TAIWANESE JOURNAL OF MATHEMATICS
Vol. 21, No. 6, pp. 1355–1379, December 2017
DOI: 10.11650/tjm/7995

Weak Solutions for Nonlinear Neumann Boundary Value Problems with $p(x)$-Laplacian Operators

Lingju Kong

Abstract. We study the nonlinear Neumann boundary value problem with a $p(x)$-Laplacian operator

$$\begin{cases} \Delta_{p(x)}u + a(x)|u|^{p(x)-2}u = f(x,u) & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial \nu} = |u|^{q(x)-2}u + \lambda|u|^{w(x)-2}u & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, with $N \geq 2$, is a bounded domain with smooth boundary and $q(x)$ is critical in the context of variable exponent $p_*(x) = (N-1)p(x)/(N-p(x))$. Using the variational method and a version of the concentration-compactness principle for the Sobolev trace immersion with variable exponents, we establish the existence and multiplicity of weak solutions for the above problem.

1. Introduction

In this paper, we are concerned with the existence of weak solutions of the following nonlinear Neumann boundary value problem with a $p(x)$-Laplacian operator

$$\begin{cases} \Delta_{p(x)}u + a(x)\left|u\right|^{p(x)-2}u = f(x,u) & \text{in } \Omega, \\ \left|\nabla u\right|^{p(x)-2}\dfrac{\partial u}{\partial \nu} = \left|u\right|^{q(x)-2}u + \lambda\left|u\right|^{w(x)-2}u & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, with $N \geq 2$, is a bounded domain with smooth boundary, $\frac{\partial}{\partial \nu}$ is the outer normal derivative, $\lambda > 0$ is a parameter, $a \in C(\overline{\Omega})$ with $a(x) > 0$ on $\overline{\Omega}$, $p \in C^{0,1}(\overline{\Omega})$ (i.e., p is Lipschitz on $\overline{\Omega}$) with $1 < p(x) < N$ on $\overline{\Omega}$, $q, w \in C(\partial\Omega)$, $\Delta_{p(x)}u = \operatorname{div}(\left|\nabla u\right|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplacian operator, and $f \in C(\Omega \times \mathbb{R})$.

In recent years, the study of differential equations and variational problems with variable exponents has received considerable attention. For instance, the reader may refer to [1–4, 8, 10, 11, 14–16, 18, 19, 21, 27] and the references therein for a small sample of some recent work on this subject. These problems have many interesting applications in mathematical physics such as in the mathematical modelling of electrorheological fluids [17, 23]

Received August 16, 2016; Accepted February 9, 2017.
Communicated by Eiji Yanagida.
2010 *Mathematics Subject Classification.* 35J20, 35J25, 35J92.
Key words and phrases. nonlinear boundary conditions, weak solutions, concentration-compactness principle, variable exponent spaces, critical growth, mountain pass lemma, dual fountain theorem.

and of other phenomena related to image processing, elasticity, and the flow in porous media [30]. It is well known that the study of variational problems with variable exponents relies heavily on the theory for variable exponent Lebesgue and Sobolev spaces. We refer the reader to Section 2 for a brief review of some related results and to the monograph [12] for more information on these spaces. One fundamental point in the study of these spaces is the generalization of the well known Sobolev immersion theorems. It has been shown in [14, 16] that the following immersions hold

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega) \quad \text{and} \quad W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$$

if $r \in C(\overline{\Omega})$ and $q \in C(\partial\Omega)$ satisfy

$$1 \le r(x) \le p^*(x) := \frac{Np(x)}{N-p(x)} \text{ on } \overline{\Omega} \quad \text{and} \quad 1 \le q(x) \le p_*(x) := \frac{(N-1)p(x)}{N-p(x)} \text{ on } \partial\Omega.$$

The exponents $p^*(x)$ and $p_*(x)$ are called the critical Sobolev exponent and the critical Sobolev trace exponent, respectively. Moreover, if $\{x \in \Omega : r(x) = p^*(x)\} \neq \emptyset$ and $\{x \in \partial\Omega : q(x) = p_*(x)\} \neq \emptyset$, then the above immersions are not compact. We refer the reader to [14, 16] for more details.

A number of variations of problem (1.1) have been investigated in the literature. See, for example, [6, 9, 22, 24, 29] for problems when $p(x)$ is a constant and [1–4, 10, 11, 14, 21, 27] for problems with variable $p(x)$. In particular, paper [10] proved, among others, the existence of infinitely many eigenvalues for the Steklov eigenvalue problem

$$\begin{cases} \Delta_{p(x)} u = |u|^{p(x)-2}\, u & \text{in } \Omega, \\ |\nabla u|^{p(x)-2} \dfrac{\partial u}{\partial \nu} = \lambda\, |u|^{p(x)-2}\, u & \text{on } \partial\Omega, \end{cases}$$

and papers [1, 4, 21] studied the existence of multiple weak solutions of the problem

$$\begin{cases} \Delta_{p(x)} u = |u|^{p(x)-2}\, u & \text{in } \Omega, \\ |\nabla u|^{p(x)-2} \dfrac{\partial u}{\partial \nu} = g(x,u) & \text{on } \partial\Omega, \end{cases}$$

where $g(x,t)$ is required to satisfy some suitable subcritical conditions. For nonlinear Neumann boundary value problems with variable exponents, to the best of the author's knowledge, no work has been done when the nonlinearities in the boundary conditions grow critically. In this paper, thanks to a recent concentration-compactness principle for the Sobolev trace immersion with variable exponents obtained in [7], we are able to study problem (1.1) when $q(x)$ is critical. To be precise, we assume throughout, and without further mention, that the following condition holds:

(H) The exponent $q(x)$ satisfies

$$p(x) < q(x) \le p_*(x) \quad \text{on } \partial\Omega,$$

and is critical in the sense that $\mathcal{S} := \{x \in \partial\Omega : q(x) = p_*(x)\} \neq \emptyset$.

As pointed out earlier, under condition (H), the compactness of the immersion $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$ fails. So, in general, the Palais-Smale (PS, for short) condition is not satisfied. To overcome this difficulty, we apply a version of the concentration-compactness principle for the Sobolev trace immersion with variable exponents [7], so that a local PS condition holds, which is sufficient for our proofs. In this paper, we consider the existence of weak solutions of problem (1.1) in two cases:

$$p(x) < w(x) < p_*(x) \text{ on } \partial\Omega \quad \text{and} \quad 1 < w(x) < p(x) \text{ on } \partial\Omega.$$

For the first case, the mountain pass lemma is used to show problem (1.1) has at least one nontrivial weak solution, and for the second case, a version of the dual fountain theorem is utilized to prove that problem (1.1) has infinitely many weak solutions. Our results improve and extend many known results in the literature.

The rest of this paper is organized as follows. Section 2 contains some preliminary results, Section 3 contains the main results, and the proofs of the main results are given in Section 4.

2. Preliminary results

In this section, we review some basic results for Lebesgue and Sobolev spaces with variable exponents. These results can be found in, for example, [6, 7, 12–16, 20].

For any bounded domain $\Omega \subset \mathbb{R}^N$ and $p \in C(\Omega)$ with $p(x) \geq 1$ on Ω, the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by

$$L^{p(x)}(\Omega) = \left\{ u \colon \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\}.$$

Equipped with the so-called Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega)} = \|u\|_{p(x)} = \inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)} dx \leq 1 \right\},$$

$L^{p(x)}(\Omega)$ is a separable and reflexive Banach space. It is clear that, when $p(x) = p > 1$ (a positive constant), the space $L^{p(x)}(\Omega)$ becomes the well-known Lebesgue space $L^p(\Omega)$ and the norm $\|u\|_{p(x)}$ reduces to the standard norm $\|u\|_p = \left(\int_\Omega |u|^p\right)^{1/p}$ in $L^p(\Omega)$.

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\}.$$

Equipped with the norm

$$\|u\| = \|u\|_{p(x)} + \|\nabla u\|_{p(x)},$$

$W^{1,p(x)}(\Omega)$ is also a separable and reflexive Banach space. For any $u \in W^{1,p(x)}(\Omega)$, let

$$\|u\|_{1,p(x)} = \inf\left\{\lambda > 0 : \int_\Omega \left(\left|\frac{\nabla u}{\lambda}\right|^{p(x)} + a(x)\left|\frac{u}{\lambda}\right|^{p(x)}\right) dx \leq 1\right\}. \tag{2.1}$$

In view of $a(x) > 0$ on $\overline{\Omega}$, we see that $\|u\|$ and $\|u\|_{1,p(x)}$ are equivalent norms in $W^{1,p(x)}(\Omega)$. In this paper, for the convenience of the discussion, we use the norm $\|u\|_{1,p(x)}$ for $W^{1,p(x)}(\Omega)$.

Throughout this paper, we use the notations

$$h^+(\Omega) := \sup_{x\in\Omega} h(x) \quad \text{and} \quad h^-(\Omega) := \inf_{x\in\Omega} h(x) \qquad \text{for any } h \in C(\Omega),$$
$$h^+(\partial\Omega) := \sup_{x\in\Omega} h(x) \quad \text{and} \quad h^-(\partial\Omega) := \inf_{x\in\Omega} h(x) \qquad \text{for any } h \in C(\partial\Omega).$$

In the sequel, for simplicity, we often write both $h^+(\Omega)$ and $h^+(\partial\Omega)$ as h^+, and both $h^-(\Omega)$ and $h^-(\partial\Omega)$ as h^- if their meanings are clear from the context of the content.

Proposition 2.1. *Let*

$$\rho(u) = \int_\Omega |u|^{p(x)}\, dx \quad \textit{for any } u \in L^{p(x)}(\Omega)$$

and

$$\rho_{1,p(x)}(u) = \int_\Omega \left(|\nabla u|^{p(x)} + a(x)\,|u|^{p(x)}\right) dx \quad \textit{for any } u \in W^{1,p(x)}(\Omega).$$

Then, we have

(a) *if $\|u\|_{p(x)} \geq 1$, then $\|u\|_{p(x)}^{p^-} \leq \rho(u) \leq \|u\|_{p(x)}^{p^+}$;*

(b) *if $\|u\|_{p(x)} \leq 1$, then $\|u\|_{p(x)}^{p^+} \leq \rho(u) \leq \|u\|_{p(x)}^{p^-}$;*

(c) *if $\|u\|_{1,p(x)} \geq 1$, then $\|u\|_{1,p(x)}^{p^-} \leq \rho_{1,p(x)}(u) \leq \|u\|_{1,p(x)}^{p^+}$;*

(d) *if $\|u\|_{1,p(x)} \leq 1$, then $\|u\|_{1,p(x)}^{p^+} \leq \rho_{1,p(x)}(u) \leq \|u\|_{1,p(x)}^{p^-}$.*

Parts (a) and (b) of Proposition 2.1 were proved in [16, Theorem 1.3]. Parts (c) and (d) can be proved similarly. Proposition 2.2 below can be found in [15, Proposition 2.4].

Proposition 2.2. *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p_1(x)}(\Omega)$, where $p_1(x)$ is the conjugate of $p(x)$, i.e., $1/p(x)+1/p_1(x) = 1$. Moreover, for any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p_1(x)}(\Omega)$, we have the following Hölder-type inequality*

$$\left|\int_\Omega uv\, dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p_1^-}\right) \|u\|_{p(x)}\, \|v\|_{p_1(x)} \leq 2\, \|u\|_{p(x)}\, \|v\|_{p_1(x)}\,.$$

The following proposition is a special case of [16, Theorem 2.3].

