ and (3.11), we have

$$\frac{1}{2}\int_{\mathbb{R}^N} (|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2)\, dx = \frac{1}{2^*}\int_{\mathbb{R}^N} |G^{-1}(v_n)|^{2^*}\, dx + c_\kappa + o(1) \tag{3.12}$$

and from $\langle J'_\kappa(v_n), v_n\rangle = o(1)\|v_n\|$, (3.11) and Lemma 2.1(3) and (5), it follows that

$$\int_{\mathbb{R}^N} \left[|\nabla v_n|^2 + V(x)\frac{G^{-1}(v_n)v_n}{g(G^{-1}(v_n))}\right] dx = \int_{\mathbb{R}^N} \frac{|G^{-1}(v_n)|^{2^*-2}G^{-1}(v_n)v_n}{g(G^{-1}(v_n))}\, dx + o(1). \tag{3.13}$$

We claim that

$$\int_{\mathbb{R}^N} V(x)\left[\frac{G^{-1}(v_n)v_n}{g(G^{-1}(v_n))} - |G^{-1}(v_n)|^2\right] dx = o(1). \tag{3.14}$$

In fact, by Lemma 2.1(1), for any $\varepsilon > 0$, there exists $\delta > 0$, such that for $|v_n(x)| < \delta$, $\forall\, n$, there holds

$$\left|\frac{G^{-1}(v_n)v_n}{g(G^{-1}(v_n))} - |G^{-1}(v_n)|^2\right| < \varepsilon v_n^2. \tag{3.15}$$

On the other hand, by Lemma 2.1(3), for $|v_n(x)| \geq \delta$, $\forall\, n$, we have

$$\left| \frac{G^{-1}(v_n)v_n}{g(G^{-1}(v_n))} - |G^{-1}(v_n)|^2 \right| \leq C|v_n|^2 \ leqC\delta^{2-p}|v_n|^p. \tag{3.16}$$

Combining (3.10), (3.15) and (3.16), recalling $V(x)$ is bounded, we get (3.14).

Next, we prove

$$\int_{\mathbb{R}^N} \left[|G^{-1}(v_n)|^{2^*} - (2\sqrt{2})^{2^*}|v_n|^{2^*} \right] dx = o(1), \tag{3.17}$$

$$\int_{\mathbb{R}^N} \left[\frac{|G^{-1}(v_n)|^{2^*-2}G^{-1}(v_n)v_n}{g(G^{-1}(v_n))} - (2\sqrt{2})^{2^*}|v_n|^{2^*} \right] dx = o(1). \tag{3.18}$$

This is a consequence of (3.10) and Lemma 2.1(2).

Let $\ell \geq 0$ be such that

$$\int_{\mathbb{R}^N} \left[|\nabla v_n|^2 + V(x)\frac{G^{-1}(v_n)v_n}{g(G^{-1}(v_n))} \right] dx \to \ell.$$

Then, from (3.13), (3.17) and (3.18), we have

$$\begin{aligned} \ell &= \int_{\mathbb{R}^N} \frac{|G^{-1}(v_n)|^{2^*-2}G^{-1}(v_n)v_n}{g(G^{-1}(v_n))}\, dx + o(1) \\ &= \int_{\mathbb{R}^N} |G^{-1}(v_n)|^{2^*}\, dx + o(1) = (2\sqrt{2})^{2^*} \int_{\mathbb{R}^N} |v_n|^{2^*}\, dx + o(1). \end{aligned} \tag{3.19}$$

Moreover, by (3.19), we get $\ell > 0$ otherwise we have $v_n \top 0$ in $H^1(\mathbb{R}^N)$ which contradicts $c_\kappa > 0$.

