TAIWANESE JOURNAL OF MATHEMATICS
Vol. xx, No. x, pp. 1–21, xx 20xx
DOI: 10.11650/tjm/231203

Blow-up and Decay for a Pseudo-parabolic Equation with Nonstandard Growth Conditions

Quach Van Chuong, Le Cong Nhan* and Le Xuan Truong

Abstract. This paper deals with a pseudo-parabolic equation involving variable exponents under homogeneous Dirichlet boundary value condition. The authors first develop the potential well method to prove a threshold result on the existence or nonexistence of global solutions to the equation when the initial energy is less than the mountain pass level d. In the case of high energy initial data, a new characterization for the nonexistence of global solution is also given. These results extend and improve some recent results obtained by Di et al. [9].

1. Introduction

In this paper, we consider the following initial-boundary value problem:

$$\begin{cases} u_t - \Delta u_t - \operatorname{div}(|\nabla u|^{m(x)-2}\nabla u) = |u|^{p(x)-2}u, & (x,t) \in Q_T, \\ u(x,t) = 0, & (x,t) \in \Gamma_T, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and $Q_T := \Omega \times [0,T]$, $\Gamma_T := \partial\Omega \times [0,T]$. It will also be assumed throughout this paper that the exponents $m(x)$ and $p(x)$ are two continuous functions on $\overline{\Omega}$ such that

$$2 \le m^- \le m(x) \le m^+ < p^- \le p(x) \le p^+ < \infty, \tag{1.2}$$

$$\operatorname*{ess\,inf}_{x\in\Omega}(m^*(x) - p(x)) > 0, \tag{1.3}$$

where $m^- = \operatorname{ess\,inf}_{x\in\Omega} m(x)$, $m^+ = \operatorname{ess\,sup}_{x\in\Omega} m(x)$, p^-, p^+ are similarly defined, and

$$m^*(x) := \begin{cases} \frac{Nm(x)}{N-m(x)} & \text{if } m(x) < N, \\ \infty & \text{if } m(x) \ge N. \end{cases}$$

Received August 27, 2022; Accepted December 7, 2023.
Communicated by François Hamel.
2020 *Mathematics Subject Classification.* 35A01, 35B44, 35K70.
Key words and phrases. pseudo-parabolic equation, global existence, blow-up, decay, variable exponent.
*Corresponding author.

In addition $m(x)$ satisfies the logarithmic module of continuity:

$$|m(x) - m(y)| \leq -\frac{C}{\log|x-y|} \quad \text{for all } x, y \in \Omega,\ |x-y| < \delta, \tag{1.4}$$

where $C > 0$, $0 < \delta < 1$.

Equations of type (1.1) with one time derivative appearing in the highest order term are called pseudo-parabolic or Sobolev equations, and can be used to describe many important physical phenomena such as flows of fluids through fissured rock [5], thermodynamics [36], the unidirectional propagation of nonlinear, dispersive, long waves [6], the aggregation of populations [25], semiconductors in physics [3, 16, 17]. In mathematics, the study of this type of equations originated from the work of Ting and Showalter [34, 35]. Subsequently, nonlinear pseudo-parabolic equations have been investigated by many authors, see [2, 7, 12, 15–19, 22, 27, 33, 39] and references therein.

Recently the equations with nonlinearities of variable exponent type, also referred as equations with nonstandard growth conditions, have been studied extensively because of its applications in various physical phenomena such as the flows of electrorheological fluids, processes of filtration through a porous media and image processing and so on, see [1, 4, 8, 10, 13, 28, 30, 31] and the further references therein.

Regarding the global existence and nonexistence results of (1.1), in [37] Xu and Su studied a special case of (1.1) (when $m(x) = 2$ and $p(x) = p$). The authors proved global existence, non-existence, and asymptotic behavior of solutions with initial energy $J(u_0) \leq d$ by using potential well method and obtained finite-time blow-up with high initial energy by the comparison principle. Then Luo [24] obtained an upper bound and a lower bound of the blow-up time and exponential decay under some appropriate conditions. This problem was studied by Qu et al. [29] for the more general case (when $m(x) = 2$ and $p(x)$ instead of p), in which the authors established a sharp threshold result on global existence or blow-up of solution by using the modified potential well method with initial energy $J(u_0) \leq d$. The case of high initial energy was concerned.

