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On Adjacent Vertex-distinguishing Total Chromatic Number of Generalized Mycielski Graphs

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Abstract. The adjacent vertex-distinguishing total chromatic number of a graph G, denoted by $\chi_{at}(G)$, is the smallest k for which G has a proper total k-coloring such that any two adjacent vertices have distinct sets of colors appearing on the vertex and its incident edges. In regard of this number, there is a famous conjecture (AVDTCC) which states that for any simple graph G, $\chi_{at}(G) \leq \Delta(G)+3$. In this paper, we study this number for the generalized Mycielski graph $\mu_m(G)$ of a graph G. We prove that the satisfiability of the conjecture AVDTCC in G implies its satisfiability in $\mu_m(G)$. Particularly we give the exact values of $\chi_{at}(\mu_m(G))$ when G is a graph with maximum degree less than 3 or a complete graph. Moreover, we investigate $\chi_{at}(G)$ for any graph G with only one maximum degree vertex by showing that $\chi_{at}(G) \leq \Delta(G) + 2$ when $\Delta(G) \leq 4$.

1. Introduction

In this paper we confine our attention to graphs that are finite, simple, connected and undirected. For a graph G, we denote by $V(G)$, $E(G)$, $d_G(v)$ and $\Delta(G)$ the *vertex set*, *edge set*, *degree* of $v \in V(G)$ and *maximum degree* of G, respectively. We use $[a,b]$ to denote the set $\{a, a+1, a+2, \ldots, b\}$ for two integers a and b with $a < b$. Notations and terminologies undefined here are followed [1].

Let G be a graph, and $V' \subseteq V(G)$, $E' \subseteq E(G)$. A *partial total k-coloring* of G regarding to $V' \cup E'$ is a coloring $f\colon (V' \cup E') \to [1,k]$, such that no incident or adjacent elements in $V' \cup E'$ receive the same color. When $V' = V$ and $E' = E$, we refer to f as a *total k-coloring* of G. Given a partial total k-coloring f of G regarding to $V' \cup E'$, for a vertex $v \in V$ and a subset $S \subseteq [1,k]$, we name $C_f^S(v) = (\{f(v)\} \cup \{f(uv) : uv \in E'\}) \cap S$ as the *color set restricted to S* of v (under f), or simply *color set* of v when $S = [1,k]$.

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Let $\overline{C}_f^S(v) = S \setminus C_f^S(v)$. Note that when $v \notin V'$, v does not receive any color under f. So $\{f(v)\} = \emptyset$ and $C_f^S(v) = \{f(uv) : uv \in E'\} \cap S$ in this case.

Let f be a total k-coloring of G, if $C_f^{[1,k]}(u) \neq C_f^{[1,k]}(v)$ for any two adjacent vertices u, v, then we call f an *adjacent vertex-distinguishing total coloring* (AVDTC) of G. The smallest k for which G has a k-AVDTC is called the *adjacent vertex-distinguishing total chromatic number* of G, denoted by $\chi_{at}(G)$. Clearly, if f is an AVDTC of G, then for each pair of adjacent vertices $u, v \in V$, $C_f^{[1,k]}(u) \triangle C_f^{[1,k]}(v) \neq \emptyset$, where $\triangle$ denotes the symmetric difference of two sets.

As an extension of vertex-distinguishing proper edge coloring of graphs [2], AVDTC was first examined by Zhang et al. [23], where $\chi_{at}(G)$ for many basic families of graphs were determined and a conjecture called AVDTCC was proposed.

Conjecture 1.1 (AVDTCC)**.** *For any simple graph G, $\Delta(G) + 1 \leq \chi_{at}(G) \leq \Delta(G) + 3$.*

The lower bound in Conjecture 1.1 is easy to see. In addition, when G has two adjacent vertices with maximum degree, $\chi_{at}(G) \geq \Delta(G) + 2$. For the upper bound, there exist graphs G with $\chi_{at}(G) = \Delta(G) + 3$, for example the complete graph K_n for $n \equiv 1 \pmod 2$ [23].

Chen [4], and independently Wang [18], confirmed Conjecture 1.1 for graphs G with $\Delta(G) \leq 3$. Later, Hulgan [9] provided a more concise proof on this result. In [20] and [22], $\chi_{at}(G)$ for K_4-minor free graphs and outerplane graphs were investigated. Wang [21] and Huang [8] considered $\chi_{at}(G)$ for graphs with smaller maximum average degree and large maximum degree, respectively. A more recent work is Wang [19], which focused on $\chi_{at}(G)$ for planar graphs.

Graphs considered in this paper are *Mycielski graphs*, which were first introduced in [15]. Such kind of graphs has gained much attention in the community of graph coloring [3, 6, 11, 13, 14, 16, 17]. Also, the Mycielskian of G was generalized to the *m-Mycielskian* of G, where $m \geq 1$ [12].

Let G be a graph with vertex set $V^0 = \{v_1^0, v_2^0, \ldots, v_n^0\}$ and edge set E^0. Given an integer $m \geq 1$, the *m-Mycielskian* of G, denoted by $\mu_m(G)$, is the graph with vertex set $V^0 \cup V^1 \cup \cdots \cup V^m \cup \{u\}$, where $V^i = \left\{v_j^i : v_j^0 \in V^0\right\}$ is the ith distinct copy of V^0 for $i \in [1, m]$, and edge set $E^0 \cup \left(\bigcup_{i=0}^{m-1} \left\{v_j^i v_{j'}^{i+1} : v_j^0 v_{j'}^0 \in E^0\right\}\right) \cup \left\{v_j^m u : v_j^m \in V^m\right\}$. In what follows, we use E^i, $i \in [1, m]$, to denote the set of edges with one end in V^{i-1} and the other in V^i, and use G^i to denote the subgraph of $\mu_m(G)$ induced by E^i. Clearly, G^i, $i \in [1, m]$, is a bipartite subgraph with maximum degree $\Delta(G)$. For completeness, we also denote G by G^0.

In this paper, we investigate the adjacent vertex-distinguishing total chromatic number of generalized Mycielski graphs. We prove that if G satisfies AVDTCC, then $\mu_m(G)$ also satisfies AVDTCC. Additionally, when G is a graph with maximum degree less than

3 or a complete graph, we determine the exact values of $\chi_{at}(\mu_m(G))$. Moreover, we explore the $\chi_{at}(G)$ for any graph G with only one maximum degree vertex, and show that $\chi_{at}(G) \leq \Delta(G) + 2$ when $\Delta(G) \leq 4$.