Proposition 2.3. *Assume that $z \in C(\overline{\Omega})$ satisfies $1 \le z(x) < p^*(x)$ on $\overline{\Omega}$. Then, there exists a continuous and compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{z(x)}(\Omega)$.*

In the sequel, we denote the boundary measure on $\partial\Omega$ by dS. For any $z \in C(\partial\Omega)$ with $z(x) \ge 1$ on $\partial\Omega$, we define the variable exponent Lebesgue space $L^{z(x)}(\partial\Omega)$ on $\partial\Omega$ by

$$L^{z(x)}(\partial\Omega) = \left\{ u \colon \partial\Omega \to \mathbb{R} \text{ is measurable and } \int_{\partial\Omega} |u(x)|^{z(x)}\, dS < \infty \right\}.$$

The corresponding Luxemburg norm is given by

$$\|u\|_{r(x),\partial\Omega} = \inf \left\{ \lambda > 0 : \int_{\partial\Omega} \left| \frac{u}{\lambda} \right|^{z(x)} dS \le 1 \right\}.$$

Remark 2.4. Similar to Propositions 2.1 and 2.2, the following results are true.

(i) Let $\rho(u, \partial\Omega) = \int_{\partial\Omega} |u|^{z(x)}\, dS$ for any $u \in L^{z(x)}(\partial\Omega)$. Then, we have (see, for example, [10, Proposition 2.4])

  (a) if $\|u\|_{z(x),\partial\Omega} \ge 1$, then $\|u\|^{z^-}_{z(x),\partial\Omega} \le \rho(u,\partial\Omega) \le \|u\|^{z^+}_{z(x),\partial\Omega}$;

  (b) if $\|u\|_{z(x),\partial\Omega} \le 1$, then $\|u\|^{z^+}_{z(x),\partial\Omega} \le \rho(u,\partial\Omega) \le \|u\|^{z^-}_{z(x),\partial\Omega}$.

(ii) For any $u \in L^{p(x)}(\partial\Omega)$ and $v \in L^{p_1(x)}(\partial\Omega)$, where $1/p(x) + 1/p_1(x) = 1$, the following Hölder-type inequality holds

$$\left| \int_{\partial\Omega} uv\, dS \right| \le \left(\frac{1}{p^-} + \frac{1}{p_1^-} \right) \|u\|_{p(x),\partial\Omega} \|v\|_{p_1(x),\partial\Omega} \le 2\, \|u\|_{p(x),\partial\Omega} \|v\|_{p_1(x),\partial\Omega}\,.$$

Parts (a) and (b) of Proposition 2.5 below give the Sobolev trace theorems for variable exponent spaces and are taken from [14, Corollaries 2.1 and 2.2], respectively.

Proposition 2.5. *Let $\Omega \subseteq \mathbb{R}^N$ be an open bounded domain with Lipschitz boundary. Then,*

(a) *if $p \in C(\overline{\Omega})$ is such that $p \in W^{1,\gamma}(\Omega)$ with $1 \le p^- \le p^+ < N < \gamma$, then there is a continuous boundary trace embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p_*(x)}(\partial\Omega)$;*

(b) *if $p \in C(\overline{\Omega})$ is such that $1 < p(x) < N$ on $\overline{\Omega}$, then, for any $z \in C(\partial\Omega)$ with $1 \le z(x) < p_*(x)$ on $\partial\Omega$, there is a compact boundary trace embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{z(x)}(\partial\Omega)$.*

Consider $\Gamma \subset \partial\Omega$, $\Gamma \ne \partial\Omega$, a (possibly empty) closed set, and define

$$W_{\Gamma}^{1,p(x)} = \overline{\left\{ \psi \in C^\infty(\overline{\Omega}) : \psi \text{ vanishes in a neighbourhood of } \Gamma \right\}},$$

where the closure is taken with respect to the norm defined in (2.1). Let $q \in C(\partial\Omega)$ satisfy condition (H). As in [7], we define the Sobolev trace constant in $W_\Gamma^{1,p(x)}$ as

$$T(p(\cdot), q(\cdot), \Omega, \Gamma) := \inf_{v \in W_\Gamma^{1,p(x)}} \frac{\|v\|_{1,p(x)}}{\|v\|_{q(x),\partial\Omega}} = \inf_{v \in W_\Gamma^{1,p(x)}} \frac{\|v\|_{1,p(x)}}{\|v\|_{q(x),\partial\Omega\setminus\Gamma}}.$$

The following concentration-compactness principle was proved in [7, Theorem 4.1].

Proposition 2.6. *Let $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,p(x)}(\Omega)$ be a sequence such that $u_n \rightharpoonup u$ weakly in $W^{1,p(x)}(\Omega)$. Then, there exist a countable set I, positive numbers $\{\mu_i\}_{i\in I}$ and $\{\nu_i\}_{i\in I}$, and points $\{x_i\}_{i\in I} \subset \mathcal{S} \subset \partial\Omega$ such that*

$$|u_n|^{q(x)}\, dS \rightharpoonup d\nu = |u|^{q(x)}\, dS + \sum_{i\in I} \nu_i \delta_{x_i} \qquad \text{weakly in the sense of measures,}$$

$$|\nabla u_n|^{p(x)}\, dx \rightharpoonup d\mu \geq |\nabla u|^{p(x)}\, dx + \sum_{i\in I} \mu_i \delta_{x_i} \quad \text{weakly in the sense of measures,}$$

$$S_i \nu_i^{\frac{1}{q(x_i)}} \leq \mu_i^{\frac{1}{p(x_i)}},$$

where $S_i = \sup_{\epsilon>0} T(p(\cdot), q(\cdot), \Omega_{\epsilon,i}, \Gamma_{\epsilon,i})$ is the localized Sobolev trace constant with

$$\Omega_{\epsilon,i} = \Omega \cap B_\epsilon(x_i) \quad \text{and} \quad \Gamma_{\epsilon,i} = \partial B_\epsilon(x_i) \cap \Omega.$$

Remark 2.7. Let $S = \min\{1, \inf_{i\in I} S_i\}$. Then it is easy to see that $0 < S \leq 1$ and

$$S \nu_i^{\frac{1}{q(x_i)}} \leq \mu_i^{\frac{1}{p(x_i)}}, \quad i \in I.$$

Here, we just want to show that $S > 0$ since the rest part is obvious. It is clear that $T(p(\cdot), q(\cdot), \Omega, \emptyset) > 0$ by Proposition 2.5(a). Since Ω is bounded, there exits $\epsilon > 0$ large enough so that $\Omega_{\epsilon,i} = \Omega$ and $\Gamma_{\epsilon,i} = \emptyset$. Thus, $S_i \geq T(p(\cdot), q(\cdot), \Omega, \emptyset) > 0$ for all $i \in I$. Assume, to the contrary, that $S = 0$. Then, there exists $i_0 \in I$ such that $S_{i_0} < T(p(\cdot), q(\cdot), \Omega, \emptyset)$, which is a contradiction. Hence, $S > 0$.

The following result can be approved by an argument similar to that of [13, Lemma 2.1].

Proposition 2.8. *Let $s_1 \in C(\partial\Omega)$ and $s_2 \in L^\infty(\partial\Omega)$ be such that $1 \leq s_1(x)s_2(x) \leq \infty$ for a.e. $x \in \partial\Omega$. Let $u \in L^{s_1(x)}(\partial\Omega)$, $u \neq 0$. Then, we have*

(a) *if $\|u\|_{s_1(x)s_2(x),\partial\Omega} \leq 1$, then* $\|u\|_{s_1(x)s_2(x),\partial\Omega}^{s_2^+} \leq \left\| |u|^{s_2(x)} \right\|_{s_1(x),\partial\Omega} \leq \|u\|_{s_1(x)s_2(x),\partial\Omega}^{s_2^-}$;

(b) *if $\|u\|_{s_1(x)s_2(x),\partial\Omega} \geq 1$, then* $\|u\|_{s_1(x)s_2(x),\partial\Omega}^{s_2^-} \leq \left\| |u|^{s_2(x)} \right\|_{s_1(x),\partial\Omega} \leq \|u\|_{s_1(x)s_2(x),\partial\Omega}^{s_2^+}$.

3. Main results

In this paper, we need the following conditions:

(H1) Either $p^+(\Omega) \geq p^-(\Omega) \geq 2$ or $1 < p^-(\Omega) \leq p^+(\Omega) < 2$;

(H2) $p(x) < w(x) < p_*(x)$ on $\partial\Omega$ and $p^+(\Omega) < \min\{q^-(\partial\Omega), w^-(\partial\Omega)\}$;

(H3) $1 < w(x) < p(x)$ on $\partial\Omega$ and $w^+(\partial\Omega) < p^-(\Omega)$;

(H4) there exist $C > 0$ and $r \in C(\overline{\Omega})$ with $p(x) < r(x) < p^*(x)$ on $\overline{\Omega}$ such that $p^+(\Omega) < r^-(\Omega)$ and

$$|f(x,t)| \leq C\left(1 + |t|^{r(x)-1}\right) \quad \text{for any } x \in \Omega \text{ and } t \in \mathbb{R};$$

(H5) there exists $\alpha \in C(\overline{\Omega})$ with $\alpha^-(\Omega) > p^+(\Omega)$ such that

$$0 < \alpha(x)F(x,t) \leq tf(x,t) \quad \text{for any } x \in \Omega \text{ and } t \in \mathbb{R}\setminus\{0\},$$

where $F(x,t) = \int_0^t f(x,s)\,ds$;

(H6) $\lim_{t\to 0} f(x,t)/|t|^{p^+(\Omega)} = 0$ uniformly for $x \in \Omega$;

(H7) $f(x,t)$ is odd with respect to t.

As mentioned in Section 2, in the remainder of the paper, we write $p^+(\Omega)$ as p^+, $p^-(\Omega)$ as p^-, $r^-(\Omega)$ as r^-, $q^-(\partial\Omega)$ as q^-, $w^+(\partial\Omega)$ as w^+, $w^-(\partial\Omega)$ as w^- for the sake of simplicity.

Define a functional $\phi\colon W^{1,p(x)}(\Omega) \to \mathbb{R}$ by

$$\begin{aligned}\phi(u) = {} & \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + a(x)\,|u|^{p(x)}\right)dx - \int_\Omega F(x,u)\,dx \\ & - \int_{\partial\Omega}\left(\frac{1}{q(x)}\,|u|^{q(x)} + \frac{\lambda}{w(x)}\,|u|^{w(x)}\right)dS.\end{aligned} \tag{3.1}$$

Then, under conditions (H), (H1) either (H2) or (H3), and (H4), we can see that $\phi \in C^1(W^{1,p(x)}(\Omega),\mathbb{R})$ and

$$\begin{aligned}\langle \phi'(u), v\rangle = {} & \int_\Omega \left(|\nabla u|^{p(x)-2}\,\nabla u \nabla v + a(x)\,|u|^{p(x)-2}\,uv\right)dx - \int_\Omega f(x,u)v\,dx \\ & - \int_{\partial\Omega}\left(|u|^{q(x)-2}\,uv + \lambda\,|u|^{w(x)-2}\,uv\right)dS\end{aligned} \tag{3.2}$$

for any $u, v \in W^{1,p(x)}(\Omega)$. We say that u is a weak solution of problem (1.1) if u is the critical point of ϕ on $W^{1,p(x)}(\Omega)$.