When both $m(x)$ and $p(x)$ are functions, Di et al. [9] showed that the solutions to (1.1) fail to exist globally when the initial energy is non-positive and gave an upper bound for the maximal existence time. Later, Liao [20] obtained the non-global existence of solutions when the initial energy $J(u_0)$ is positive. And then Liao et al. [21] proved the decay estimates due to a key integral inequality. In a recent result, Zhu et al. [38] obtained the global existence and blow-up results of solutions in case $J(u_0) > d$.

Inspired by these papers, we develop the potential well method due to Payne and Sattinger [26, 32] to treat the problem (1.1). Note that our method is technically different from in [9, 20, 21], in which the authors obtained the results on existence and non-existence of solution under initial energy is non-positive or positive with assuming its smallness. The main advantage of our method is to obtain a sharp result on the existence or nonexistence of global solutions in case $J(u_0) < d$. The main difficulty here is the presence of variable exponents that causes the lack of homogeneous properties, the gap between the norm and the integral in variable exponent spaces, and hence it requires more delicate technique to overcome these difficulties, for example see Lemmas 2.3 and 2.4 for constructing the potential well, the proofs of Theorems 3.4 and 3.5. In case $J(u_0) > d$, we give a new characterization for the nonexistence of global solutions, see Theorem 3.8. It is also noticed that the authors [38] only stated a such result for the special case $m(x) = m$, see Proposition 2.2 in [38]. We also show the asymptotic behavior of global solution, see Theorem 3.7.

The rest of this paper is organized as follows: In Section 2 we recall some facts about the function spaces of Orlicz–Sobolev type and construct the stable and unstable sets; Section 3 states our main results and the rest of the paper is devoted to the proofs of main results.

Notations. Let $\|\cdot\|_q$ denote the usual norm of the space $L^q(\Omega)$ for $1 \leq q \leq \infty$. Moreover, $\langle\,\cdot\,,\cdot\,\rangle$ denote the usual inner product of the Lebesgue space $L^2(\Omega)$ and $\langle u, v\rangle_{H_0^1} = \langle u, v\rangle + \langle \nabla u, \nabla v\rangle$ for $u, v \in H_0^1(\Omega)$. We also denote the norm of $H_0^1(\Omega)$ by $\|\cdot\|_{H_0^1}$, that is, $\|u\|_{H_0^1} = \sqrt{\|u\|_2^2 + \|\nabla u\|_2^2}$. Finally, constants are denoted generically by C although they can change in line or line by line.

2. Preliminaries

2.1. Functional spaces

In this section, we introduce some definitions and basic results on Lebesgue and Sobolev spaces with variable exponents (see [11, 14] and references therein), which will be used in the next sections.

Let $m\colon \Omega \to [1,\infty)$ be a measurable function, where Ω is a domain of $\mathbb{R}^N$. The Lebesgue space with a variable exponent $m(\cdot)$ is defined by

$$L^{m(\cdot)}(\Omega) = \left\{ u\colon \Omega \to \mathbb{R} \text{ is measurable}, \rho_{m(\cdot)}(u) := \int_\Omega |u(x)|^{m(x)}\,\mathrm{d}x < \infty \right\},$$

which is equipped with the Luxemburg-type norm

$$\|u\|_{m(\cdot)} = \inf\left\{ \lambda > 0 : \rho_{m(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1 \right\}$$

is a Banach space. We call it a generalized Lebesgue space.

We next define variable exponent Sobolev spaces $W^{1,m(\cdot)}(\Omega)$ as follows:

$$W^{1,m(\cdot)}(\Omega) = \{u \in L^{m(\cdot)}(\Omega) : |\nabla u| \in L^{m(\cdot)}(\Omega)\},$$

endowed with the norm

$$\|u\|_{W^{1,m(\cdot)}(\Omega)} = \|u\|_{m(\cdot)} + \|\nabla u\|_{m(\cdot)}.$$

The space $W_0^{1,m(\cdot)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,m(\cdot)}(\Omega)$.