From $J_\kappa(v_n) = c_\kappa + o(1)$, (3.12), (3.14) and (3.19), we have

$$c_\kappa = \frac{1}{2}\ell - \frac{1}{2^*}\ell. \tag{3.20}$$

By the definition of S, we have

$$\int_{\mathbb{R}^N} \left[|\nabla v_n|^2 + V(x)\frac{G^{-1}(v_n)v_n}{g(G^{-1}(v_n))} \right] dx \geq S \left(\int_{\mathbb{R}^N} v_n^{2^*}\, dx \right)^{2/2^*}. \tag{3.21}$$

Taking the limit in (3.21), we get

$$\ell \geq \frac{1}{8} S \ell^{2/2^*}. \tag{3.22}$$

Finally, combining (3.20) and (3.22), it follows that

$$c_\kappa \geq \frac{1}{N} \left(\frac{\sqrt{2}}{4} \right)^N S^{N/2},$$

which contradicts Lemma 3.2. □

By Lemma 3.3, up to subsequence, we may assume that there is $v_\kappa \in E$ such that $v_n \rightharpoonup v_\kappa$ in E, $v_n \to v_\kappa$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ and $v_n \to v_\kappa$ a.e. in $\mathbb{R}^N$. We now show that $\langle J'_\kappa(v_\kappa), \psi\rangle = 0$ for any $\psi \in C_0^\infty(\mathbb{R}^N)$, i.e., v_κ is a critical point of J_κ. In fact, we have

$$\begin{aligned}
&\langle J'_\kappa(v_n), \psi\rangle - \langle J'_\kappa(v_\kappa), \psi\rangle \\
&= \int_{\mathbb{R}^N} \nabla(v_n - v_\kappa)\nabla\psi\,dx + \int_{\mathbb{R}^N} V(x)\left[\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} - \frac{G^{-1}(v_\kappa)}{g(G^{-1}(v_\kappa))}\right]\psi\,dx \\
&\quad - \int_{\mathbb{R}^N}\left[\frac{|G^{-1}(v_n)|^{p-2}G^{-1}(v_n)}{g(G^{-1}(v_n))} - \frac{|G^{-1}(v_\kappa)|^{p-2}G^{-1}(v_\kappa)}{g(G^{-1}(v_\kappa))}\right]\psi\,dx \\
&\quad - \int_{\mathbb{R}^N}\left[\frac{|G^{-1}(v_n)|^{2^*-2}G^{-1}(v_n)}{g(G^{-1}(v_n))} - \frac{|G^{-1}(v_\kappa)|^{2^*-2}G^{-1}(v_\kappa)}{g(G^{-1}(v_\kappa))}\right]\psi\,dx.
\end{aligned}$$

Since $v_n \rightharpoonup v_\kappa$ in E, $v_n \to v_\kappa$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ and $v_n \to v_\kappa$ a.e. in $\mathbb{R}^N$, it follows that $v_n \to v_\kappa$ a.e. on $\mathcal{O} := \operatorname{supp}\psi$ and there exists $w_p \in L^p(\mathcal{O})$ such that for any n, $|v_n(x)| \le |w_p(x)|$ a.e. on $\mathcal{O}$. Consequently, as $n \to \infty$, we get

$$\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} \to \frac{G^{-1}(v_\kappa)}{g(G^{-1}(v_\kappa))}, \quad \text{a.e. on } \mathcal{O}; \tag{3.23}$$

$$\frac{|G^{-1}(v_n)|^{p-2}G^{-1}(v_n)}{g(G^{-1}(v_n))} \to \frac{|G^{-1}(v_\kappa)|^{p-2}G^{-1}(v_\kappa)}{g(G^{-1}(v_\kappa))}, \quad \text{a.e. on } \mathcal{O}; \tag{3.24}$$

$$\frac{|G^{-1}(v_n)|^{2^*-2}G^{-1}(v_n)}{g(G^{-1}(v_n))} \to \frac{|G^{-1}(v_\kappa)|^{2^*-2}G^{-1}(v_\kappa)}{g(G^{-1}(v_\kappa))}, \quad \text{a.e. on } \mathcal{O}. \tag{3.25}$$

Furthermore, by Lemma 2.1(3),

$$\left|V(x)\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}\psi\right| \le V_\infty|v_n||\psi| \le V_\infty|w_2||\psi|, \quad \text{a.e. on } \mathcal{O}; \tag{3.26}$$

$$\left|\frac{|G^{-1}(v_n)|^{p-2}G^{-1}(v_n)}{g(G^{-1}(v_n))}\psi\right| \le C|v_n|^{p-1}|\psi| \le C|w_{p-1}|^{p-1}|\psi|, \quad \text{a.e. on } \mathcal{O}; \tag{3.27}$$

$$\left|\frac{|G^{-1}(v_n)|^{2^*-2}G^{-1}(v_n)}{g(G^{-1}(v_n))}\psi\right| \le C|v_n|^{2^*-1}|\psi| \le C|w_{2^*-1}|^{2^*-1}|\psi|, \quad \text{a.e. on } \mathcal{O}. \tag{3.28}$$