Now, we state our main theorems.

Theorem 3.1. *Assume that* (H1), (H2), (H4), (H5) *and* (H6) *hold. Then there exists* $\underline{\lambda} > 0$, *depending only on* p, q, w, N *and* Ω, *such that for all* $\lambda > \underline{\lambda}$, *problem* (1.1) *has at least one nontrivial solution in* $W^{1,p(x)}(\Omega)$.

Theorem 3.2. *Assume that* (H1), (H3), (H4), (H5) *and* (H7) *hold. Then there exists* $\overline{\lambda} > 0$, *depending only on* p, q, w, N *and* Ω, *such that for all* $0 < \lambda < \overline{\lambda}$, *problem* (1.1) *has infinitely many solutions* $u_n \in W^{1,p(x)}(\Omega)$, $n \in \mathbb{N}$, *such that* $\phi(u_n) < 0$ *and* $\phi(u_n) \to 0$ *as* $n \to \infty$.

Remark 3.3. We want to emphasize that condition (H6) is not needed in Theorem 3.2.

4. Proofs of main results

4.1. Proof of Theorem 3.1

In this subsection, we assume all the conditions of Theorem 3.1 are satisfied. To prove Theorem 3.1, we need the following version of the mountain pass lemma whose proof can be found in [5]. Below, let $(X, \|\cdot\|)$ be a Banach space and X^* its dual space.

Lemma 4.1. *Let* ϕ *be a functional on* X, $\phi \in C^1(X, \mathbb{R})$. *Assume that there exist* $r, R > 0$ *such that*

(i) $\phi(u) > r$ *for any* $u \in X$ *with* $\|u\| = R$,

(ii) $\phi(0) = 0$ *and* $\phi(v_0) < r$ *for some* $v_0 \in X$ *with* $\|v_0\| > R$.

Define $\mathcal{C} = \{g \in C([0,1], X) : g(0) = 0, g(1) = v_0\}$ *and*

$$c = \inf_{g \in \mathcal{C}} \max_{t \in [0,1]} \phi(g(t)).$$

Then, there exists a sequence $\{u_n\} \subset X$ *such that* $\phi(u_n) \to c$ *and* $\phi'(u_n) \to 0$ *in* X^*.

Definition 4.2. Let $c \in \mathbb{R}$. A functional $\phi \in C^1(X, \mathbb{R})$ is said to satisfy the $(\mathrm{PS})_c$ condition if any sequence $\{u_n\} \subset X$ such that

$$\phi(u_n) \to c, \ \phi'(u_n) \to 0 \quad \text{in } X^*, \text{ as } n \to \infty, \tag{4.1}$$

contains a subsequence converging to a critical point of ϕ. Any sequence $\{u_n\}$ satisfying (4.1) is called a $(\mathrm{PS})_c$ sequence for ϕ with energy level c.

Lemma 4.3. *Let* $\{u_n\} \subset W^{1,p(x)}(\Omega)$ *be a* $(PS)_c$ *sequence for* ϕ *with energy level* c, *where* ϕ *is defined by* (3.1). *Then,* $\{u_n\}$ *is bounded in* $W^{1,p(x)}(\Omega)$.

Proof. Let $\{u_n\} \subset W^{1,p(x)}(\Omega)$ be a $(\mathrm{PS})_c$ sequence for ϕ, i.e.,

$$\phi(u_n) \to c,\ \phi'(u_n) \to 0 \quad \text{in } (W^{1,p(x)}(\Omega))^*, \text{ as } n \to \infty.$$

Choose $\beta \in (p^+, \min\{q^-, w^-, \alpha^-\})$. Then, for n sufficiently large, from Proposition 2.1, (H5), (3.1) and (3.2), we obtain that

$$\begin{aligned}
&|c| + 1 + \|u_n\|_{1,p(x)} \\
&\geq \phi(u_n) - \frac{1}{\beta}\langle \phi'(u_n), u_n\rangle \\
&= \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{\beta}\right)\left(|\nabla u_n|^{p(x)} + a(x)|u_n|^{p(x)}\right) dx + \int_\Omega \left(\frac{1}{\beta} f(x,u_n)u_n - F(x,u_n)\right) dx \\
&\quad + \int_{\partial\Omega}\left(\frac{1}{\beta} - \frac{1}{q(x)}\right)|u_n|^{q(x)}\, dS + \lambda \int_{\partial\Omega}\left(\frac{1}{\beta} - \frac{1}{w(x)}\right)|u_n|^{w(x)}\, dS \\
&\geq \left(\frac{1}{p^+} - \frac{1}{\beta}\right)\int_\Omega \left(|\nabla u_n|^{p(x)} + a(x)|u_n|^{p(x)}\right) dx + \left(\frac{1}{\beta} - \frac{1}{q^-}\right)\int_{\partial\Omega}|u_n|^{q(x)}\, dS \\
&\quad + \lambda\left(\frac{1}{\beta} - \frac{1}{w^-}\right)\int_{\partial\Omega}|u_n|^{w(x)}\, dS \\
&\geq \left(\frac{1}{p^+} - \frac{1}{\beta}\right)\min\left\{\|u_n\|_{1,p(x)}^{p^-}, \|u_n\|_{1,p(x)}^{p^+}\right\} + \left(\frac{1}{\beta} - \frac{1}{q^-}\right)\int_{\partial\Omega}|u_n|^{q(x)}\, dS \\
&\quad + \lambda\left(\frac{1}{\beta} - \frac{1}{w^-}\right)\int_{\partial\Omega}|u_n|^{w(x)}\, dS.
\end{aligned}$$

Since

$$p^+ \geq p^- > 1, \quad \frac{1}{p^+} - \frac{1}{\beta} > 0, \quad \frac{1}{\beta} - \frac{1}{q^-} > 0 \quad \text{and} \quad \frac{1}{\beta} - \frac{1}{w^-} > 0,$$

we see that $\{u_n\}$ is bounded in $W^{1,p(x)}(\Omega)$. This completes the proof of the lemma. $\square$

In the remainder of the paper, let $p_1(x)$, $q_1(x)$, $w_1(x)$ and $r_1(x)$ be the conjugates of $p(x)$, $q(x)$, $w(x)$ and $r(x)$, respectively.

Lemma 4.4. *Let $\{u_n\} \subset W^{1,p(x)}(\Omega)$ be a $(PS)_c$ sequence for ϕ with energy level c and $\theta \in (p^+, \min\{q^-, w^-, \alpha^-\})$ be fixed, where ϕ is defined by* (3.1). *If $c < (\frac{1}{\theta} - \frac{1}{q^-})S^{(N-1)p^+/(p^- - 1)}$, where S is given in Remark* 2.7, *then there exists a subsequence of $\{u_n\}$ that converges strongly in $W^{1,p(x)}(\Omega)$.*

Proof. Let $\{u_n\}$ be given as in the lemma. By Lemma 4.3, $\{u_n\}$ is bounded in $W^{1,p(x)}(\Omega)$. Then, in view of Propositions 2.3, 2.5, 2.6 and Remark 2.7, up to a subsequence, we may assume that

$$
\begin{cases}
u_n \rightharpoonup u & \text{weakly in } W^{1,p(x)}(\Omega), \\
u_n \to u & \text{strongly in } L^{r(x)}(\Omega),\ 1 < r(x) < p^*(x) \text{ on } \overline{\Omega}, \\
u_n \to u & \text{strongly in } L^{w(x)}(\partial\Omega),\ 1 < w(x) < p_*(x) \text{ on } \partial\Omega, \\
u_n \to u & \text{a.e. on } \Omega, \\
|u_n|^{q(x)}\, dS \rightharpoonup d\nu = |u|^{q(x)}\, dS + \sum_{i\in I} \nu_i \delta_{x_i}, \quad \nu_i > 0, \\
|\nabla u_n|^{p(x)}\, dx \rightharpoonup d\mu \geq |\nabla u|^{p(x)}\, dx + \sum_{i\in I} \mu_i \delta_{x_i}, \quad \mu_i > 0, \\
S\nu_i^{\frac{1}{q(x_i)}} \leq \mu_i^{\frac{1}{p(x_i)}}, \quad i \in I.
\end{cases}
\tag{4.2}
$$

In the following, we show that $I = \emptyset$. Assume, to the contrary, that $I \neq \emptyset$. Let $j \in I$ be fixed and $\phi \in C_0^\infty(\mathbb{R}^N)$ with the support in the unit ball of $\mathbb{R}^N$. For any $\epsilon > 0$, let $\phi_{j,\epsilon}(x) = \phi(\frac{x-x_j}{\epsilon})$. Since $\phi'(u_n) \to 0$ in $(W^{1,p(x)}(\Omega))^*$, we have

$$
\lim_{n\to\infty} \langle \phi'(u_n), \phi_{j,\epsilon} u_n \rangle = 0.
$$

On the other hand, note from (3.2) that

$$
\begin{aligned}
\langle \phi'(u_n), \phi_{j,\epsilon} u_n \rangle &= \int_\Omega \left(|\nabla u_n|^{p(x)-2} \nabla u_n \nabla(\phi_{j,\epsilon} u_n) + a(x) |u_n|^{p(x)} \phi_{j,\epsilon} \right) dx \\
&\quad - \int_\Omega f(x,u_n) u_n \phi_{j,\epsilon}\, dx - \int_{\partial\Omega} \left(|u_n|^{q(x)} \phi_{j,\epsilon} + \lambda |u_n|^{w(x)} \phi_{j,\epsilon} \right) dS \\
&= \int_\Omega \left(|\nabla u_n|^{p(x)} \phi_{j,\epsilon} + |\nabla u_n|^{p(x)-2} \nabla u_n \nabla(\phi_{j,\epsilon}) u_n + a(x) |u_n|^{p(x)} \phi_{j,\epsilon} \right) dx \\
&\quad - \int_\Omega f(x,u_n) u_n \phi_{j,\epsilon}\, dx - \int_{\partial\Omega} \left(|u_n|^{q(x)} \phi_{j,\epsilon} + \lambda |u_n|^{w(x)} \phi_{j,\epsilon} \right) dS.
\end{aligned}
$$

Then, passing to the limit as $n \to \infty$, from (4.2), we see that

$$
\begin{aligned}
&\lim_{n\to\infty} \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \nabla(\phi_{j,\epsilon}) u_n\, dx \\
&= -\int_\Omega \phi_{j,\epsilon}\, d\mu - \int_\Omega a(x) |u|^{p(x)} \phi_{j,\epsilon}\, dx + \int_\Omega f(x,u) u \phi_{j,\epsilon}\, dx \\
&\quad + \int_{\partial\Omega} \phi_{j,\epsilon}\, d\nu + \lambda \int_{\partial\Omega} |u|^{w(x)} \phi_{j,\epsilon}\, dS.
\end{aligned}
\tag{4.3}
$$