To prove the main results of this paper, we need to quote the following two theorems.

Theorem 1.2. [10] *Every bipartite graph G has a $\Delta(G)$-edge-coloring.*

An *L-edge-coloring* of a graph G is a proper edge-coloring f of G such that $f(e) \in L(e)$ for each edge e, where $L(e)$ called the *list* of e, is a set of colors of e, and $f(e)$ denotes the color assigned to e under f. We say G is *L-edge-colorable*, if it admits an L-edge-coloring. For an integer k, if G is L-edge-colorable for every list assignment with $|L(e)| \geq k$ for each $e \in E(G)$, then G is *k-edge-choosable*. The following Theorem 1.3 will be used to prove the main results later in the next section.

Theorem 1.3. [7] *Every bipartite multigraph G is $\Delta(G)$-edge-choosable.*

2. Generalized Mycielski graphs

In this section, we study the adjacent vertex-distinguishing chromatic number of generalized Mycielski graphs. Observe that when $\Delta(G) = 1$, G is a complete graph on 2 vertices. Then $\mu_m(G)$ is a cycle, and $\chi_{at}(\mu_m(G)) = 4$ [23]. In the following we assume $\Delta(G) \geq 2$.

Let G be a graph, and $V' \subseteq V(G)$, $E' \subseteq E(G)$. For a partial total coloring f of G regarding to $V' \cup E'$ and a vertex $v \in V'$, we define $E_f(v) = \{uv : uv \in E'\}$, and $\overline{E}_f(v) = \{uv : uv \in E(G)\} \setminus E_f(v)$. We first have the following observation for latter use.

Lemma 2.1. *Let f be a partial total k-coloring of a graph G regarding to $V' \cup E'$, where $V' \subseteq V(G)$ and $E' \subseteq E(G)$. Suppose that V' contains two adjacent vertices u and v satisfying $C_f^{[1,k]}(u) \neq C_f^{[1,k]}(v)$. Let $S \subseteq [1,k]$ be a color set such that $C_f^{[1,k]}(u) \triangle C_f^{[1,k]}(v) \not\subseteq S$. If there exist two edge colorings, $f_1 \colon \overline{E}_f(u) \to S$ and $f_2 \colon \overline{E}_f(v) \to S$, then $C_{f\cup f_1\cup f_2}^{[1,k]}(u) \neq C_{f\cup f_1\cup f_2}^{[1,k]}(v)$.*

Proof. Let c be a color in $C_f^{[1,k]}(u) \triangle C_f^{[1,k]}(v)$ and not in S. Then we have $c \in C_{f\cup f_1\cup f_2}^{[1,k]}(u) \triangle C_{f\cup f_1\cup f_2}^{[1,k]}(v)$. Hence, the result holds. □

Lemma 2.2. [5] *For a generalized Mycielski graph $\mu_m(G)$ of a graph G, there exists a matching in G^i for any $i \in [1,m]$, which saturates all of the maximum degree vertices of G^i.*

Sun et al. [16] studied the adjacent vertex-distinguishing chromatic number of $\mu_m(G)$ for $m = 1$, i.e., the Mycielskian of G. They proved that if $\chi_{at}(G) \leq \Delta(G) + k$ and $\Delta(G) + k \geq |V(G)|$, then $\chi_{at}(\mu_1(G)) \leq 2\Delta(G) + k$. The theorem below gives a characterization of $\mu_m(G)$ for $m = 2$, which is followed by cases of $m \geq 3$.

Theorem 2.3. *Let G be a graph with $\chi_{at}(G) = k$ and $\Delta(G) = \Delta$ (≥ 2). Then $\chi_{at}(\mu_2(G))$ $\leq \max\{k + \Delta + 1, n + 1\}$.*

Proof. Let $f\colon V(G^0) \cup E(G^0) \to [1, k]$, be a k-AVDTC of G^0. Based on f, color uv_j^2 with j for $j \in [1, n]$, and by Theorem 1.2 properly color E^1 by the set $[k + 1, k + \Delta]$. Color u and V^1 by k^*, where $k^* = \max\{k + \Delta + 1, n + 1\}$. For any edge $e = v_{j_1}^1 v_{j_2}^2 \in E^2$, let $L(e) = [1, k] \setminus \{j_2\}$. Then, $|L(e)| \geq \Delta$. So by Theorem 1.3 E^2 can be properly colored by the set $[1, k]$. In addition, $\Delta \geq 2$ implies $[k + 1, k^*] \geq 3$. Therefore, v_j^2 can be colored with an arbitrary color in $[k + 1, k^*] \setminus \{j, k^*\}$. Denote by f' the resulting coloring. By Lemma 2.1, any two vertices of V^0 have different color sets under f'. Since k^* is in the color sets of vertices in $V^1 \cup \{u\}$, but not in those of vertices in $V^0 \cup V^2$, it follows that $C_{f'}^{[1,k^*]}(x) \neq C_{f'}^{[1,k^*]}(y)$ for any two vertices $x \in V^1 \cup \{u\}$ and $y \in V^0 \cup V^2$. Hence, f' is a k^*-AVDTC of $\mu_2(G)$. □

Theorem 2.4. *Let G be a graph on n (≥ 3) vertices with $\Delta(G) = \Delta$ (≥ 2). When m (≥ 3) is an odd integer, we have*

$$\chi_{at}(\mu_m(G)) \leq \begin{cases} \max\{\chi_{at}(G) + \Delta, n + 1\} & \text{if } \chi_{at}(G) \geq \Delta + 2, \\ \max\{2\Delta + 2, n + 1\} & \text{if } \chi_{at}(G) = \Delta + 1. \end{cases}$$

Proof. Let $\chi_{at}(G) = k$, and $g\colon V(G) \cup E(G) \to [1, k]$, be a k-AVDTC of G. We first define a partial total $(k + \Delta)$-coloring of $\mu_m(G)$ regarding to $\bigcup_{i=0}^{m-1}(V^i \cup E^i)$, denoted by f^*.