Now, combining (3.23)–(3.28), the Lebesgue Dominated Convergence Theorem and the weak convergence $v_n \rightharpoonup v_\kappa$ in $H^1(\mathbb{R}^N)$, we have $\langle J'_\kappa(v_n), \psi\rangle \to \langle J'_\kappa(v_\kappa), \psi\rangle$ as $n \to \infty$. Since $J'_\kappa(v_n) \to 0$ as $n \to \infty$, we conclude that $J'_\kappa(v_\kappa) = 0$. If $v_\kappa \neq 0$, then v_κ is a nontrivial critical point. Thus, we assume that $v_\kappa \equiv 0$. First, we show that $\{v_n\}$ is also a (PS) sequence for the functional $J_{\kappa,\infty}\colon H^1(\mathbb{R}^N) \to \mathbb{R}$:

$$\begin{aligned}
J_{\kappa,\infty}(v_n) &= \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla v_n|^2 + V_\infty|G^{-1}(v_n)|^2)\,dx - \frac{1}{p}\int_{\mathbb{R}^N}|G^{-1}(v_n)|^p\,dx \\
&\quad - \frac{1}{2^*}\int_{\mathbb{R}^N}|G^{-1}(v_n)|^{2^*}\,dx.
\end{aligned}$$

It suffices to show

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}[V(x)-V_\infty]|G^{-1}(v_n)|^2\,dx=0. \tag{3.29}$$

In fact, from (V1), for any $\varepsilon>0$, there exists $R>0$ such that for $|x|>R$, it follows that $|V(x)-V_\infty|<\varepsilon$. Thus,

$$\left|\int_{|x|>R}[V(x)-V_\infty]|G^{-1}(v_n)|^2\,dx\right|\le\varepsilon\int_{\mathbb{R}^N}|G^{-1}(v_n)|^2\,dx\le C\varepsilon. \tag{3.30}$$

On the other hand, since $v_n\to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$,

$$\left|\int_{|x|\le R}[V(x)-V_\infty]|G^{-1}(v_n)|^2\,dx\right|\le 2C|V_\infty|\int_{|x|\le R}|v_n|^2\,dx\to 0,\quad n\to\infty. \tag{3.31}$$

Combining (3.30) and (3.31), we get (3.29).

By Lemma 3.4, $\{v_n\}$ does not vanish and there exist $\beta,R>0$, and $\{z_n\}\subset\mathbb{R}^N$ such that

$$\lim_{n\to\infty}\int_{B_R(z_n)}v_n^2\,dx\ge\beta>0. \tag{3.32}$$

Define $\widetilde{v}_n(x)=v_n(x+z_n)$. Since $\{v_n\}$ is a (PS) sequence for $J_{\kappa,\infty}$, $\{\widetilde{v}_n\}$ is a (PS) sequence for $J_{\kappa,\infty}$. Arguing as in the case of $\{v_n\}$ we get that $\widetilde{v}_n\rightharpoonup\widetilde{v}_\kappa$ in $H^1(\mathbb{R}^N)$ with $J'_{\kappa,\infty}(\widetilde{v}_\kappa)=0$. From (3.32), we have $\widetilde{v}_\kappa\ne 0$. The last limits together with the lower semicontinuity of convex functional and Fatou's Lemma lead to