In virtue of Proposition 2.2, we obtain that

$$
\begin{aligned}
\left| \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \nabla(\phi_{j,\epsilon}) u_n\, dx \right| &\leq \frac{2}{\epsilon} \|\nabla \phi\|_\infty \left\| |\nabla u_n|^{p(x)-1} \right\|_{L_\epsilon^{p_1(x)}} \|u_n\|_{L_\epsilon^{p(x)}} \\
&\leq \frac{d}{\epsilon} \|u_n\|_{L_\epsilon^{p(x)}},
\end{aligned}
$$

where $d = 2\|\nabla\phi\|_\infty \sup_{n\in\mathbb{N}} \left\| |\nabla u_n|^{p(x)-1} \right\|_{L_\epsilon^{p_1(x)}}$, $L_\epsilon^{p_1(x)} = L^{p_1(x)}(B_\epsilon(x_j))$ and $L_\epsilon^{p(x)} =$

$L^{p(x)}(B_\epsilon(x_j))$ with $B_\epsilon(x_j)$ being the ball of center x_j and radius ϵ. From Propositions 2.1 and 2.2, and the fact that $p^- \le p^+ < N$, it follows that

$$\begin{aligned}
&\lim_{n\to\infty}\left|\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla(\phi_{j,\epsilon})u_n\,dx\right| \\
&\le \frac{d}{\epsilon}\|u\|_{L_\epsilon^{p(x)}} \\
&\le \frac{d}{\epsilon}\left(\int_{B_\epsilon(x_j)}|u|^{p(x)}\right)^{1/p^+} + \frac{d}{\epsilon}\left(\int_{B_\epsilon(x_j)}|u|^{p(x)}\right)^{1/p^-} \\
&= \frac{d}{\epsilon}O(\epsilon^{N/p^+}) + \frac{d}{\epsilon}O(\epsilon^{N/p^-}) \to 0 \quad \text{as } \epsilon \to 0.
\end{aligned}$$

Hence, letting $\epsilon \to 0$ in (4.3), we have

$$0 \le -\mu_j + \nu_j,$$

i.e., $\nu_j \ge \mu_j$. Thus, from the last inequality in (4.2), we get that

$$S\nu_j^{\frac{1}{q(x_j)}} \le \nu_j^{\frac{1}{p(x_j)}},$$

and so,

$$\nu_j \ge S^{\frac{p(x_j)q(x_j)}{q(x_j)-p(x_j)}}.$$

Since $x_j \in \mathcal{S} \subset \partial\Omega$, we have $q(x_j) = p_*(x_j)$. This, together with the fact that $0 < S \le 1$, implies that

$$\nu_j \ge S^{\frac{(N-1)p(x_j)}{p(x_j)-1}} \ge S^{\frac{(N-1)p^+}{p^--1}}. \tag{4.4}$$

In view of $\theta \in (p^+, \min\{q^-, w^-, \alpha^-\})$, from (H5), (3.1), (3.2), (4.2) and (4.4), we obtain that

$$\begin{aligned}
c &= \lim_{n\to\infty}\phi(u_n) = \lim_{n\to\infty}\left(\phi(u_n) - \frac{1}{\theta}\langle\phi'(u_n), u_n\rangle\right) \\
&= \lim_{n\to\infty}\Bigg[\int_\Omega\left(\frac{1}{p(x)} - \frac{1}{\theta}\right)\left(|\nabla u_n|^{p(x)} + a(x)|u_n|^{p(x)}\right)dx \\
&\qquad + \int_\Omega\left(\frac{1}{\theta}f(x,u_n)u_n - F(x,u_n)\right)dx + \int_{\partial\Omega}\left(\frac{1}{\theta} - \frac{1}{q(x)}\right)|u_n|^{q(x)}\,dS \\
&\qquad + \lambda\int_{\partial\Omega}\left(\frac{1}{\theta} - \frac{1}{w(x)}\right)|u_n|^{w(x)}\,dS\Bigg] \\
&\ge \left(\frac{1}{\theta} - \frac{1}{q^-}\right)\left(\int_{\partial\Omega}|u|^{q(x)}\,dS + \sum_{i\in I}\nu_i\delta_{x_i}\right) \\
&\ge \left(\frac{1}{\theta} - \frac{1}{q^-}\right)\nu_j \ge \left(\frac{1}{\theta} - \frac{1}{q^-}\right)S^{\frac{(N-1)p^+}{p^--1}},
\end{aligned}$$

which contradicts the assumption that $c < (\frac{1}{\theta} - \frac{1}{q^-})S^{(N-1)p^+/(p^- -1)}$. Hence, $I = \emptyset$. Therefore, we have

$$\int_{\partial\Omega} |u_n|^{q(x)}\, dS \to \int_{\partial\Omega} |u|^{q(x)}\, dS \quad \text{as } n \to \infty, \tag{4.5}$$

i.e., $u_n \to u$ in $L^{q(x)}(\partial\Omega)$.

In the sequel, we let

$$\begin{aligned}\Theta_n(u_n, u) = &- \int_\Omega (f(x, u_n) - f(x, u))\,(u_n - u)\, dx \\ &- \int_{\partial\Omega} \left(|u_n|^{q(x)-2}\, u_n - |u|^{q(x)-2}\, u\right)(u_n - u)\, dS \\ &- \lambda \int_{\partial\Omega} \left(|u_n|^{w(x)-2}\, u_n - |u|^{w(x)-2}\, u\right)(u_n - u)\, dS\end{aligned}$$

and $C_1 = 2\sup_{n\in\mathbb{N}} \|f(x, u_n) - f(x, u)\|_{r_1(x)}$. Then, $0 \le C_1 < \infty$ by (H4). By Proposition 2.2, Remark 2.4, (4.2) and (4.5), we have

$$\begin{aligned}\left|\int_\Omega (f(x, u_n) - f(x, u))\,(u_n - u)\, dx\right| &\le 2\, \|f(x, u_n) - f(x, u)\|_{r_1(x)}\, \|u_n - u\|_{r(x)} \\ &\le C_1\, \|u_n - u\|_{r(x)} \to 0 \quad \text{as } n \to \infty,\end{aligned}$$

$$\begin{aligned}&\left|\int_{\partial\Omega} \left(|u_n|^{q(x)-2}\, u_n - |u|^{q(x)-2}\, u\right)(u_n - u)\, dS\right| \\ &\le \left\||u_n|^{q(x)-2}\, u_n - |u|^{q(x)-2}\, u\right\|_{q_1(x),\partial\Omega} \|u_n - u\|_{q(x),\partial\Omega} \to 0 \quad \text{as } n \to \infty\end{aligned}$$

and

$$\begin{aligned}&\left|\int_{\partial\Omega} \left(|u_n|^{w(x)-2}\, u_n - |u|^{w(x)-2}\, u\right)(u_n - u)\, dS\right| \\ &\le \left\||u_n|^{w(x)-2}\, u_n - |u|^{w(x)-2}\, u\right\|_{w_1(x),\partial\Omega} \|u_n - u\|_{w(x),\partial\Omega} \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Hence,

$$\Theta_n(u_n, u) \to 0 \quad \text{as } n \to \infty. \tag{4.6}$$

Moreover, it is easy to check that

$$\left\langle \phi'(u_n) - \phi'(u), u_n - u\right\rangle \to 0 \quad \text{as } n \to \infty. \tag{4.7}$$

On the other hand, from (3.2), we see that

$$\begin{aligned}&\left\langle \phi'(u_n) - \phi'(u), u_n - u\right\rangle \\ &= \int_\Omega \left(|\nabla u_n|^{p(x)-2}\, \nabla u_n - |\nabla u|^{p(x)-2}\, \nabla u\right)(\nabla u_n - \nabla u)\, dx \\ &\quad + \int_\Omega a(x) \left(|u_n|^{p(x)-2}\, u_n - |u|^{p(x)-2}\, u\right)(u_n - u)\, dx + \Theta_n(u_n, u).\end{aligned} \tag{4.8}$$

Recall the following well known inequalities (see, for example (2.2) in [25])

$$(4.9)\quad \left(|x|^{p-2}x-|y|^{p-2}y,x-y\right)\geq\begin{cases}\dfrac{1}{2^p}|x-y|^p & \text{if } p\geq 2,\\ \dfrac{(p-1)|x-y|^2}{(|x|+|y|)^{2-p}} & \text{if } 1<p<2,\end{cases}\qquad \text{for } x,y\in\mathbb{R}^k \text{ with } k\in\mathbb{N},$$

where $(\cdot,\cdot)$ is the inner product in $\mathbb{R}^k$.

If $p^+\geq p^-\geq 2$, then, from Proposition 2.1, (4.8) and (4.9), we have

$$\begin{aligned}\left\langle\phi'(u_n)-\phi'(u),u_n-u\right\rangle &\geq\frac{1}{2^{p(x)}}\int_\Omega\left(|\nabla u_n-\nabla u|^{p(x)}+a(x)|u_n-u|^{p(x)}\right)dx+\Theta_n(u_n,u)\\ &\geq\frac{1}{2^{p(x)}}\min\left\{\|u_n-u\|_{1,p(x)}^{p^-},\|u_n-u\|_{1,p(x)}^{p^+}\right\}+\Theta_n(u_n,u).\end{aligned}$$

Then, from (4.6) and (4.7), we have $\|u_n-u\|_{1,p(x)}\to 0$ as $n\to\infty$, i.e., $u_n\to u$ in $W^{1,p(x)}(\Omega)$.

If $1<p^-\leq p^+<2$, then, in view of Propositions 2.1 and 2.2, we obtain that

$$\begin{aligned}&\int_\Omega a(x)|u_n-u|^{p(x)}dx\\ &=\int_\Omega\left(\frac{(a(x))^{p(x)/2}|u_n-u|^{p(x)}}{(|u_n|+|u|)^{p(x)(2-p(x))/2}}\right)\left((a(x))^{(2-p(x))/2}(|u_n|+|u|)^{p(x)(2-p(x))/2}\right)dx\\ &\leq\left\|\frac{a^{p(x)/2}|u_n-u|^{p(x)}}{(|u_n|+|u|)^{p(x)(2-p(x))/2}}\right\|_{L^{2/p(x)}(\Omega)}\left\|a^{(2-p(x))/2}(|u_n|+|u|)^{p(x)(2-p(x))/2}\right\|_{L^{2/(2-p(x))}(\Omega)}\\ &\leq\left(\int_\Omega\frac{a(x)|u_n-u|^2}{(|u_n|+|u|)^{2-p(x)}}dx\right)^{\kappa_1}\left(\int_\Omega a(x)(|u_n|+|u|)^{p(x)}dx\right)^{\kappa_2}\qquad(4.10)\\ &\leq 2^{\kappa_2p^+}\left(\int_\Omega\frac{a(x)|u_n-u|^2}{(|u_n|+|u|)^{2-p(x)}}dx\right)^{\kappa_1}\left(\int_\Omega a(x)(|u_n|^{p(x)}+|u|^{p(x)})dx\right)^{\kappa_2}\\ &\leq 2^{\kappa_2p^+}\left(\int_\Omega\frac{a(x)|u_n-u|^2}{(|u_n|+|u|)^{2-p(x)}}dx\right)^{\kappa_1}\left(\rho_{1,p(x)}(u_n)+\rho_{1,p(x)}(u)\right)^{\kappa_2}\\ &\leq 2^{\kappa_2p^+}\left(\int_\Omega\frac{a(x)|u_n-u|^2}{(|u_n|+|u|)^{2-p(x)}}dx\right)^{\kappa_1}(\nu(u_n,u))^{\kappa_2},\end{aligned}$$

where

$$\kappa_1=\begin{cases}\dfrac{p^+}{2} & \text{if } \left\|\frac{a^{p(x)/2}|u_n-u|^{p(x)}}{(|u_n|+|u|)^{p(x)(2-p(x))/2}}\right\|_{L^{2/p(x)}(\Omega)}\geq 1,\\ \dfrac{p^-}{2} & \text{if } \left\|\frac{a^{p(x)/2}|u_n-u|^{p(x)}}{(|u_n|+|u|)^{p(x)(2-p(x))/2}}\right\|_{L^{2/p(x)}(\Omega)}<1,\end{cases}$$

$$\kappa_2=\begin{cases}\dfrac{2-p^-}{2} & \text{if } \left\|a^{(2-p(x))/2}(|u_n|+|u|)^{p(x)(2-p(x))/2}\right\|_{L^{2/(2-p(x))}(\Omega)}\geq 1,\\ \dfrac{2-p^+}{2} & \text{if } \left\|a^{(2-p(x))/2}(|u_n|+|u|)^{p(x)(2-p(x))/2}\right\|_{L^{2/(2-p(x))}(\Omega)}<1\end{cases}$$

and

$$\nu(u_n,u)=\max\left\{\|u_n\|_{1,p(x)}^{p^-},\|u_n\|_{1,p(x)}^{p^+}\right\}+\max\left\{\|u\|_{1,p(x)}^{p^-},\|u\|_{1,p(x)}^{p^+}\right\}.$$