(1) Let $f^*(v_j^i) = g(v_j^0)$ for $i \in [0, m - 1]$, $j \in [1, n]$.

(2) Let $f^*(e) = g(e)$ for $e \in E^0$.

(3) When i is odd, we properly color E^i with the set of $[k + 1, k + \Delta]$ by Theorem 1.2 (since G^i is a bipartite graph with maximum degree Δ for $i \geq 1$). When i is even, for each edge $e = v_j^{i-1} v_{j'}^i \in E^i$, let $f^*(e) = g(v_j^0 v_{j'}^0)$.

Thus, we obtain a partial total $(k + \Delta)$-coloring of $\mu(G)$ with only elements in $V^m \cup \{u\} \cup E^m \cup \left\{uv_j^m : j \in [1, n]\right\}$ uncolored. As for f^*, since $m \geq 3$ is odd, it follows that $C_{f^*}^{[1,k]}(v_j^i) = C_{f^*}^{[1,k]}(v_j^{i'})$ for $i, i' \in [0, m-1]$ and $j \in [1, n]$, and $C_{f^*}^{[k+1,k+\Delta]}(v_j^{m-1}) = \emptyset$. Given that g is a k-AVDTC of G, we have $C_{f^*}^{[1,k]}(x) \neq C_{f^*}^{[1,k]}(y)$ for any $xy \in \bigcup_{i=0}^{m-1} E^i$. To complete our proof, we consider the following two cases.

Case 1: $k \geq \Delta + 2$. Let $k^* = \max\{k + \Delta, n + 1\}$. Then, we can modify and extend f^* to a k^*-AVDTC of $\mu_m(G)$ as follows.

Since $k \geq \Delta + 2$, it has that $\left|\overline{C}_{f^*}^{[1,k]}(v_j^{m-1})\right| \geq 1$ for any $j \in [1, n]$. By Lemma 2.2, let M be a matching of G^m which saturates every maximum degree vertex of G^m. Color each edge $e = v_x^{m-1} v_y^m \in M$ with a color $c_x \in \overline{C}_{f^*}^{[1,k]}(v_x^{m-1})$, and denote the resulting coloring by f. We claim that there exists a bijection, from $E_u = \left\{uv_j^m : j \in [1,n]\right\}$ to $[1,n]$, say f', such that any pair of two incident edges in $M \cup E_u$ have different colors.

Suppose this is not the case. Choose an f' of E_u with the fewest pairs of incident edges in $M \cup E_u$ receiving the same color under $f \cup f'$. Let e_1 and e_2 be two edges in $M \cup E_u$ with the same color. Obviously, $\{e_1, e_2\} \not\subset E_u$ and $\{e_1, e_2\} \nsubseteq M$. Without loss of generality, assume $e_1 \in M$ and $e_2 \in E_u$, and let $e_1 = v_x^{m-1}v_y^m$ and $e_2 = uv_y^m$. If there exists a $v_{y'}^m$ not saturated by M, then we interchange the colors of e_2 and $uv_{y'}^m$. Now, e_1 and e_2 have distinct colors, a contradiction with the choice of f'. If all of vertices in V^m are saturated by M, then $|M| = n$ and there exists an edge $v_{x'}^{m-1}v_{y'}^m \in M$ such that $c_{x'} \neq c_x$. (When G is not regular, let $v_{x'}^0$ be a vertex with $d_G(v_{x'}^0) < \Delta(G)$. Evidently, $\left|\overline{C}_{f^*}^{[1,k]}(v_{x'}^{m-1})\right| \geq 2$. So, such a $c_{x'}$ can be chosen from $\overline{C}_{f^*}^{[1,k]}(v_{x'}^{m-1})$. When G is a regular graph, such an edge $v_{x'}^{m-1}v_{y'}^m$ also exists because if every $\overline{C}_{f^*}^{[1,k]}(v_j^{m-1}) = \{c_x\}$ for $j \in [1,n]$, then there are two adjacent vertices in V^0 with the same color set under g, and a contradiction with g.) We interchange the colors of e_2 and $uv_{y'}^m$. Then e_1 and e_2 have distinct colors and $v_{x'}^{m-1}v_{y'}^m$, and $uv_{y'}^m$ also have distinct colors. This contradicts to the choice of f'.

After we have properly colored edges of $M \cup \left\{uv_j^m : j \in [1,n]\right\}$ by $f \cup f'$ defined above, our purpose is to color elements in $V^m \cup \{u\} \cup (E^m \setminus M)$. Let $f'' = f \cup f'$. For any $v_y^m \in V^m$, when $f''(uv_y^m) \leq k$, color v_y^m with $k+1$, and let $L(e) = [k+2, k^*]$ for each edge $e = v_x^{m-1}v_y^m \in E^m \setminus M$. When $f''(uv_y^m) \geq k+1$, since $k \geq \Delta + 2$, we color v_y^m by one color in $[1,k] \setminus \left(\left\{f''(v_x^{m-1}) : v_x^{m-1}v_y^m \in E^m\right\} \cup \left\{c_{x'} : v_{x'}^{m-1}v_y^m \in M\right\}\right)$, and let $L(e) = [k+1, k^*] \setminus \left\{f''(uv_y^m)\right\}$ for each edge $e = v_x^{m-1}v_y^m \in E^m \setminus M$. Clearly, $|L(e)| \geq \Delta - 1$, so by Theorem 1.3, we properly color $E^m \setminus M$ by the set $[k+1, k^*]$ based on f'' (since $G^m - M$ is a bipartite graph with maximum degree $\Delta - 1$). Finally, we color u with one color in $[1, k^*] \setminus \left\{f''(uv_j^m) : j \in [1,n]\right\}$. This gives a total k^*-coloring of $\mu_m(G)$, denoted by f'''.