Proposition 2.1. [11] *For any $u \in L^{m(\cdot)}(\Omega)$. Then the following inequality holds*

$$\min\left\{\|u\|_{m(\cdot)}^{m^-}, \|u\|_{m(\cdot)}^{m^+}\right\} \le \rho_{m(\cdot)}(u) \le \max\left\{\|u\|_{m(\cdot)}^{m^-}, \|u\|_{m(\cdot)}^{m^+}\right\}.$$

Proposition 2.2. [11,14] *Let $p, m\colon \Omega \to [1,\infty)$ be measurable functions. Then we have*

(i) *If Ω is bounded and $m(x) \le p(x)$ for a.e. $x \in \Omega$, then $L^{p(\cdot)}(\Omega) \hookrightarrow L^{m(\cdot)}(\Omega)$ and the embedding is continuous.*

(ii) *If $m \in C(\overline{\Omega})$ satisfies (1.3) then the Sobolev imbedding $W_0^{1,m(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous and compact.*

(iii) *If Ω is bounded and m satisfies (1.4), then the $m(\cdot)$-Poincaré inequality*

$$\|u\|_{m(\cdot)} \le C\|\nabla u\|_{m(\cdot)}$$

holds for all $u \in W_0^{1,m(\cdot)}(\Omega)$. Here C is a positive constant depending only on m and Ω.

By Proposition 2.2(iii), the space $W_0^{1,m(\cdot)}(\Omega)$ also has a norm $\|\nabla u\|_{m(\cdot)}$, which is equivalent to $\|u\|_{W^{1,m(\cdot)}(\Omega)}$. So we can use this norm to replace $\|u\|_{W^{1,m(\cdot)}(\Omega)}$ in the next sections.

2.2. Stationary state and potential well

Consider the stationary solutions of (1.1) which solve the problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{m(x)-2}\nabla u) = |u|^{p(x)-2}u & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where m and p satisfy (1.2)–(1.4). Define the energy functional J and the Nehari functional I by

$$J(u) = \int_\Omega \frac{1}{m(x)}|\nabla u|^{m(x)}\,\mathrm{d}x - \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,\mathrm{d}x,$$
$$I(u) = \int_\Omega |\nabla u|^{m(x)}\,\mathrm{d}x - \int_\Omega |u|^{p(x)}\,\mathrm{d}x.$$

Then J and I are of class C^1 over $W_0^{1,m(\cdot)}(\Omega)$ and critical points of J are weak solutions of (2.1). Moreover, it is easily seen that

$$J(u) \geq \left(\frac{1}{m^+} - \frac{1}{p^-}\right) \int_\Omega |\nabla u|^{m(x)}\,\mathrm{d}x + \frac{1}{p^-} I(u), \tag{2.2}$$

$$J(u) \leq \left(\frac{1}{m^-} - \frac{1}{p^+}\right) \int_\Omega |\nabla u|^{m(x)}\,\mathrm{d}x + \frac{1}{p^+} I(u). \tag{2.3}$$

Let $u \in W_0^{1,m(\cdot)}(\Omega) \setminus \{0\}$ and consider the fibering map $s \mapsto j(s) := J(su)$ for $s > 0$,

$$j(s) = \int_\Omega \frac{s^{m(x)}}{m(x)} |\nabla u|^{m(x)}\,\mathrm{d}x - \int_\Omega \frac{s^{p(x)}}{p(x)} |u|^{p(x)}\,\mathrm{d}x.$$

We then have the following lemma.

Lemma 2.3. *Let m, p satisfy* (1.2)–(1.4), *and $u \in W_0^{1,m(\cdot)}(\Omega) \setminus \{0\}$. Then*

(i) $\lim_{s\to 0^+} j(s) = 0$ *and* $\lim_{s\to\infty} j(s) = -\infty$.