Similarly, we can obtain that

$$\int_\Omega |\nabla u_n - \nabla u|^{p(x)}\,dx \le 2^{\kappa_2 p^+}\left(\int_\Omega \frac{|\nabla u_n - \nabla u|^2}{(|\nabla u_n| + |\nabla u|)^{2-p(x)}}\,dx\right)^{\kappa_1}(\nu(u_n,u))^{\kappa_2}. \tag{4.11}$$

From (4.8), (4.9), (4.10), (4.11) and Proposition 2.1, it follows that

$$\begin{aligned}
&\langle \phi'(u_n) - \phi'(u), u_n - u\rangle \\
&\ge (p-1)\int_\Omega \left(\frac{|\nabla u_n - \nabla u|^2}{(|\nabla u_n| + |\nabla u|)^{2-p(x)}} + \int_\Omega \frac{a(x)\,|u_n - u|^2}{(|u_n| + |u|)^{2-p(x)}}\right)dx + \Theta_n(u_n,u) \\
&\ge \frac{(p-1)}{2^{\frac{\kappa_2}{\kappa_1}p^+}} \frac{\left(\int_\Omega |\nabla u_n - \nabla u|^{p(x)}\,dx\right)^{1/\kappa_1} + \left(\int_\Omega a(x)\,|u_n - u|^{p(x)}\,dx\right)^{1/\kappa_1}}{\nu(u_n,u))^{\kappa_2/\kappa_1}} + \Theta_n(u_n,u) \\
&\ge \frac{(p-1)}{2^{\frac{1}{\kappa_1}(1+\kappa_2 p^+)}} \frac{\left(\int_\Omega |\nabla u_n - \nabla u|^{p(x)} + a(x)\,|u_n - u|^{p(x)}\,dx\right)^{1/\kappa_1}}{(\nu(u_n,u))^{\kappa_2/\kappa_1}} + \Theta_n(u_n,u) \\
&\ge \frac{(p-1)}{2^{\frac{1}{\kappa_1}(1+\kappa_2 p^+)}} \frac{\left(\min\left\{\|u_n - u\|_{1,p(x)}^{p^-}, \|u_n - u\|_{1,p(x)}^{p^+}\right\}\right)^{1/\kappa_1}}{(\nu(u_n,u))^{\kappa_2/\kappa_1}} + \Theta_n(u_n,u).
\end{aligned}$$

Combining this with (4.6) and (4.7), we have $\|u_n - u\|_{1,p(x)} \to 0$ as $n \to \infty$, i.e., $u_n \to u$ in $W^{1,p(x)}(\Omega)$. This completes the proof of the lemma. □

Now, we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. We will apply Lemmas 4.1 and 4.4 to prove Theorem 3.1. To apply Lemma 4.1, we need to verify that there exist $r, R > 0$ such that

(i) $\phi(u) > r$ for any $u \in W^{1,p(x)}(\Omega)$ with $\|u\| = R$,

(ii) $\phi(0) = 0$ and $\phi(v_0) < r$ for some $v_0 \in W^{1,p(x)}(\Omega)$ with $\|v_0\| > R$.

We first check condition (i). By (H4) and (H6), there exists $C_2 > 0$ such that

$$F(x,t) \le \epsilon\,|t|^{p^+} + C_2\,|t|^{r(x)} \quad \text{for all } x \in \Omega \text{ and } t \in \mathbb{R}, \tag{4.12}$$

where $\epsilon \to 0$ as $t \to 0$. For any $u \in W^{1,p(x)}(\Omega)$ with $\|u\|_{1,p(x)} < 1$, we may assume that $\|u\|_{r(x)} < 1$, $\|u\|_{q(x),\partial\Omega} < 1$ and $\|u\|_{w(x),\partial\Omega} < 1$. The other cases can be treated similarly. Then, in view of Propositions 2.1, 2.3 and 2.5, Remark 2.4, (3.1) and (4.12), there exist positive constants C_3, C_4, C_5, C_6 such that

$$\begin{aligned}
\phi(u) \ge{}& \frac{1}{p^+}\,\|u\|_{1,p(x)}^{p^+} - \epsilon\int_\Omega |u|^{p^+}\,dx - C_2\int_\Omega |u|^{r(x)}\,dx \\
&- \frac{1}{q^-}\int_{\partial\Omega} |u|^{q(x)}\,dx - \frac{\lambda}{w^-}\int_{\partial\Omega} |u|^{w(x)}\,dx
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{p^+}\|u\|_{1,p(x)}^{p^+} - \epsilon C_3\|u\|_{1,p(x)}^{p^+} - C_2\|u\|_{r(x)}^{r^-} - \frac{1}{q^-}\|u\|_{q(x),\partial\Omega}^{q^-} - \frac{\lambda}{w^-}\|u\|_{w(x),\partial\Omega}^{w^-} \\
&\geq \left(\frac{1}{p^+} - \epsilon C_3\right)\|u\|_{1,p(x)}^{p^+} - C_4\|u\|_{1,p(x)}^{r^-} - C_5\|u\|_{1,p(x)}^{q^-} - C_6\|u\|_{1,p(x)}^{w^-}.
\end{aligned}$$

Let $h(t) = (\frac{1}{p^+} - \epsilon C_3)t^{p^+} - C_3 t^{r^-} - C_4 t^{q^-} - C_5 t^{w^-}$. Since $p^+ < \min\{r^-, q^-, w^-\}$, we see that there exist $r, R > 0$ such that $h(R) > r$. This proves (i).

On the other hand, from $p^+ < \max\{r^+, q^+, w^+\}$, we have $\lim_{t\to\infty}\phi(tu_0) = -\infty$ for a fixed $u_0 \in W^{1,p(x)}(\Omega)$ with $\|u_0\|_{1,p(x)} \neq 0$. This, together with the fact that $\phi(0) = 0$, implies that condition (ii) holds.

By Lemma 4.1, there exists a sequence $\{u_n\} \subset W^{1,p(x)}(\Omega)$ such that $\phi(u_n) \to c$ and $\phi'(u_n) \to 0$ in $(W^{1,p(x)}(\Omega))^*$, where $c = \inf_{g\in\mathcal{C}}\max_{t\in[0,1]}\phi(g(t))$ with $\mathcal{C} = \{g \in C([0,1], W^{1,p(x)}(\Omega)) : g(0) = 0, g(1) = v_0\}$.

Now, in virtue of Lemma 4.4, the proof will be finished if we can show that

$$c < \left(\frac{1}{\theta} - \frac{1}{q^-}\right) S^{(N-1)p^+/(p^- - 1)}. \tag{4.13}$$

Below, we show that (4.13) holds for all large λ. For $v \in W^{1,p(x)}(\Omega)$ such that $\|v\|_{1,p(x)} \leq 1$. Then, for $0 < t < 1$, from Proposition 2.1 and (3.1), we have

$$\begin{aligned}
\phi(tv) &\leq \frac{1}{p^-}\int_\Omega t^{p(x)}\left(|\nabla v|^{p(x)} + a(x)|v|^{p(x)}\right)dx - \frac{\lambda}{w^+}\int_{\partial\Omega} t^{w(x)}|v|^{w(x)}\,dS \\
&\leq \frac{t^{p^-}}{p^-}\|v\|_{1,p(x)} - \frac{\lambda t^{w^+}}{w^+}\int_{\partial\Omega}|v|^{w(x)}\,dS = h(t),
\end{aligned}$$

where $h(t) = a_1 t^{p^-} - \lambda a_2 t^{w^+}$ with $a_1 = \frac{1}{p^-}\|v\|_{1,p(x)}$ and $a_2 = \frac{\lambda}{w^+}\int_{\partial\Omega}|v|^{w(x)}\,dS$. Note that the maximum of $h(t)$ is achieved at $t_\lambda = (\frac{p^- a_1}{\lambda a_2 w^+})^{1/(w^+ - p^-)}$. Then, there exists $\underline{\lambda} > 0$ such that (4.13) holds for all $\lambda > \underline{\lambda}$. This completes the proof of the theorem. □

4.2. Proof of Theorem 3.2

In this subsection, we assume all the conditions of Theorem 3.2 are satisfied. To prove Theorem 3.2, we need to recall the dual fountain theorem. To this end, let X be a reflexive and separable Banach space with the norm $\|\cdot\|_X$ and X^* its dual space. Then, it is well known (see, for example, [28, p. 233]) that there exist $e_i \in X$ and $e_i^* \in X^*$ such that

$$X = \overline{\operatorname{span}\{e_i : i = 1, 2, \ldots\}}, \quad X^* = \overline{\operatorname{span}\{e_i^* : i = 1, 2, \ldots\}}^{w^*}$$

and

$$\langle e_i^*, e_j\rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

For convenience, we write

$$X_i = \operatorname{span}\{e_i\}, \quad Y_k = \bigoplus_{i=1}^{k} X_i \quad \text{and} \quad Z_k = \overline{\bigoplus_{i=k}^{\infty} X_i}. \tag{4.14}$$

Definition 4.5. Let $c \in \mathbb{R}$. A functional $\phi \in C^1(X, \mathbb{R})$ is said to satisfy the $(\mathrm{PS})_c^*$ condition (with respect to $\{Y_n\}$) if any sequence $\{u_{n_j}\} \subset X$ such that

$$\phi(u_{n_j}) \to c, \ \phi'|_{Y_{n_j}}(u_{n_j}) \to 0 \quad \text{in } X^*, \text{ as } n_j \to \infty,$$

contains a subsequence converging to a critical point of ϕ.

Obviously, $(\mathrm{PS})_c$ condition implies $(\mathrm{PS})_c^*$ condition. The following dual fountain theorem can be found in [26, Theorem 3.18].