We now show that f''' is a k^*-AVDTC of $\mu_m(G)$. Since g is a k-AVDTC of G, it follows that $\left|C_{f^*}^{[1,k]}(u) \triangle C_{f^*}^{[1,k]}(v)\right| \geq 2$ if $uv \in E^0$ and $d_{G^0}(u) = d_{G^0}(v)$. Then, by Lemma 2.1 each pair of adjacent vertices in $V^{m-1} \cup V^{m-2}$ have different color sets under f'''. For two adjacent vertices $v_x^{m-1} \in V^{m-1}$ and $v_y^m \in V^m$, since $\left|C_{f'''}^{[1,k]}(v_{j_2}^m)\right| \leq 2$ and $\left|C_{f'''}^{[1,k]}(v_{j_1}^{m-1})\right| \geq \ell + 1$ when $\left|C_{f'''}^{[1,k]}(v_{j_2}^m)\right| = \ell$, $\ell = 1, 2$, we deduce that v_x^{m-1} and v_y^m have distinct color sets under f'''. Additionally, $n \geq 3$ implies that $\left|C_{f'''}^{[1,k]}(u)\right| \geq 3$. Therefore, $C_{f'''}^{[1,k]}(u) \neq C_{f'''}^{[1,k]}(v_j^m)$ for any $j \in [1,n]$.

Case 2: $k = \Delta + 1$. Let $k^* = \max\{2\Delta + 2, n+1\}$. We now define a k^*-AVDTC of $\mu(G)$ based on f^*.

Color u with k^*, and uv_j^m with j for $j \in [1,n]$. Color v_j^m with $k^* - 1$ for $j \in [1,k]$, and with one color in $[1,k] \setminus \left\{f^*(v_{j'}^{m-1}) : v_j^m v_{j'}^{m-1} \in E^m\right\}$ for $j \in [k+1,n]$ (note that $\left|\left\{f^*(v_{j'}^{m-1}) : v_j^m v_{j'}^{m-1} \in E^m\right\}\right| \le \Delta$). For any edge $v_{j_1}^{m-1}v_{j_2}^m \in E^m$, if $j_2 \le k$, let $L(v_{j_1}^{m-1}v_{j_2}^m) = [\Delta+2, 2\Delta] \cup \{k^*\}$. If $j_2 \ge k+1$, let $L(v_{j_1}^{m-1}v_{j_2}^m) = [\Delta+2, k^*] \setminus \{j_2\}$. Clearly, $\left|L(v_{j_1}^{m-1}v_{j_2}^m)\right| \ge \Delta$, and by Theorem 1.3 we can properly color E^m by the set $[\Delta+2, k^*]$. This gives a total k^*-coloring of $\mu(G)$, denoted by f.

We now show that f is a k^*-AVDTC of $\mu(G)$. First, according to Lemma 2.1, any two adjacent vertices in $V^{m-1} \cup V^{m-2}$ have different color sets. In addition, it is easy to see that $\left|C_{f^*}^{[1,k]}(v)\right| \ge 2$ for each vertex $v \in V^{m-1} \cup \{u\}$, and $\left|C_f^{[1,k]}(v_j^m)\right| = 1$ for $j \in [1, n]$. Therefore, the color set of $v_j^m \in V^m$ is different from those of its adjacent vertices under f. Hence, f is a k^*-AVDTC of $\mu_m(G)$ in this case. □

Theorem 2.5. *Let G be a graph on n (≥ 3) vertices with $\Delta(G) = \Delta$ (≥ 2), and m be an integer. If m (≥ 4) is even, then*

$$\chi_{at}(\mu_m(G)) \le \begin{cases} \max\{\chi_{at}(G) + \Delta, n+1\} & \text{if } \chi_{at}(G) \ge \Delta + 2, \\ \max\{2\Delta + 2, n+1\} & \text{if } \chi_{at}(G) = \Delta + 1. \end{cases}$$

Proof. Let f^* be the partial total $(k+\Delta)$-coloring of $\mu(G)$ regarding to $\bigcup_{i=0}^{m-1}(V^i \cup E^i)$, defined in Theorem 2.4. Then, under f^*, any two adjacent vertices in G^i have different color sets for $i \in [0, m-2]$, and when $m \ge 4$ is even, $E_{f^*}(x)$ are colored by the set $[k+1, k+\Delta]$ for any vertex $x \in V^{m-1}$. Based on f^*, we consider the following two cases to complete our proof.

Case 1: $k \ge \Delta + 2$. Let $k^* = \max\{k+\Delta, n+1\}$. In this case, we first erase the colors appearing on $E^{m-1} \cup V^{m-1}$ under f^*. According to Lemma 2.2 suppose M is a matching of G^{m-1} which saturates every maximum degree vertex of G^{m-1}. Then for each $v_x^{m-2}v_y^{m-1} \in M$, color $v_x^{m-2}v_y^{m-1}$ with one color in $\overline{C}_{f^*}^{[1,k]}(v_x^{m-2})$ (since $k \ge \Delta+2$, $\left|\overline{C}_{f^*}^{[1,k]}(v_x^{m-2})\right| \ge 1$). Given that $G^{m-1} - M$ is a bipartite graph with maximum degree $\Delta - 1$, we can properly color edges of $E^{m-1} \setminus M$ by the set $[k+1, k+\Delta-1]$ according to Theorem 1.2, and then color vertices of V^{m-1} by k^*. We also denote the resulting coloring by f^*. According to Lemma 2.1, it is easy to see that any two adjacent vertices in V^{m-3} and V^{m-2} have different color sets under f^*. So, any two adjacent vertices in $\bigcup_{i=0}^{m-2}$ have distinct color sets under f^*.

We now based on f^* color elements in $E^m \cup V^m \cup \left\{uv_j^m : j = 1, 2, \ldots, n\right\} \cup \{u\}$ as follows. Color u with k^* and uv_j^m with j for $j \in [1, n]$. For any edge $v_{j_1}^{m-1}v_{j_2}^m \in E^m$, let $L(v_{j_1}^{m-1}v_{j_2}^m) = [1, k] \setminus \left\{j_2, f^*(v_{j_1}^{m-1}v_{j'}^{m-2})\right\}$, where $v_{j_1}^{m-1}v_{j'}^{m-2} \in M$. Since $k \ge \Delta + 2$, it has that $\left|L(v_{j_1}^{m-1}v_{j_2}^m)\right| \ge \Delta$. By Theorem 1.3, we can properly color E^m by the set $[1, k]$. Finally, properly color v_j^m by one color in $[1, k] \setminus \left(\{j\} \cup \left\{f'(v_j^m v_{j'}^{m-1}) : v_j^m v_{j'}^{m-1} \in E^m\right\}\right)$ because of $k \ge \Delta + 2$. This gives a total k^*-coloring of $\mu_m(G)$, denoted by f. It is easy to see that $k^* \in C_f^{[1,k^*]}(x)$ for any $x \in V^{m-1} \cup \{u\}$, and $k^* \notin C_f^{[1,k^*]}(y)$ for any $y \in V^m \cup V^{m-2}$.