(ii) *There exists a unique $s^* = s^*(u) > 0$ such that $I(su) > 0$ for $s \in (0, s^*)$, $I(s^*u) = 0$ and $I(su) < 0$ for $s \in (s^*, \infty)$. In addition, we have that $j(s)$ is strictly increasing on $(0, s^*)$, strictly decreasing on (s^*, ∞), and attains the maximum at $s = s^*$.*

Proof. (i) Elementary calculations imply that

$$j(s) \leq \max\left\{s^{m^+}, s^{m^-}\right\} \int_\Omega \frac{1}{m(x)} |\nabla u|^{m(x)}\,\mathrm{d}x - \min\left\{s^{p^+}, s^{p^-}\right\} \int_\Omega \frac{1}{p(x)} |u|^{p(x)}\,\mathrm{d}x$$

and

$$j(s) \geq \min\left\{s^{m^+}, s^{m^-}\right\} \int_\Omega \frac{1}{m(x)} |\nabla u|^{m(x)}\,\mathrm{d}x - \max\left\{s^{p^+}, s^{p^-}\right\} \int_\Omega \frac{1}{p(x)} |u|^{p(x)}\,\mathrm{d}x.$$

Since $m^+ < p^-$ and $u \neq 0$, we obtain (i).

(ii) It is also noticed that $j(s) > 0$ for small $s > 0$ and j is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$. Combining this facts and (i), we imply that j attains its maximum value at some number $s^* := s^*(u) > 0$. By Fermat's theorem, one has $j'(s^*) = 0$. On the other hand, since $I(su) = sj'(s)$, we obtain $I(s^*u) = 0$. Then for any $s > 0$, we have

$$\begin{aligned} I(su) &= I(su) - \left(\frac{s}{s^*}\right)^{m^+} I(s^*u) \\ &= \int_\Omega \left(\left(\frac{s}{s^*}\right)^{m(x)} - \left(\frac{s}{s^*}\right)^{m^+}\right) |\nabla s^*u|^{m(x)}\,\mathrm{d}x \\ &\quad + \int_\Omega \left(\left(\frac{s}{s^*}\right)^{m^+} - \left(\frac{s}{s^*}\right)^{p(x)}\right) |s^*u|^{p(x)}\,\mathrm{d}x. \end{aligned}$$

Since $m(x) \leq m^+ < p(x)$ for a.e. $x \in \Omega$, we derive that $I(su) > 0$ for $s \in (0, s^*)$, $I(s^*u) = 0$ and $I(su) < 0$ for $s \in (s^*, \infty)$. Hence $j(s)$ is strictly increasing on $(0, s^*)$, strictly decreasing on (s^*, ∞), and attains the maximum at $s = s^*$ due to the relation $j'(s) = \frac{1}{s} I(su)$. The proof is complete. □

Let us define the so-called mountain pass level

$$d = \inf_{u \in W_0^{1,m(\cdot)}(\Omega) \setminus \{0\}} \sup_{s>0} J(su).$$

We also define the Nehari manifold defined by

$$\mathcal{N} = \big\{ u \in W_0^{1,m(\cdot)}(\Omega) \setminus \{0\} : I(u) = 0 \big\}.$$

The next lemma gives the variational characterization of d.

Lemma 2.4. *Let m, p satisfy* (1.2)–(1.4)*. Then there exists non-negative function $u_* \in \mathcal{N}$ such that*

$$d = \min_{u \in \mathcal{N}} J(u) = J(u_*) > 0.$$

Proof. We first prove that $d = \inf_{u \in \mathcal{N}} J(u)$. By Lemma 2.3, we can rewrite the mountain pass level d as follows

$$d = \inf_{u \in W_0^{1,m(\cdot)}(\Omega) \setminus \{0\}} J(s^* u),$$

which implies that $d \geq \inf_{u \in \mathcal{N}} J(u)$ since $s^* u \in \mathcal{N}$. For any $u \in \mathcal{N}$, by using Lemma 2.3 again, we have $s^* = 1$ and so $d = \inf_{u \in W_0^{1,m(\cdot)}(\Omega) \setminus \{0\}} J(u) \leq \inf_{u \in \mathcal{N}} J(u)$. Hence $d = \inf_{u \in \mathcal{N}} J(u)$.