Lemma 4.6. *Let $\phi \in C^1(X, \mathbb{R})$ be an even functional. Assume that, for any $k \geq k_0 \in \mathbb{N}$, there exist $\rho_k > \gamma_k > 0$ such that*

(B1) $a_k := \inf_{u \in Z_k, \|u\|_X = \rho_k} \phi(u) \geq 0$;

(B2) $b_k := \sup_{u \in Y_k, \|u\|_X = \gamma_k} \phi(u) < 0$;

(B3) $d_k := \inf_{u \in Z_k, \|u\|_X \leq \rho_k} \phi(u) \to 0$ *as* $k \to \infty$;

(B4) ϕ *satisfies the* $(PS)_c^*$ *condition for every* $c \in [d_{k_0}, 0)$.

Then, I has a sequence of negative critical values converging to 0.

In the sequel, we let Y_k and Z_k be defined by (4.14) with $X = W^{1,p(x)}(\Omega)$.

Lemma 4.7. *The following conclusions are true.*

(a) *For $k \in \mathbb{N}$, define*

$$\beta_k = \sup_{\substack{u \in Z_k \\ \|u\|_{1,p(x)} = 1}} \|u\|_{L^1(\Omega)}.$$

Then, $\lim_{k \to \infty} \beta_k = 0$.

(b) *For $k \in \mathbb{N}$, define*

$$\sigma_k = \sup_{\substack{u \in Z_k \\ \|u\|_{1,p(x)} = 1}} \|u\|_{w(x), \partial\Omega}.$$

Then, $\lim_{k \to \infty} \sigma_k = 0$.

Proof. The proofs of parts (a) and (b) are similar. Below, we just prove part (b). Obviously, $0 < \sigma_{k+1} < \sigma_k$. Then, there exists $\sigma \geq 0$ such that $\lim_{k\to\infty} \sigma_k = \sigma$. For any $k \in \mathbb{N}$, let $u_k \in Z_n$ be such that

$$\|u_k\|_{1,p(x)} = 1 \quad \text{and} \quad 0 \leq \sigma_k - \|u_k\|_{w(x),\partial\Omega} < \frac{1}{k}. \tag{4.15}$$

Since $W^{1,p(x)}(\Omega)$ is reflexive, there exists a subsequence of $\{u_k\}$ (which is still denoted by $\{u_k\}$) and $u \in W^{1,p(x)}(\Omega)$ such that $u_k \rightharpoonup u$. We claim $u = 0$. In fact, for any $e_m^* \in \{e_k^* : k \in \mathbb{N}\}$, we have $\langle e_m^*, u_k\rangle = 0$ when $k > m$. Hence,

$$\langle e_m^*, u\rangle = \lim_{k\to\infty} \langle e_m^*, u_k\rangle = 0 \quad \text{for } m \in \mathbb{N}.$$

Therefore, $u = 0$, i.e., $u_k \rightharpoonup 0$. By Proposition 2.5, $W^{1,p(x)}(\Omega) \hookrightarrow L^{w(x)}(\partial\Omega)$ is compact. Hence, $u_k \to 0$ in $L^{w(x)}(\partial\Omega)$. Then, from (4.15), $\lim_{k\to\infty} \sigma_k = 0$. This completes the proof of the lemma. □

Lemma 4.8. *Let $\{u_n\} \subset W^{1,p(x)}(\Omega)$ be a (PS)$_c$ sequence for ϕ with energy level c, where ϕ is defined by* (3.1)*. Then, $\{u_n\}$ is bounded in $W^{1,p(x)}(\Omega)$.*

The proof of this lemma is similar to that of Lemma 4.3.

Proof. Let $\{u_n\} \subset W^{1,p(x)}(\Omega)$ be a (PS)$_c$ sequence for ϕ, i.e.,

$$\phi(u_n) \to c,\ \phi'(u_n) \to 0 \quad \text{in } (W^{1,p(x)}(\Omega))^* \text{ as } n \to \infty.$$

Choose $\beta \in (p^+, \min\{q^-, \alpha^-\})$. Then, for n sufficiently large, from Propositions 2.1 and 2.5, Remark 2.4, (H5), (3.1) and (3.2), we obtain that

$$\begin{aligned}
&|c| + 1 + \|u_n\|_{1,p(x)} \\
&\geq \phi(u_n) - \frac{1}{\beta}\langle \phi'(u_n), u_n\rangle \\
&= \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{\beta}\right)\left(|\nabla u_n|^{p(x)} + a(x)|u_n|^{p(x)}\right) dx + \int_\Omega \left(\frac{1}{\beta} f(x,u_n)u_n - F(x,u_n)\right) dx \\
&\quad + \int_{\partial\Omega} \left(\frac{1}{\beta} - \frac{1}{q(x)}\right)|u_n|^{q(x)}\, dS + \lambda \int_{\partial\Omega}\left(\frac{1}{\beta} - \frac{1}{w(x)}\right)|u_n|^{w(x)}\, dS \\
&\geq \left(\frac{1}{p^+} - \frac{1}{\beta}\right)\int_\Omega \left(|\nabla u_n|^{p(x)} + a(x)|u_n|^{p(x)}\right) dx + \left(\frac{1}{\beta} - \frac{1}{q^-}\right)\int_{\partial\Omega} |u_n|^{q(x)}\, dS \\
&\quad + \lambda\left(\frac{1}{\beta} - \frac{1}{w^-}\right)\max\left\{\|u_n\|_{w(x),\partial\Omega}^{w^-}, \|u_n\|_{w(x),\partial\Omega}^{w^+}\right\} \\
&\geq \left(\frac{1}{p^+} - \frac{1}{\beta}\right)\min\left\{\|u_n\|_{1,p(x)}^{p^-}, \|u_n\|_{1,p(x)}^{p^+}\right\} + \left(\frac{1}{\beta} - \frac{1}{q^-}\right)\int_{\partial\Omega} |u_n|^{q(x)}\, dS \\
&\quad + \lambda C_6\left(\frac{1}{\beta} - \frac{1}{w^-}\right)\max\left\{\|u_n\|_{1,p(x)}^{w^-}, \|u_n\|_{1,p(x)}^{w^+}\right\},
\end{aligned}$$

where $C_6 > 0$ is an appropriate constant. Since

$$p^+ \ge p^- > w^+ \ge w^- > 1, \quad \frac{1}{p^+} - \beta > 0 \quad \text{and} \quad \frac{1}{\beta} - \frac{1}{q^-} > 0,$$

we get that $\{u_n\}$ is bounded in $W^{1,p(x)}(\Omega)$. This completes the proof of the lemma. □

Now, let $\theta \in (p^+, \min\{q^-, \alpha^-\})$ be fixed. To present the next lemma, we first introduce the following notations:

$$b_1 = \frac{1}{\theta} - \frac{1}{q^-} > 0, \quad b_2 = \left(\frac{1}{w^-} - \frac{1}{\theta}\right) \|1\|_{q(x)/(q(x)-w(x)),\partial\Omega} > 0,$$
$$K_1 = \frac{b_1 b_2 (w^+ - q^-)}{q^- b_1} \left(\frac{w^+ b_2}{q^- b_1}\right)^{q^-/(q^- - w^+)} < 0,$$
$$K_2 = \frac{b_1 b_2 (w^- - q^+)}{q^+ b_1} \left(\frac{w^- b_2}{q^+ b_1}\right)^{q^+/(q^+ - w^-)} < 0$$

and

$$K = \min\{K_1, K_2\} < 0.$$

Clearly, K depends only on q, w, θ and Ω.

Lemma 4.9. *Let $\{u_n\} \subset W^{1,p(x)}(\Omega)$ be a (PS)$_c$ sequence for ϕ with energy level c and $\theta \in (p^+, \min\{q^-, \alpha^-\})$ be fixed, where ϕ is defined by* (3.1)*. If*

$$c < \left(\frac{1}{\theta} - \frac{1}{q^-}\right) S^{(N-1)p^+/(p^- - 1)} + K \max\left\{\lambda^{q^-/(q^- - w^+)}, \lambda^{q^+/(q^+ - w^-)}\right\}, \tag{4.16}$$

where S is given in Remark 2.4*, then there exists a subsequence of $\{u_n\}$ that converges strongly in $W^{1,p(x)}(\Omega)$.*

The proof of Lemma 4.9 is similar to that of Lemma 4.4. In what follows, we give a sketch of the proof.

Proof. Let $\{u_n\}$ be given as in the lemma. By Lemma 4.8, $\{u_n\}$ is bounded in $W^{1,p(x)}(\Omega)$. Then, up to a subsequence, we may assume that (4.2) holds. Let I be the set appeared in (4.2). Below, we show that $I = \emptyset$. Assume, to the contrary, that $I \ne \emptyset$. Let $j \in I$ be fixed and $\phi \in C_0^\infty(\mathbb{R}^N)$ with the support in the unit ball of $\mathbb{R}^N$. Then, arguing as in the proof of Lemma 4.4, we can show that (4.4) holds. From (H5), (3.1), (3.2), (4.2), (4.4) and the fact that $\theta \in (p^+, \min\{q^-, \alpha^-\})$, it follows that

$$c = \lim_{n\to\infty} \phi(u_n) = \lim_{n\to\infty} \left(\phi(u_n) - \frac{1}{\theta}\langle \phi'(u_n), u_n\rangle\right)$$
$$= \lim_{n\to\infty} \left[\int_\Omega \left(\frac{1}{p(x)} - \frac{1}{\theta}\right)\left(|\nabla u_n|^{p(x)} + a(x)|u_n|^{p(x)}\right) dx\right.$$

$$
\begin{aligned}
&+\int_\Omega \left(\frac{1}{\theta} f(x,u_n)u_n - F(x,u_n)\right) dx + \int_{\partial\Omega} \left(\frac{1}{\theta} - \frac{1}{q(x)}\right) |u_n|^{q(x)}\, dS \\
&+\lambda \int_{\partial\Omega} \left(\frac{1}{\theta} - \frac{1}{w(x)}\right) |u_n|^{w(x)}\, dS\Bigg] \\
\geq{}& \left(\frac{1}{\theta} - \frac{1}{q^-}\right) \int_{\partial\Omega} |u|^{q(x)}\, dS + \left(\frac{1}{\theta} - \frac{1}{q^-}\right)\nu_j + \lambda \left(\frac{1}{\theta} - \frac{1}{w^-}\right) \int_{\partial\Omega} |u|^{w(x)}\, dS \\
\geq{}& \left(\frac{1}{\theta} - \frac{1}{q^-}\right) \int_{\partial\Omega} |u|^{q(x)}\, dS + \left(\frac{1}{\theta} - \frac{1}{q^-}\right) S^{(N-1)p^+/(p^--1)} \\
&+\lambda \left(\frac{1}{\theta} - \frac{1}{w^-}\right) \int_{\partial\Omega} |u|^{w(x)}\, dS.
\end{aligned}
$$

Now, in view of Remark 2.4, we obtain that

$$
\begin{aligned}
c \geq{}& \left(\frac{1}{\theta} - \frac{1}{q^-}\right) \int_{\partial\Omega} |u|^{q(x)}\, dS + \left(\frac{1}{\theta} - \frac{1}{q^-}\right) S^{(N-1)p^+/(p^--1)} \\
&+\lambda \left(\frac{1}{\theta} - \frac{1}{w^-}\right) \|1\|_{q(x)/(q(x)-w(x)),\partial\Omega} \left\|u^{w(x)}\right\|_{q(x)/w(x),\partial\Omega}.
\end{aligned} \tag{4.17}
$$