This shows that any two adjacent vertices in $V^{m-2} \cup V^{m-1} \cup V^m \cup \{u\}$ have different color sets. Therefore, f is a k^*-AVDTC of $\mu_m(G)$.

Case 2: $k = \Delta + 1$. Let $k^* = \max\{2\Delta + 2, n + 1\}$. We now extend and modify f^* to a k^*-AVDTC.

We first recolor vertices of V^{m-1} with k^*. Then, color u with k^*. And for any $j \in [1, n]$, color uv_j^m with j, color v_j^m with $2\Delta + 1$ if $j \neq 2\Delta + 1$ and with 2Δ if $j = 2\Delta + 1$. For each edge $v_{j_1}^{m-1} v_{j_2}^m \in E^m$, let $L(v_{j_1}^{m-1} v_{j_2}^m) = [1, k] \setminus \left\{ f^*(uv_{j_2}^m) \right\}$. Since $k = \Delta + 1$, we have $\left| L(v_{j_1}^{m-1} v_{j_2}^m) \right| \geq \Delta$. By Theorem 1.3, E^m can be properly colored by the set $[1, k]$. This gives a total k^*-coloring of $\mu_m(G)$, denoted by f. Since k^* is in the color sets of vertices in $V^{m-1} \cup \{u\}$, but not in the color sets of vertices in V^{m-2} or V^m, it follows that f is a k^*-AVDTC of $\mu_m(G)$. □

Let G be a graph with $\Delta(G) = 2$, $n = |V(G)| \geq 4$. Then, $\chi_{at}(G) = 4$ [23], and by Theorems 2.4 and 2.5 $\chi_{at}(\mu_m(G)) \leq \max\{6, n + 1\}$ when $m \geq 3$. On the other hand, when $n \geq 4$, it has that $\chi_{at}(\mu_m(G)) \geq \max\{6, n + 1\}$. (When $n = 4$, $\chi_{at}(\mu_m(G)) \geq 6$ because $\mu_m(G)$ has two adjacent vertices with maximum degree 4. When $n \geq 5$, $\chi_{at}(\mu_m(G)) \geq n + 1$ since $\Delta(\mu_m(G)) = n$ in this case.) Thus, $\chi_{at}(\mu_m(G)) = \max\{6, n + 1\}$ when $m \geq 3$. Moreover, when $m = 1, 2$, one can easily give a k^*-AVDTC of $\mu_m(G)$, where $k^* = \max\{6, n + 1\}$. So, we have the following result.

Corollary 2.6. *Let G be an n vertices graph with $\Delta(G) = 2$, $n \geq 4$. Then $\chi_{at}(\mu_m(G)) = \max\{6, n + 1\}$.*

For a graph G, when $\Delta(G) = 3$, Hulgan [9] proved that G satisfies the AVDTCC, and showed that G has a 6-AVDTC with the properties in the following lemma.

Lemma 2.7. [9] *Let G be a graph with $\Delta(G) = 3$. If $G \neq K_4$, then G has a 6-AVDTC with the following properties:*

(1) *the vertices of G are colored* $1, 2, 3$;

(2) *the edges of G are colored* $3, 4, 5, 6$.

Corollary 2.8. *Let G be an n vertices graph with $\Delta(G) = 3$, $n \geq 8$. Then $\chi_{at}(\mu_m(G)) = n + 1$.*

Proof. $n \geq 8$ implies that $\mu_m(G)$ contains only one maximum degree vertex u with $d_{\mu_m(G)}(u) = \Delta(\mu_m(G)) = n$. So, $\chi_{at}(\mu_m(G)) \geq n+1$. In order to show $\chi_{at}(\mu_m(G)) = n+1$, it suffices to give an $(n + 1)$-AVDTC of $\mu_m(G)$.

When $m \geq 3$, such a coloring does exist by $\chi_{at}(G) \leq 6$ and Theorems 2.4 and 2.5. When $m \leq 2$, let f be a 6-AVDTC of G^0 with the properties in Lemma 2.7, and let

$V_i^0 = \left\{v_j^0 : f(v_j^0) = i\right\}$ for $i = 1,2,3$. Then, $2 \in \overline{C}_f^{[1,6]}(v_j^0)$ for each $v_j^0 \in V_1^0$, $1 \in \overline{C}_f^{[1,6]}(v_j^0)$ for each $v_j^0 \in V_2^0$, and $\{1,2\} \subseteq \overline{C}_f^{[1,6]}(v_j^0)$ for each $v_j^0 \in V_3^0$. We now extend f to an $(n+1)$-AVDTC of $\mu_m(G)$.

Color uv_j^m with j for $j \in [1,n]$, and color u by $n+1$.

According to Lemma 2.2 suppose that M is a matching of G^1 which saturates every maximum degree vertex of G^1. For each edge $e = v_x^0 v_y^1 \in M$, color e with 2 when $v_x^0 \in V_1^0$, with one color in $\overline{C}_f^{[3,6]}(v_x^0)$ when $v_x^0 \in V_2^0$, and with 1 when $v_x^0 \in V_3^0$. We now denote the resulting coloring still by f. Since $2 \in C_f^{[1,6]}(v_x^0)$ for $v_x^0 \in (V_1^0 \cup V_2^0)$ but $2 \notin C_f^{[1,6]}(v_x^0)$ for $v_x^0 \in V_3^0$, and $1 \in C_f^{[1,6]}(v_x^0)$ for $v_x^0 \in V_1^0$ but $1 \notin C_f^{[1,6]}(v_x^0)$ for $v_x^0 \in V_2^0$, it has that any two adjacent vertices in V^0 have different color sets under f.