$$\begin{aligned}
2c_\kappa&=\lim_{n\to\infty}\big[2J_{\kappa,\infty}(\widetilde{v}_n)-\langle J'_{\kappa,\infty}(\widetilde{v}_n),G^{-1}(\widetilde{v}_n)g(G^{-1}(\widetilde{v}_n))\rangle\big]\\
&=-\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{G^{-1}(\widetilde{v}_n)g'(G^{-1}(\widetilde{v}_n))}{g(G^{-1}(\widetilde{v}_n))}|\nabla\widetilde{v}_n|^2\,dx\\
&\quad-\frac{2-p}{p}\lim_{n\to\infty}\int_{\mathbb{R}^N}|G^{-1}(\widetilde{v}_n)|^p\,dx-\frac{2-2^*}{2^*}\lim_{n\to\infty}\int_{\mathbb{R}^N}|G^{-1}(\widetilde{v}_n)|^{2^*}\,dx\\
&\ge-\int_{\mathbb{R}^N}\frac{G^{-1}(\widetilde{v}_\kappa)g'(G^{-1}(\widetilde{v}_\kappa))}{g(G^{-1}(\widetilde{v}_\kappa))}|\nabla\widetilde{v}_\kappa|^2\,dx\\
&\quad-\frac{2-p}{p}\int_{\mathbb{R}^N}|G^{-1}(\widetilde{v}_\kappa)|^p\,dx-\frac{2-2^*}{2^*}\int_{\mathbb{R}^N}|G^{-1}(\widetilde{v}_\kappa)|^{2^*}\,dx\\
&=2J_{\kappa,\infty}(\widetilde{v}_\kappa)-\langle J'_{\kappa,\infty}(\widetilde{v}_\kappa),G^{-1}(\widetilde{v}_\kappa)g(G^{-1}(\widetilde{v}_\kappa))\rangle\\
&=2J_{\kappa,\infty}(\widetilde{v}_\kappa),
\end{aligned} \tag{3.33}$$

that is, $J_{\kappa,\infty}(\widetilde{v}_\kappa)\le c_\kappa$. It follows the argument used in [11], we get a path $\gamma(t)\colon[0,L]\to H^1(\mathbb{R}^N)$ such that

$$\max_{t\in[0,L]}J_{\kappa,\infty}(\gamma(t))=J_{\kappa,\infty}(\widetilde{v}_\kappa).$$

In fact, we define

$$\widetilde{v}_{\kappa,t}(x) = \begin{cases} \widetilde{v}_\kappa(x/t) & \text{if } t > 0, \\ 0 & \text{if } t = 0. \end{cases}$$

Then,

$$\int_{\mathbb{R}^N} |\nabla \widetilde{v}_{\kappa,t}|^2\,dx = t^{N-2}\int_{\mathbb{R}^N} |\nabla \widetilde{v}_\kappa|^2\,dx, \quad \int_{\mathbb{R}^N} |G^{-1}(\widetilde{v}_{\kappa,t})|^2\,dx = t^N \int_{\mathbb{R}^N} |G^{-1}(\widetilde{v}_\kappa)|^2\,dx,$$

and

$$\int_{\mathbb{R}^N} |G^{-1}(\widetilde{v}_{\kappa,t})|^p\,dx = t^N \int_{\mathbb{R}^N} |G^{-1}(\widetilde{v}_\kappa)|^p\,dx, \quad \int_{\mathbb{R}^N} |G^{-1}(\widetilde{v}_{\kappa,t})|^{2^*}\,dx = t^N \int_{\mathbb{R}^N} |G^{-1}(\widetilde{v}_\kappa)|^{2^*}\,dx.$$

Since $J'_{\kappa,\infty}(\widetilde{v}_\kappa) = 0$, elliptic regularity implies that $\widetilde{v}_\kappa \in C^2(\mathbb{R}^N)$. Hence, by

$$\frac{d}{dt} J_{\kappa,\infty}(\widetilde{v}_{\kappa,t})\Big|_{t=1} = 0,$$

it follows that

$$\begin{aligned} \frac{N-2}{2N}\int_{\mathbb{R}^N} |\nabla \widetilde{v}_\kappa|^2\,dx = &-\frac{V_\infty}{2}\int_{\mathbb{R}^N} |G^{-1}(\widetilde{v}_\kappa)|^2\,dx \\ &+ \frac{1}{p}\int_{\mathbb{R}^N} |G^{-1}(\widetilde{v}_\kappa)|^p\,dx + \frac{1}{2^*}\int_{\mathbb{R}^N} |G^{-1}(\widetilde{v}_\kappa)|^{2^*}\,dx. \end{aligned}$$