If $\|u\|_{q(x),\partial\Omega} \geq 1$, from Remark 2.4, Proposition 2.8 and (4.17), we have

$$
\begin{aligned}
c \geq{}& \left(\frac{1}{\theta} - \frac{1}{q^-}\right) S^{(N-1)p^+/(p^--1)} + \left(\frac{1}{\theta} - \frac{1}{q^-}\right) \|u\|_{q(x),\partial\Omega}^{q^-} \\
&+\lambda \left(\frac{1}{\theta} - \frac{1}{w^-}\right) \|1\|_{q(x)/(q(x)-w(x)),\partial\Omega} \|u\|_{q(x),\partial\Omega}^{w^+} \\
={}& \left(\frac{1}{\theta} - \frac{1}{q^-}\right) S^{(N-1)p^+/(p^--1)} + b_1 \|u\|_{q(x),\partial\Omega}^{q^-} - \lambda b_2 \|u\|_{q(x),\partial\Omega}^{w^+}.
\end{aligned}
$$

Let

$$
h_1(t) = b_1 t^{q^-} - \lambda b_2 t^{w^+}, \quad t > 0.
$$

Since $h_1(t)$ achieves its absolute minimum at $t_0 = \left(\frac{\lambda w^+ b_2}{q^- b_1}\right)^{1/(q^- - w^+)}$, we have

$$
h_1(t) \geq h_1(t_0) = \frac{b_1 b_2 (w^+ - q^-)}{q^- b_1} \left(\frac{w^+ b_2}{q^- b_1}\right)^{q^-/(q^- - w^+)} \lambda^{q^-/(q^- - w^+)} = K_1 \lambda^{q^-/(q^- - w^+)}.
$$

Hence,

$$
c \geq \left(\frac{1}{\theta} - \frac{1}{q^-}\right) S^{(N-1)p^+/(p^--1)} + K_1 \lambda^{q^-/(q^- - w^+)}. \tag{4.18}
$$

On the other hand, if $\|u\|_{q(x),\partial\Omega} \leq 1$, again from Remark 2.4, Proposition 2.8 and (4.17),

$$
\begin{aligned}
c \geq{}& \left(\frac{1}{\theta} - \frac{1}{q^-}\right) S^{(N-1)p^+/(p^--1)} + \left(\frac{1}{\theta} - \frac{1}{q^-}\right) \|u\|_{q(x),\partial\Omega}^{q^+} \\
&+\lambda \left(\frac{1}{\theta} - \frac{1}{w^-}\right) \|1\|_{q(x)/(q(x)-w(x)),\partial\Omega} \|u\|_{q(x),\partial\Omega}^{w^-} \\
={}& \left(\frac{1}{\theta} - \frac{1}{q^-}\right) S^{(N-1)p^+/(p^--1)} + b_1 \|u\|_{q(x),\partial\Omega}^{q^+} - \lambda b_2 \|u\|_{q(x),\partial\Omega}^{w^-}.
\end{aligned}
$$

Let

$$h_2(t) = b_1 t^{q^+} - \lambda b_2 t^{w^-}, \quad t > 0.$$

Since $h_2(t)$ achieves its absolute minimum at $t_0 = \left(\frac{\lambda w^- b_2}{q^+ b_1}\right)^{1/(q^+ - w^-)}$, we have

$$h_1(t) \geq h_1(t_0) = \frac{b_1 b_2 (w^- - q^+)}{q^+ b_1} \left(\frac{w^- b_2}{q^+ b_1}\right)^{q^+/(q^+ - w^-)} \lambda^{q^+/(q^+ - w^-)} = K_2 \lambda^{q^+/(q^+ - w^-)}.$$

Thus,

$$c \geq \left(\frac{1}{\theta} - \frac{1}{q^-}\right) S^{(N-1)p^+/(p^- - 1)} + K_2 \lambda^{q^+/(q^+ - w^-)}. \tag{4.19}$$

From (4.18) and (4.19), we see that

$$c \geq \left(\frac{1}{\theta} - \frac{1}{q^-}\right) S^{(N-1)p^+/(p^- - 1)} + K \max\left\{\lambda^{q^-/(q^- - w^+)}, \lambda^{q^+/(q^+ - w^-)}\right\},$$

which contradicts (4.16). Therefore, $I = \emptyset$. This, in turn, implies that (4.5) holds. Now, by following exactly the same argument as in the proof of Lemma 4.4, we can finish the proof of the lemma. □

Lemma 4.10. *There exists $\overline{\lambda} > 0$ such that if $0 < \lambda < \overline{\lambda}$, then ϕ satisfies a local (PS)$_c$ condition for any $c \leq 0$.*

Proof. Let $c \leq 0$. Clearly, there exists $\overline{\lambda} > 0$ such that if $0 < \lambda < \overline{\lambda}$, then

$$c \leq 0 < \left(\frac{1}{\theta} - \frac{1}{q^-}\right) S^{(N-1)p^+/(p^- - 1)} + K \max\left\{\lambda^{q^-/(q^- - w^+)}, \lambda^{q^+/(q^+ - w^-)}\right\}.$$

Then, by Lemma 4.8, ϕ satisfies a local (PS)$_c$ condition when $0 < \lambda < \overline{\lambda}$. □

Now, we are in a position to prove Theorem 3.2.

Proof of Theorem 3.2. Let $\overline{\lambda}$ be given as in Lemma 4.10 and $0 < \lambda < \overline{\lambda}$. We will apply Lemma 4.6 to prove Theorem 3.2. From (H7), ϕ is even. Below, we verify the conditions (B1)–(B4) of Lemma 4.6.

From (H4), there exists $D_1 > 0$ such that

$$F(x, u) \leq D_1 \left(|u| + |u|^{r(x)}\right) \quad \text{for all } x \in \Omega \text{ and } u \in W^{1,p(x)}(\Omega).$$

Let $k \in \mathbb{N}$ and $u \in Z_k$. Then, from Proposition 2.1, Remark 2.4 and (3.1), we have

$$\begin{aligned}
\phi(u) \geq\ & \frac{1}{p^+} \min\left\{\|u\|_{1,p(x)}^{p^-}, \|u\|_{1,p(x)}^{p^+}\right\} - D_1 \int_\Omega \left(|u| + |u|^{r(x)}\right) dx \\
& - \frac{1}{q^-} \int_{\partial\Omega} |u|^{q(x)}\, dS - \frac{\lambda}{w^-} \int_{\partial\Omega} |u|^{w(x)}\, dS \\
\geq\ & \frac{1}{p^+} \min\left\{\|u\|_{1,p(x)}^{p^-}, \|u\|_{1,p(x)}^{p^+}\right\} - D_1 \|u\|_{L^1(\Omega)} - D_1 \max\left\{\|u\|_{r(x)}^{r^-}, \|u\|_{r(x)}^{r^+}\right\} \\
& - \frac{1}{q^-} \max\left\{\|u\|_{q(x),\partial\Omega}^{q^-}, \|u\|_{q(x),\partial\Omega}^{q^+}\right\} - \frac{\lambda}{w^-} \max\left\{\|u\|_{w(x),\partial\Omega}^{w^-}, \|u\|_{w(x),\partial\Omega}^{w^+}\right\}.
\end{aligned}$$

Now, by Propositions 2.3 and 2.5 and Lemma 4.7, there exist $D_2, D_3 > 0$ such that

$$\begin{aligned}
\phi(u) &\geq \frac{1}{p^+} \min\left\{\|u\|_{1,p(x)}^{p^-}, \|u\|_{1,p(x)}^{p^+}\right\} - D_1\beta_k \|u\|_{1,p(x)} - D_2 \max\left\{\|u\|_{1,p(x)}^{r^-}, \|u\|_{1,p(x)}^{r^+}\right\} \\
&\quad - D_3 \max\left\{\|u\|_{1,p(x)}^{q^-}, \|u\|_{1,p(x)}^{q^+}\right\} - \frac{\lambda}{w^-} \max\left\{\sigma_k^{w^-} \|u\|_{1,p(x)}^{w^-}, \sigma_k^{w^+} \|u\|_{1,p(x)}^{w^+}\right\} \\
&= \left(\frac{1}{4p^+} \min\left\{\|u\|_{1,p(x)}^{p^-}, \|u\|_{1,p(x)}^{p^+}\right\} - D_1\beta_k \|u\|_{1,p(x)}\right) \\
&\quad + \left(\frac{1}{4p^+} \min\left\{\|u\|_{1,p(x)}^{p^-}, \|u\|_{1,p(x)}^{p^+}\right\} - \frac{\lambda}{w^-} \max\left\{\sigma_k^{w^-} \|u\|_{1,p(x)}^{w^-}, \sigma_k^{w^+} \|u\|_{1,p(x)}^{w^+}\right\}\right) \\
&\quad + \left(\frac{1}{4p^+} \min\left\{\|u\|_{1,p(x)}^{p^-}, \|u\|_{1,p(x)}^{p^+}\right\} - D_2 \max\left\{\|u\|_{1,p(x)}^{r^-}, \|u\|_{1,p(x)}^{r^+}\right\}\right) \\
&\quad + \left(\frac{1}{4p^+} \min\left\{\|u\|_{1,p(x)}^{p^-}, \|u\|_{1,p(x)}^{p^+}\right\} - D_3 \max\left\{\|u\|_{1,p(x)}^{q^-}, \|u\|_{1,p(x)}^{q^+}\right\}\right).
\end{aligned}$$

Since $p^- \leq p^+ < r^- \leq r^+$ and $p^- \leq p^+ < q^- \leq q^+$, there exists $0 < \rho < 1$ such that

$$\frac{1}{4p^+} \min\left\{\|u\|_{1,p(x)}^{p^-}, \|u\|_{1,p(x)}^{p^+}\right\} - D_2 \max\left\{\|u\|_{1,p(x)}^{r^-}, \|u\|_{1,p(x)}^{r^+}\right\} \geq 0$$

and

$$\frac{1}{4p^+} \min\left\{\|u\|_{1,p(x)}^{p^-}, \|u\|_{1,p(x)}^{p^+}\right\} - D_3 \max\left\{\|u\|_{1,p(x)}^{q^-}, \|u\|_{1,p(x)}^{q^+}\right\} \geq 0$$

for any $\|u\|_{1,p(x)} \leq \rho$. From Lemma 4.7, $\lim_{k\to\infty} \sigma_k = 0$. Then, there exists $k_1 \in \mathbb{N}$ such that $0 < \sigma_k < 1$ for all $k \geq k_1$. Thus, if $\|u\|_{1,p(x)} \leq \rho < 1$ and $k \geq k_1$, we have

$$\begin{aligned}
\phi(u) &\geq \left(\frac{1}{4p^+} \min\left\{\|u\|_{1,p(x)}^{p^-}, \|u\|_{1,p(x)}^{p^+}\right\} - D_1\beta_k \|u\|_{1,p(x)}\right) \\
&\quad + \left(\frac{1}{4p^+} \min\left\{\|u\|_{1,p(x)}^{p^-}, \|u\|_{1,p(x)}^{p^+}\right\} - \frac{\lambda}{w^-} \max\left\{\sigma_k^{w^-} \|u\|_{1,p(x)}^{w^-}, \sigma_k^{w^+} \|u\|_{1,p(x)}^{w^+}\right\}\right) \\
&\geq \left(\frac{1}{4p^+} \|u\|_{1,p(x)}^{p^+} - D_1\beta_k \|u\|_{1,p(x)}\right) + \left(\frac{1}{4p^+} \|u\|_{1,p(x)}^{p^+} - \frac{\lambda}{w^-}\sigma_k^{w^-} \|u\|_{1,p(x)}^{w^-}\right).
\end{aligned} \tag{4.20}$$