Consider $G^1 - M$. It is a bipartite graph with maximum degree 2. When $m = 1$, for any edge $e = v_{j_1}^0 v_{j_2}^1 \in E^1 \setminus M$, let $L(e) = [8,9]$ when $j_2 \notin [7,9]$, and $L(e) = [7,9] \setminus \{j_2\}$ when $j_2 \in [7,9]$. Clearly, $|L(e)| = 2$. By Theorem 1.3, we can properly color $E^1 \setminus M$ by the set $[7,9]$. For each vertex $v_j^1 \in V^1$, color it with 7 when $j \notin [7,9]$ and with one color in $[4,6] \setminus \{c\}$ when $j \in [7,9]$, where c is the color appearing on a possible edge $v_j^1 v_{j'}^0 \in M$. Thus, we obtain a total $(n+1)$-coloring of $\mu_1(G)$, say f'. By Lemma 2.1, any two vertices of V^0 have different color sets under f'. In addition, that u is the unique maximum degree vertex of $\mu_1(G)$ shows that $C_{f'}^{[1,n+1]}(u) \neq C_{f'}^{[1,n+1]}(v_j^1)$ for any $j \in [1,n]$. Finally, for two vertices $v_x^0 \in V^0$ and $v_y^1 \in V^1$, one can readily check that $\left|C_{f'}^{[1,6]}(v_x^0)\right| \neq \left|C_{f'}^{[1,6]}(v_y^1)\right|$. So, v_x^0 and v_y^1 have different color sets under f'. This shows that f' is an $(n+1)$-AVDTC of $\mu_1(G)$.

When $m = 2$, we properly color $E^1 \setminus M$ with colors 7 and 8 by Theorem 1.2, and color V^1 with $n+1$. For each edge $e = v_{j_1}^1 v_{j_2}^2 \in E^2$, let $L(e) = [2,6] \setminus \{c, j_2\}$, where c is the color appearing on a possible edge $v_{j_1}^1 v_{j_1'}^0 \in M$. Clearly, $|L(e)| \geq 3$, and by Theorem 1.3, we can properly color E^2 by the set $[2,6]$. Additionally, there are at least two colors of $[1,6]$ available for each $v_j^2 \in V^2$. Thus, we obtain a total $(n+1)$-coloring of $\mu_2(G)$, denoted by f'. Obviously, for $j \in [1,n]$, $n+1 \in C_{f'}^{[1,n+1]}(v_j^1)$, $n+1 \in C_{f'}^{[1,n+1]}(u)$, $n+1 \notin C_{f'}^{[1,n+1]}(v_j^0)$, and $n+1 \notin C_{f'}^{[1,n+1]}(v_j^2)$. Therefore, f' is an $(n+1)$-AVDTC of $\mu_2(G)$. □

Corollary 2.9. *For a graph G on n (≥ 2) vertices, if $n \geq \chi_{at}(G) + \Delta(G)$, then $\chi_{at}(\mu_m(G)) = n+1$.*

Proof. That G is nontrivial, $\Delta(G) \geq 1$. Since $n \geq \chi_{at}(G) + \Delta(G)$, it follows that $\mu_m(G)$ contains only one maximum degree vertex u with $d_{\mu_m(G)}(u) = n$. Obviously, $\chi_{at}(\mu_m(G)) \geq n+1$. On the other hand, when $m \geq 2$, by Theorems 2.3, 2.4 and 2.5, we have $\chi_{at}(\mu_m(G)) \leq n+1$. When $m = 1$, let f be a $\chi_{at}(G)$-AVDTC of G^0. Then we can easily extend f to an $(n+1)$-AVDTC as follows. First, color u with $n+1$ and uv_j^1 with j for any $j \in [1,n]$. Then, for each vertex v_j^1, color it by $\chi_{at}(G)+1$ when $j \in [1, \chi_{at}(G)]$, and by one color of $[1, \chi_{at}(G)] \setminus \left\{f(v_j^1 v_{j'}^0) : v_j^1 v_{j'}^0 \in E^1\right\}$ when $j \in [\chi_{at}(G)+1, n]$. Finally, for each edge $v_x^0 v_y^1 \in E^1$, let $L(v_x^0 v_y^1) = [\chi_{at}(G)+2, n+1]$ when $y \leq \chi_{at}(G)$ and $L(v_x^0 v_y^1) = [\chi_{at}(G)+1, n+1] \setminus \{y\}$ when $y \geq \chi_{at}(G)+1$. Since $n \geq \chi_{at}(G) + \Delta(G)$, it has that $\left|L(v_x^0 v_y^1)\right| \geq \Delta(G)$. So by Theorem 1.3 we can properly color E^1 by the set $[\chi_{at}(G)+1, n+1]$. This gives an $(n+1)$-AVDTC of $\mu_1(G)$. □

Let K_n be a complete graph on n (≥ 3) vertices. Then, $\Delta(\mu_m(K_n)) = 2n-2$, and $\mu_m(K_n)$ contains two adjacent vertices with maximum degree. Therefore, $\chi_{at}(\mu_m(K_n)) \geq 2n$. On the other hand, when n is even and $m \geq 3$, it has that $\chi_{at}(K_n) = n+1$ [23] and $\chi_{at}(\mu_m(K_n)) \leq 2n$ by Theorem 2.5. Additionally, when $m \leq 2$, we can easily obtain a $2n$-AVDTC of $\mu_m(K_n)$ based on an $(n+1)$-AVDTC f of K_n^0 as follows: Color vertices v_j^1 (and v_j^2 when $m=2$) with $f(v_j^0)$ for $j \in [1,n]$. Color E^1 by the set $[n+2, 2n]$ according to Theorem 1.2, and color each $v_x^1 v_y^2 \in E^2$ with $f(v_x^0 v_y^0)$ when $m=2$. For $j \in [1,n]$, color uv_j^m with $\overline{C}_f^{[1,n+1]}(v_j^0)$. Color u by $2n$. It is easy to see that such a coloring is a $2n$-AVDTC of $\mu_m(K_n)$. Hence, $\chi_{at}(\mu_m(K_n)) = 2n$ when n is even. We now prove that this result also holds when n is odd.

Theorem 2.10. *Let K_n be a complete graph on n vertices, $n \geq 3$. Then $\chi_{at}(\mu_m(K_n)) = 2n$.*

Proof. It is sufficient to assume n is odd and give a $2n$-AVDTC of $\mu_m(K_n)$. We first give a total $(n+2)$-coloring of K_{n+2}, denoted by f. Let $V(K_{n+2}) = \{v_1, v_2, \ldots, v_{n+2}\}$. For $i \in [1, n+2]$, color v_i and edges of F_i by i, where $F_i = \{v_{i-j}v_{i+j} : j = 1, 2, \ldots, (n+1)/2\}$ according to modulo $n+2$ (here we denote 0 by $n+2$). Clearly, such a coloring is a total $(n+2)$-coloring of K_{n+2}. We now construct a $2n$-AVDTC, denoted by f', of $\mu_m(K_n)$ according to f.