Setting $\gamma(t)(x) = \widetilde{v}_{\kappa,t}(x)$, we see that

$$\begin{aligned} &J_{\kappa,\infty}(\gamma(t)) \\ &= \frac{t^{N-2}}{2}\int_{\mathbb{R}^N} |\nabla \widetilde{v}_\kappa|^2\,dx \\ &\quad - t^N\left[-\frac{V_\infty}{2}\int_{\mathbb{R}^N} |G^{-1}(\widetilde{v}_\kappa)|^2\,dx + \frac{1}{p}\int_{\mathbb{R}^N} |G^{-1}(\widetilde{v}_\kappa)|^p\,dx + \frac{1}{2^*}\int_{\mathbb{R}^N} |G^{-1}(\widetilde{v}_\kappa)|^{2^*}\,dx\right]. \end{aligned}$$

Thus $\gamma \in C([0,\infty), H^1(\mathbb{R}^N))$ and

$$\begin{aligned} &\frac{d}{dt} J_{\kappa,\infty}(\gamma(t)) \\ &= \frac{N-2}{2} t^{N-3}\int_{\mathbb{R}^N} |\nabla \widetilde{v}_\kappa|^2\,dx \\ &\quad - N t^{N-1}\left[-\frac{V_\infty}{2}\int_{\mathbb{R}^N} |G^{-1}(\widetilde{v}_\kappa)|^2\,dx + \frac{1}{p}\int_{\mathbb{R}^N} |G^{-1}(\widetilde{v}_\kappa)|^p\,dx + \frac{1}{2^*}\int_{\mathbb{R}^N} |G^{-1}(\widetilde{v}_\kappa)|^{2^*}\,dx\right] \\ &= \frac{N-2}{2} t^{N-3}(1-t^2)\int_{\mathbb{R}^N} |\nabla \widetilde{v}_\kappa|^2\,dx. \end{aligned}$$

So, $\frac{d}{dt} J_{\kappa,\infty}(\gamma(t)) > 0$ for $t \in (0,1)$ and $\frac{d}{dt} J_{\kappa,\infty}(\gamma(t)) < 0$ for $t > 1$. Thus for sufficiently large $L > 1$, we get the desired path. Define the set

$$\Gamma_\infty = \{\gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) \neq 0, J_\infty(\gamma(1)) < 0\}.$$

After a suitable scale change in t, we can assume $\gamma(t) \in \Gamma_\infty$.

Thereby, since $V(x)$ is continuous, by (V2), and since $\gamma \in \Gamma_\infty \subset \Gamma$, we have

$$c_\kappa \le \max_{t\in[0,1]} J_\kappa(\gamma(t)) := J_\kappa(\gamma(\bar{t})) < J_{\kappa,\infty}(\gamma(\bar{t})) \le \max_{t\in[0,1]} J_{\kappa,\infty}(\gamma(t)) = J_{\kappa,\infty}(\widetilde{v}_\kappa) \le c_\kappa,$$

which is a contradiction. This way, v_κ is a nontrivial critical point for J_κ. Moreover, repeating the same type of arguments explored in (3.33), we have that $J_\kappa(v_\kappa) \le c_\kappa$.

4. L^∞ estimate of the solution

In the following, we will prove an L^∞ estimate dependent of $\kappa > 0$. To this end, first we need to give an uniform boundedness of the Sobolev norm independent on $\kappa > 0$ for v_κ.

Lemma 4.1. *The solution v_κ satisfies*

$$\|v_\kappa\|^2 \le \frac{2p}{N(p-2)} \left(\frac{\sqrt{2}}{4}\right)^N S^{N/2}.$$

Proof. Using the hypothesis that v_κ is a critical point of J_κ,

$$\begin{aligned} pc_\kappa &= pJ_\kappa(v_\kappa) - \langle J'_\kappa(v_\kappa), G^{-1}(v_\kappa)g(G^{-1}(v_\kappa))\rangle \\ &\ge \frac{p-2}{2}\int_{\mathbb{R}^N} |\nabla v_\kappa|^2\,dx + \frac{p-2}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(v_\kappa)|^2\,dx, \end{aligned}$$

from which it follows that,

$$\|v_\kappa\|^2 \le \frac{2pc_\kappa}{p-2}.$$

By Lemma 3.2, we get

$$\|v_\kappa\|^2 \le \frac{2p}{N(p-2)} \left(\frac{\sqrt{2}}{4}\right)^N S^{N/2}.$$