Let

$$\rho_k = \max\left\{(4p^+ D_1\beta_k)^{1/(p^+-1)}, \left(\frac{4\lambda p^+ \sigma_k^{w^-}}{w^-}\right)^{1/(p^+-w^-)}\right\}.$$

Then, $\rho_k > 0$, and for any $u \in Z_k$ with $\|u\|_{1,p(x)} = \rho_k$, it is easy to verify that

$$\begin{aligned}
\frac{1}{4p^+} \|u\|_{1,p(x)}^{p^+} - D_1\beta_k \|u\|_{1,p(x)} &\geq 0, \\
\frac{1}{4p^+} \|u\|_{1,p(x)}^{p^+} - \frac{\lambda}{w^-}\sigma_k^{w^-} \|u\|_{1,p(x)}^{w^-} &\geq 0.
\end{aligned} \tag{4.21}$$

Moreover, $\rho_k \to 0$ by Lemma 4.7. Thus, there exists $k_2 \in \mathbb{N}$ such that $0 < \rho_k < \rho < 1$ for $k \geq k_2$. Let $k_0 = \max\{k_1, k_2\}$. Then, for all $k \geq k_0$ and $u \in Z_k$ with $\|u\|_{1,p(x)} = \rho_k$, from (4.20) and (4.21), we see that $\phi(u) \geq 0$, i.e., condition (B1) holds.

From (H5), there exists $D_4 > 0$ such that $F(x,u) \geq D_4(|u|^{\alpha(x)} - 1)$ for all $x \in \Omega$ and $u \in W^{1,p(x)}(\Omega)$. Then, for any $u \in W^{1,p(x)}(\Omega)$, from Proposition 2.1 and (3.1),

$$\begin{aligned}
\phi(u) &\leq \frac{1}{p^-} \max\left\{\|u\|_{1,p(x)}^{p^-}, \|u\|_{1,p(x)}^{p^+}\right\} - D_4 \int_\Omega |u|^{\alpha(x)}\,dx - D_4 \operatorname{meas}(\Omega) \\
&\quad - \frac{1}{q^+} \int_{\partial\Omega} |u|^{q(x)}\,dS - \frac{\lambda}{w^+} \int_{\partial\Omega} |u|^{w(x)}\,dS \\
&\leq \frac{1}{p^+} \max\left\{\|u\|_{1,p(x)}^{p^-}, \|u\|_{1,p(x)}^{p^+}\right\} - D_4 \int_\Omega |u|^{\alpha(x)}\,dx - D_4 \operatorname{meas}(\Omega) \\
&\quad - \frac{1}{q^-} \int_{\partial\Omega} |u|^{q(x)}\,dS - \frac{\lambda}{w^-} \min\left\{\|u\|_{w(x)}^{w^-}, \|u\|_{w(x)}^{w^+}\right\},
\end{aligned} \tag{4.22}$$

where meas$(\cdot)$ stands for the Lebesgue measure of a set. Note that all norms of the finite dimensional Y_k are equivalent. Then, since $w^- \leq w^+ < p^- \leq p^+$, from (4.22), we can choose $\gamma_k > 0$ small enough so that $\gamma_k < \rho_k$ and $\phi(u) < 0$ for all $u \in Y_k$ with $\|u\|_{1,p(x)} = \gamma_k$. This verifies condition (B2).

From (4.20), we see that, for all $k \geq k_0$ and $u \in Z_k$ with $\|u\|_{1,p(x)} \leq \rho_k$,

$$\phi(u) \geq -D_1\beta_k\rho_k - \frac{\lambda}{w^-}\sigma_k^{w^-}\rho_k^{w^-}.$$

As $\beta_k \to 0$, $\sigma_k \to 0$ and $\rho_k \to 0$ by Lemma 4.7, condition (B3) is also satisfied.

Finally, by Lemma 4.10, ϕ satisfies $(\mathrm{PS})_c$ condition for any $c \leq 0$. This obviously implies that condition (B4) holds.

We have verified that all the conditions of Lemma 4.6 are satisfied. Therefore, the conclusion of the theorem follows from Lemma 4.6. This completes the proof of the theorem. □

References

[1] G. A. Afrouzi, A. Hadjian and S. Heidarkhani, *Steklov problems involving the $p(x)$-Laplacian*, Electron. J. Differential Equations **2014**, no. 134, 11 pp.

[2] G. A. Afrouzi, S. Heidarkhani and S. Shokooh, *Infinitely many solutions for Steklov problems associated to non-homogeneous differential operators through Orlicz-Sobolev spaces*, Complex Var. Elliptic Equ. **60** (2015), no. 11, 1505–1521.
https://doi.org/10.1080/17476933.2015.1031122

[3] M. Allaoui, *Continuous spectrum of Steklov nonhomogeneous elliptic problem*, Opuscula Math. **35** (2015), no. 6, 853–866.
https://doi.org/10.7494/opmath.2015.35.6.853

[4] M. Allaoui, A. R. El Amrouss and A. Ourraoui, *Existence of infinitely many solutions for a Steklov problem involving the $p(x)$-Laplace operator*, Electron. J. Qual. Theory Differ. Equ. **2014**, no. 20, 10 pp. https://doi.org/10.14232/ejqtde.2014.1.20

[5] J.-P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, Pure and Applied Mathematics (New York), John Wiley & Sons, New York, 1984.

[6] J. F. Bonder and J. D. Rossi, *Existence results for the p-Laplacian with nonlinear boundary conditions*, J. Math. Anal. Appl. **263** (2001), no. 1, 195–223. https://doi.org/10.1006/jmaa.2001.7609

[7] J. F. Bonder, N. Saintier and A. Silva, *On the Sobolev trace theorem for variable exponent spaces in the critical range*, Ann. Mat. Pura Appl. (4) **193** (2014), no. 6, 1607–1628. https://doi.org/10.1007/s10231-013-0346-6

[8] ______, *Existence of solution to a critical trace equation with variable exponent*, Asymptot. Anal. **93** (2015), no. 1-2, 161–185. https://doi.org/10.3233/asy-151289

[9] N. T. Chung, *Multiple solutions for quasilinear elliptic problems with nonlinear boundary conditions*, Electron. J. Differential Equations **2008**, no. 165, 6 pp.

[10] S.-G. Deng, *Eigenvalues of the $p(x)$-Laplacian Steklov problem*, J. Math. Anal. Appl. **339** (2008), no. 2, 925–937. https://doi.org/10.1016/j.jmaa.2007.07.028

[11] S.-G. Deng, Q. Wang and S. Cheng, *On the $p(x)$-Laplacian Robin eigenvalue problem*, Appl. Math. Comput. **217** (2011), no. 12, 5643–5649. https://doi.org/10.1016/j.amc.2010.12.042

[12] L. Diening, P. Harjulehto, P. Hästö and M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Mathematics **2017**, Springer, Heidelberg, 2011. https://doi.org/10.1007/978-3-642-18363-8

[13] D. Edmunds and J. Rákosník, *Sobolev embeddings with variable exponent*, Studia Math. **143** (2000), no. 3, 267–293.

[14] X. Fan, *Boundary trace embedding theorems for variable exponent Sobolev spaces*, J. Math. Anal. Appl. **339** (2008), no. 2, 1395–1412. https://doi.org/10.1016/j.jmaa.2007.08.003

[15] X. Fan and X. Han, *Existence and multiplicity of solutions for $p(x)$-Laplacian equations in $\mathbf{R}^N$*, Nonlinear Anal. **59** (2004), no. 1-2, 173–188. https://doi.org/10.1016/s0362-546x(04)00254-8

[16] X. Fan and D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. **263** (2001), no. 2, 424–446. https://doi.org/10.1006/jmaa.2000.7617

[17] T. C. Halsey, *Electrorheological Fluids*, Science **258** (1992), no. 5083, 761–766. https://doi.org/10.1126/science.258.5083.761

[18] L. Kong, *Eigenvalues for a fourth order elliptic problem*, Proc. Amer. Math. Soc. **143** (2015), no. 1, 249–258. https://doi.org/10.1090/s0002-9939-2014-12213-1

[19] ______, *Multiple solutions for fourth order elliptic problems with $p(x)$-biharmonic operators*, Opuscula Math. **36** (2016), no. 2, 252–264. https://doi.org/10.7494/opmath.2016.36.2.253

[20] O. Kováčik and Rákosník, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J. **41** (1991), no. 4, 592–618.

[21] A. Ourraoui, *Multiplicity results for Steklov problem with variable exponent*, Appl. Math. Comput. **277** (2016), 34–43. https://doi.org/10.1016/j.amc.2015.12.043

[22] C. D. Pagani and D. Pierotti, *Multiple variational solutions to nonlinear Steklov problems*, NoDEA Nonlinear Differential Equations Appl. **19** (2012), no. 4, 417–436. https://doi.org/10.1007/s00030-011-0136-z

[23] M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory*, Lecture Notes in Mathematics **1748**, Springer-Verlag, Berlin, 2000. https://doi.org/10.1007/bfb0104029

[24] J. C. Sabina de Lis and S. Segura de León, *Multiplicity of solutions to a nonlinear boundary value problem of concave-convex type*, Nonlinear Anal. **113** (2015), 283–297. https://doi.org/10.1016/j.na.2014.09.029

[25] J. Simon, *Régularité de la solution d'une équation non linéaire dans $\mathbf{R}^N$* in *Journées d'Analyse Non Linéaire* (Proc. Conf., Besançon, 1977), 205–227, Lecture Notes in Math. **665**, Springer, Berlin, 1978. https://doi.org/10.1007/bfb0061807

[26] M. Willem, *Minimax Theorems*, Progress in Nonlinear Differential Equations and Their Applications **24**, Birkhäuser Boston, Boston, MA, 1996. https://doi.org/10.1007/978-1-4612-4146-1

[27] J. Yao, *Solutions for Neumann boundary value problems involving $p(x)$-Laplace operators*, Nonlinear Anal. **68** (2008), no. 5, 1271–1238. https://doi.org/10.1016/j.na.2006.12.020

[28] J. Zhao, *Structure Theory for Banach Space*, Wuhan Univ. Press, Wuhan, 1991.

[29] J.-H. Zhao and P.-H. Zhao, *Infinitely many weak solutions for a p-Laplacian equation with nonlinear boundary conditions*, Electron. J. Differential Equations **2007**, no. 90, 14 pp.

[30] V. V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Math. USSR-Izv. **29** (1987), no. 1, 33–66.
https://doi.org/10.1070/im1987v029n01abeh000958

Lingju Kong
Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, TN 37403, USA
E-mail address: Lingju-Kong@utc.edu