(1) For G^0, let $f'(v_j^0) = f(v_j)$, $f'(v_j^0 v_{j'}^0) = f(v_j v_{j'})$, $j, j' \in [1,n]$, $j \neq j'$. For any uncolored edges and vertices of $\mu_m(K_n)$ yet, we will color them by the order $E^1, V^1, E^2, V^2, \ldots, E^m, V^m, uv_j^m, u$, $(j = 1, 2, \ldots, n)$, and denote the resulting coloring always by f' at each stage.

(2) For any $i \in [1,m]$, set $M_i = \left\{v_j^{i-1} v_{j+1}^i : j \in [1,n]\right\}$ with $v_{n+1}^i = v_1^i$. Clearly, M_i is a perfect matching of G^i. Therefore, $G^i - M_i$ is a bipartite graph with maximum degree $n-2$. For M_1, let $f'(v_j^0 v_{j+1}^1) = f(v_{n+1}v_j)$ for $j \in [1,n]$. Then we can properly color edges of $E^1 \backslash M_1$ by the set $[n+3, 2n]$ by Theorem 1.2. Now, it has that $\overline{C}_{f'}^{[1,2n]}(v_j^0) = \{f(v_{n+2}v_j)\}$, and $n+1$ does not appear on any edge in M_1. So, we color vertices in V^1 by $n+1$.

When $m=1$, color uv_j^1 with $f(v_{n+2}v_j)$ for $j = [1, n-1]$ and uv_n^1 with $(n+1)/2$, and color u with $2n$. Obviously, such a coloring is a $2n$-AVDTC of $\mu_1(K_n)$.

When $m \geq 2$, we color the remainder elements as follows.

(3) For each edge $e = v_j^1 v_{j'}^2 \in E^2$, let $L(e) = [1,n] \backslash \left\{f'(v_j^1 v_{j-1}^0)\right\}$. (Here we let $v_n^0 = v_0^0$. Moreover, consider $f'(v_2^1 v_1^0) = n+2$. We specially let $L(v_2^1 v_{j'}^2) = [1,n] \backslash \{(n+1)/2\}$.) Then

$|L(e)| = n-1$, so by Theorem 1.3 we can properly color E^2 by the set $[1,n]$. Now, we can see that for any v_j^1, $\overline{C}_{f'}^{[1,2n]}(v_j^1) = \{n+2\}$ when $j \neq 2$, and $\overline{C}_{f'}^{[1,2n]}(v_2^1) = \{(n+1)/2\}$. Therefore, any two adjacent vertices in $V^0 \cup V^1$ have different color sets under f'.

(4) For $i \in [2,m]$, we color the vertices in V^i with $n+2$ when i is even, and with $n+1$ when i is odd. And when i is odd, color $v_j^{i-1}v_{j+1}^i$ with $\overline{C}_{f'}^{[1,n]}(v_j^{i-1})$ for each $v_j^{i-1}v_{j+1}^i \in M_i$, and color $E^i \setminus M_i$ by the set $[n+3,2n]$. When i is even, for any $e = v_j^{i-1}v_{j'}^i \in E^i$, let $L(e) = [1,n] \setminus \left\{f'(v_j^{i-1}v_{j-1}^{i-2})\right\}$. Then $|L(e)| = n-1$, and by Theorem 1.3 E^i can be properly colored by the set $[1,n]$.

After the above coloring, we can see that for $i \in [2,m]$, the color sets of vertices in V^i do not contain color $n+1$ when i is even and do not contain color $n+2$ when i is odd. Additionally, when i is odd, $\{f'(e) : e \in M_i\} = [1,n]$, and when i is even, $\left\{\overline{C}_{f'}^{[1,n]}(v_j^i) : j \in [1,n]\right\} = [1,n]$. Therefore, when m is odd, we color uv_j^m with $f'(v_j^m v_{j-1}^{m-1}) + 1$ for $j \in [1,n]$ (here $v_0^{m-1} = v_n^{m-1}$) and color u with $n+2$. When m is even, color uv_j^m with $\overline{C}_{f'}^{[1,n]}(v_j^m)$ for $j \in [1,n]$ and color u with $n+1$. Since the degrees of vertices in V^m are different from those of vertices in V^{m-1}, and $C_{f'}^{[1,2n]}(u)$ does not contain the color $f(v_j^m)$, it follows that f' is a $2n$-AVDTC of $\mu_m(K_n)$. □

3. Graphs with only one maximum degree vertex

In this section, we embark on the study of $\chi_{at}(G)$ for a graph G with only one maximum degree vertex.

Theorem 3.1. *Let G be a graph with only one maximum degree vertex. If $\Delta \leq 3$, then $\chi_{at}(G) = \Delta + 1$.*

Proof. $\Delta \leq 2$ are trivial cases. So we assume $\Delta = 3$. It suffices to give a 4-AVDTC of G. Let u be the unique vertex of degree 3 in G, and v_1, v_2, v_3 be its three neighbors. Then $G - u$, obtained from G by deleting u and its incident edges is disconnected, and v_1, v_2, v_3 are not in the same component of $G - u$. Let v_1 be the one that is not in the same component with v_2 or v_3 in $G - u$. Then $G - uv_1$, obtained from G by deleting edge uv_1 has two components, say G_1 and G_2, where $v_1 \in V(G_1)$. Clearly, $\Delta(G_i) \leq 2$ and G_1 is a path. Let f be a 4-AVDTC of G_2 [23]. Without loss of generality, assume $f(u) = 1$, $f(uv_2) = 2$, $f(uv_3) = 3$. Then alternately color the vertices of G_1 with 2 and 1, and alternately color the edges of $\{uv_1\} \cup E(G_1)$ with 4 and 3, where v_1 is colored with 2 and uv_1 is colored with 4. Obviously, such a coloring of $\{uv_1\} \cup G_1$ together with f is a 4-AVDTC of G. □

Theorem 3.2. *Let G be a graph with only one maximum degree vertex. If $\Delta = 4$, then $\chi_{at}(G) \leq 6$.*

Proof. Let u be the vertex of degree 4, and v be a neighbor of u. By Lemma 2.7 $G-uv$ has a 6-AVDTC f with the properties in Lemma 2.7. We now modify and extend f to a 6-AVDTC of G.