□

Proposition 4.2. *There exists a constant $C_0 > 0$ independent of κ, such that $\|v_\kappa\|_\infty \le C_0$.*

Proof. In what follows, we denote v_κ by v. For each $m \in$ N and $\beta > 1$, let $A_m = \{x \in \mathbb{R}^N : |v|^{\beta-1} \le m\}$ and $B_m = \mathbb{R}^N \setminus A_m$. Define

$$v_m = \begin{cases} v|v|^{2(\beta-1)} & \text{in } A_m, \\ m^2 v & \text{in } B_m. \end{cases}$$

Note that $v_m \in H^1(\mathbb{R}^N)$, $v_m \le |v|^{2\beta-1}$ and

$$\nabla v_m = \begin{cases} (2\beta-1)|v|^{2(\beta-1)}\nabla v & \text{in } A_m, \\ m^2\nabla v & \text{in } B_m. \end{cases}$$

Using v_m as a test function in (2.8), we deduce that

$$\begin{aligned}(4.1)\qquad &\int_{\mathbb{R}^N}\left[\nabla v\nabla v_m+V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}v_m\right]dx\\ &=\int_{\mathbb{R}^N}\frac{|G^{-1}(v)|^{p-2}G^{-1}(v)+|G^{-1}(v)|^{2^*-2}G^{-1}(v)}{g(G^{-1}(v))}v_m\,dx.\end{aligned}$$

By (4.1),

$$\int_{\mathbb{R}^N}\nabla v\nabla v_m\,dx=(2\beta-1)\int_{A_m}|v|^{2(\beta-1)}|\nabla v|^2\,dx+m^2\int_{B_m}|\nabla v|^2\,dx. \tag{4.2}$$

Let

$$w_m=\begin{cases}v|v|^{\beta-1} & \text{in } A_m,\\ mv & \text{in } B_m.\end{cases}$$

Then $w_m^2=vv_m\le|v|^{2\beta}$ and

$$\nabla w_m=\begin{cases}\beta|v|^{\beta-1}\nabla v & \text{in } A_m,\\ m\nabla v & \text{in } B_m.\end{cases}$$

Hence,

$$\int_{\mathbb{R}^N}|\nabla w_m|^2\,dx=\beta^2\int_{A_m}|v|^{2(\beta-1)}|\nabla v|^2\,dx+m^2\int_{B_m}|\nabla v|^2\,dx. \tag{4.3}$$

Then, from (4.2) and (4.3),

$$\int_{\mathbb{R}^N}(|\nabla w_m|^2-\nabla v\nabla v_m)\,dx=(\beta-1)^2\int_{A_m}|v|^{2(\beta-1)}|\nabla v|^2\,dx. \tag{4.4}$$

Combining (4.1), (4.2) and (4.4), since $\beta>1$, we have

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla w_m|^2\,dx&\le\left[\frac{(\beta-1)^2}{2\beta-1}+1\right]\int_{\mathbb{R}^N}\nabla v\nabla v_m\,dx\\ &\le\beta^2\int_{\mathbb{R}^N}\left[\nabla v\nabla v_m+V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}v_m\right]dx\\ &=\beta^2\int_{\mathbb{R}^N}\frac{|G^{-1}(v)|^{p-2}G^{-1}(v)+|G^{-1}(v)|^{2^*-2}G^{-1}(v)}{g(G^{-1}(v))}v_m\,dx\\ &\le C\beta^2\int_{\mathbb{R}^N}(|v|^{p-1}+|v|^{2^*-1})|v_m|\,dx.\end{aligned}$$

Now, by Morse iteration and by arguments similar to [3], the result follows. □

Proof of Theorem 1.1. Combining the arguments in Section 3 and Proposition 4.2, the solution v_κ obtained in Section 3 satisfies $\|v_\kappa\|_\infty\le C_0$. Choosing $\overline{\kappa}=\min\left\{1/(16C_0^2),\widetilde{\kappa}\right\}$, it follows that

$$\|G^{-1}(v_\kappa)\|_\infty\le 2\sqrt{2}\|v_\kappa\|_\infty\le\sqrt{1/(2\kappa)},\quad\forall\,\kappa\in(0,\overline{\kappa}].$$

From this, $u=G^{-1}(v_\kappa)$ is a classical solution of (1.1). □

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