We next show that $d > 0$. Indeed, let $u \in \mathcal{N}$. If $\|\nabla u\|_{m(\cdot)} \leq 1$, then

$$\begin{aligned} \|\nabla u\|_{m(\cdot)}^{m^+} &\leq \int_\Omega |\nabla u|^{m(x)} \, \mathrm{d}x = \int_\Omega |u|^{p(x)} \, \mathrm{d}x \leq \max \big\{ \|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+} \big\} \\ &\leq \max \big\{ S_{p(\cdot)}^{p^-} \|\nabla u\|_{m(\cdot)}^{p^-}, S_{p(\cdot)}^{p^+} \|\nabla u\|_{m(\cdot)}^{p^+} \big\} \leq \max \big\{ S_{p(\cdot)}^{p^-}, S_{p(\cdot)}^{p^+} \big\} \|\nabla u\|_{m(\cdot)}^{p^-}, \end{aligned}$$

which implies that $\|\nabla u\|_{m(\cdot)} \geq (1/\delta)^{1/(p^- - m^+)}$, where $\delta := \max \big\{ S_{p(\cdot)}^{p^-}, S_{p(\cdot)}^{p^+} \big\}$ and $S_{p(\cdot)} > 0$ is the optimal constant in the embedding $W_0^{1,m(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$. Hence, for any $u \in \mathcal{N}$, one has

$$\|\nabla u\|_{m(\cdot)} \geq \min \left\{ \left(\frac{1}{\delta} \right)^{1/(p^- - m^+)}, 1 \right\} > 0. \tag{2.4}$$

On the other hand, we also deduce from $u \in \mathcal{N}$ and (2.2) that

$$J(u) \geq \left(\frac{1}{m^+} - \frac{1}{p^-} \right) \min \big\{ \|\nabla u\|_{m(\cdot)}^{m^+}, \|\nabla u\|_{m(\cdot)}^{m^-} \big\}. \tag{2.5}$$

Combining (2.4) and (2.5) we obtain $d > 0$.

We finally prove that d is actually attained by some non-negative function $u \in \mathcal{N}$. Indeed, let $\{u_k\} \subset \mathcal{N}$ be a minimizing sequence for J such that

$$J(u_k) \to d \quad \text{as } k \to \infty. \tag{2.6}$$

Note that $|u_k| \in W_0^{1,m(\cdot)}(\Omega)$ thanks to $u_k \in W_0^{1,m(\cdot)}(\Omega)$, $I(|u_k|) = I(u_k)$ and $J(|u_k|) = J(u_k)$, so we may assume that $u_k \geq 0$ a.e. in Ω for all $k \in \mathbb{N}$. From (2.5), (2.6) and the compactness of the embedding $W_0^{1,m(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ due to (1.3), we can find a function u_* and a sub-sequence of $\{u_k\}$, still denote by $\{u_k\}$, so that

$$u_k \rightharpoonup u_* \quad \text{weakly in } W_0^{1,m(\cdot)}(\Omega), \tag{2.7}$$

$$u_k \to u_* \quad \text{strongly in } L^{p(\cdot)}(\Omega) \text{ and a.e. in } \Omega. \tag{2.8}$$

Then $u_* \geq 0$ a.e. in Ω. We next show that $u_* \in \mathcal{N}$. Since $u_k \in \mathcal{N}$, then

$$\int_\Omega |\nabla u_k|^{m(x)}\,\mathrm{d}x = \int_\Omega |u_k|^{p(x)}\,\mathrm{d}x,$$

which, together with (2.4) and (2.8), implies that $u_* \neq 0$. From (2.7), (2.8) and notice that $u_k \in \mathcal{N}$, we have $I(u_*) \leq \liminf_{k\to\infty} I(u_k) = 0$. Suppose that $I(u_*) < 0$, then by Lemma 2.3 there exists $s^* \in (0,1)$ such that $I(s^*u_*) = 0$ which implies

$$\begin{aligned} d &\leq J(s^*u_*) = J(s^*u_*) - \frac{1}{p^-} I(s^*u_*) \\ &= \int_\Omega (s^*)^{m(x)} \left(\frac{1}{m(x)} - \frac{1}{p^-}\right) |\nabla u_*|^{m(x)}\,\mathrm{d}x + \int_\Omega (s^*)^{p(x)} \left(\frac{1}{p^-} - \frac{1}{p(x)}\right) |u_*|^{p(x)}\,\mathrm{d}x \\ &< \int_\Omega \left(\frac{1}{m(x)} - \frac{1}{p^-}\right) |\nabla u_*|^{m(x)}\,\mathrm{d}x + \int_\Omega \left(\frac{1}{p^-} - \frac{1}{p(x)}\right) |u_*|^{p(x)}\,\mathrm{d}x \\ &= J(u_*) - \frac{1}{p^-} I(u_*). \end{aligned} \tag{2.9}$$