When $d_G(v) \leq 2$, color uv with one color in $[1,2] \setminus \{f(u)\}$ and recolor v with one in $[4,6]$ (or $[4,6] \setminus \{f(vv')\}$ when $d_G(v) = 2$, where v' is the neighbor of v in $G-uv$). Obviously, v has at least two available colors, so we can obtain a 6-AVDTC of G in this case. In what follows, we assume $d_G(v) = 3$. Denote by u_1, u_2, u_3 the three neighbors of u in $G-uv$, and v_1, v_2 the two neighbors of v in $G-uv$. Since u is the unique maximum degree vertex in G, the color set of u is different from that of its each neighbor under any 6-coloring of G.

Case 1: $f(u) = f(v)$. If $f(u) \neq 3$, we without loss of generality assume $f(u) = f(v) = 1$. When $[3,6] \not\subseteq \{f(uu_1), f(uu_2), f(uu_3), f(u_1), f(u_2), f(u_3)\}$, we recolor u with $[3,6] \setminus \{f(uu_1), f(uu_2), f(uu_3)\}$, and color uv with 2. Thus, we obtain a total 6-coloring of G, also denoted by f. Obviously, under f, both 1 and 2 are in the color set of v but at most one of them is in the color set of v_1 or v_2, so f is a 6-AVDTC of G. When $[3,6] \subseteq \{f(uu_1), f(uu_2), f(uu_3), f(u_1), f(u_2), f(u_3)\}$, it has that $\{f(uu_1), f(uu_2), f(uu_3)\} = [4,6]$. Recolor v with $[4,6] \setminus \{f(vv_1), f(vv_2)\}$ and color uv with 2. We denote the resulting coloring still by f. If $C_f^{[1,6]}(v) \neq C_f^{[1,6]}(v_1)$ and $C_f^{[1,6]}(v) \neq C_f^{[1,6]}(v_2)$, then f is a 6-AVDTC of G. Otherwise, assume $C_f^{[1,6]}(v) = C_f^{[1,6]}(v_1)$ (which implies $f(v_1) = 2$). We then recolor vv_1 with 1 and v with $[4,6] \setminus \{f(v), f(vv_2)\}$. This gives a new 6-AVDTC of G, also denoted by f. Clearly, $C_f^{[1,6]}(v) \neq C_f^{[1,6]}(v_1)$. Moreover, since $C_f^{[1,6]}(v_1)$ and $C_f^{[1,6]}(v)$ contain both 1 and 2, it follows that v_1 has different color set with each of its neighbors and $C_f^{[1,6]}(v) \neq C_f^{[1,6]}(v_2)$. So, f is a 6-AVDTC of G.

If $f(u) = f(v) = 3$, then $\{f(vv_1), f(vv_2)\} \subseteq [4,6]$. Without loss of generality, assume $f(vv_1) = 4$ and $f(vv_2) = 5$. Recolor v with 6, and color uv with 1 or 2. If no matter when uv is colored 1 or 2, there always exists a vertex in $\{v_1, v_2\}$ with the same color set with v under the resulting coloring, then $\left\{C_f^{[1,6]}(v_1), C_f^{[1,6]}(v_2)\right\} = \{\{1,4,5,6\}, \{2,4,5,6\}\}$. Let $C_f^{[1,6]}(v_1) = \{1,4,5,6\}$, and then $f(v_1) = 1$. Since the color sets (under f) of neighbors of v_1 do not contain color 1, we can recolor vv_1 with 3 and color uv with 2 to gain a 6-AVDTC of G.

Case 2: $f(u) \neq f(v)$. If $f(v) \neq 3$, say 2, then $f(v_1) \neq 2$ and $f(v_2) \neq 2$ (i.e., $2 \notin C_f^{[1,6]}(v_1)$ or $C_f^{[1,6]}(v_2)$). Then, color uv with 2 and recolor v with one color in $[4,6] \setminus \{f(vv_1), f(vv_2)\}$ to gain a 6-AVDTC of G.

If $f(v) = 3$, assume $f(vv_1) = 4$, $f(vv_2) = 5$ and $f(u) = 1$. Color uv with 2. If $C_f^{[1,6]}(v_1) \neq C_f^{[1,6]}(v)$ and $C_f^{[1,6]}(v_2) \neq C_f^{[1,6]}(v)$, then f is a 6-AVDTC of G. Otherwise, assume $C_f^{[1,6]}(v_1) = C_f^{[1,6]}(v)$. Then $f(v_1) = 2$. We recolor vv_1 with 1 and recolor v with 6, and also denote the resulting coloring by f. Then, $C_f^{[1,6]}(v_2) \neq C_f^{[1,6]}(v)$, and

$C_f^{[1,6]}(v_1) \neq C_f^{[1,6]}(v)$, which implies that v_1 has different color set with each of its neighbors under f since $C_f^{[1,6]}(v_1)$ and $C_f^{[1,6]}(v)$ contain both 1 and 2. Hence, f is a 6-AVDTC of G. □

4. Discussion

Motivated by Corollary 2.9, Theorems 3.1 and 3.2, we propose the following problem.

Problem 4.1. If a graph G has only one vertex of maximum degree, then $\chi_{at}(G) \leq \Delta(G) + 2$.

The correctness of this problem would provide a strong support for AVDTCC, since if we have had a proof of this problem, then we can prove a weaker result of AVDTCC: For any graph G, $\chi_{at}(G) \leq \Delta(G) + 4$. To see this we first add a new vertex and connect it with a maximum degree vertex of G, say v. Denote by G' the resulting graph. Clearly, G' contains only one vertex v with the maximum degree $\Delta(G)+1$. Hence, $\chi_{at}(G') \leq \Delta(G)+3$. Let f be a $(\Delta(G)+3)$-AVDTC of G'. Then f is a $(\Delta(G)+4)$-AVDTC of G by recoloring v with $\Delta(G)+4$ in G.

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