On the other hand, since $u_k \in \mathcal{N}$, it follows from (2.7) and (2.8) that

$$\begin{aligned} d &= \liminf_{k\to\infty} \left[J(u_k) - \frac{1}{p^-} I(u_k) \right] \\ &= \liminf_{k\to\infty} \left[\int_\Omega \left(\frac{1}{m(x)} - \frac{1}{p^-}\right) |\nabla u_k|^{m(x)}\,\mathrm{d}x + \int_\Omega \left(\frac{1}{p^-} - \frac{1}{p(x)}\right) |u_k|^{p(x)}\,\mathrm{d}x \right] \\ &\geq \int_\Omega \left(\frac{1}{m(x)} - \frac{1}{p^-}\right) |\nabla u_*|^{m(x)}\,\mathrm{d}x + \int_\Omega \left(\frac{1}{p^-} - \frac{1}{p(x)}\right) |u_*|^{p(x)}\,\mathrm{d}x \\ &= J(u_*) - \frac{1}{p^-} I(u_*), \end{aligned}$$

which contradicts (2.9). Hence we get $u_* \neq 0$ and $I(u_*) = 0$, and therefore $u_* \in \mathcal{N}$. From this and the above estimate, we obtain $d \geq J(u_*)$. On the other hand, $d \leq J(u_*)$ since $u_* \in \mathcal{N}$. So $d = J(u_*)$. The proof is complete. □

We now define the so-called stable set $\mathcal{W}$ (also known as potential well) and unstable set $\mathcal{U}$ as follows:

$$\mathcal{W} = \left\{u \in W_0^{1,m(\cdot)}(\Omega) : I(u) > 0, J(u) < d\right\} \cup \{0\},$$
$$\mathcal{U} = \left\{u \in W_0^{1,m(\cdot)}(\Omega) : I(u) < 0, J(u) < d\right\}.$$

We also define

$$\mathcal{N}_+ = \{u \in W_0^{1,m(\cdot)}(\Omega) : I(u) > 0\} \quad \text{and} \quad \mathcal{N}_- = \{u \in W_0^{1,m(\cdot)}(\Omega) : I(u) < 0\},$$

and the closed sub levels of J

$$J^k = \{u \in W_0^{1,m(\cdot)}(\Omega) : J(u) \le k\} \quad \text{and} \quad \mathcal{N}_k := \mathcal{N} \cap J^k \neq \emptyset \quad \text{for } k \ge d.$$

For $k \ge d$, we define

$$\lambda_k = \inf\{\|u\|_{H_0^1} : u \in \mathcal{N}_k\} \quad \text{and} \quad \Lambda_k = \sup\{\|u\|_{H_0^1} : u \in \mathcal{N}_k\}.$$

3. Main results

Before stating our main result, we recall here some known results and definitions.

Theorem 3.1. (see [9, Theorem 3.1]) *Assume that* (1.2)–(1.4) *hold, then for any* $u_0 \in W_0^{1,m(\cdot)}(\Omega)$*, there exists a* $T_0 \in (0, T]$ *such that the problem* (1.1) *has a unique local solution*

$$u \in L^\infty(0, T_0; W_0^{1,m(\cdot)}(\Omega)), \quad u_t \in L^2(0, T_0; H_0^1(\Omega))$$

satisfying

$$\langle u_t, v\rangle + \langle \nabla u_t, \nabla v\rangle + \langle |\nabla u|^{m(x)-2}\nabla u, \nabla v\rangle = \langle |u|^{p(x)-2}u, v\rangle \tag{3.1}$$

for all $v \in W_0^{1,m(\cdot)}(\Omega)$*. In addition,* $u(t) := u(x,t)$ *satisfies the following energy identity*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{H_0^1}^2 = -2I(u(t)) \tag{3.2}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}J(u(t)) = -\|u'(t)\|_{H_0^1}^2. \tag{3.3}$$

Remark 3.2. The existence of such solution was stated in [9] while the identity (3.2) is obtained by replacing v in (3.1) by u. In addition, by integrating the identity (3.3) with respect to time variable from 0 to t we obtain

$$\int_0^t \|u'(\tau)\|_{H_0^1}^2 \,\mathrm{d}\tau + J(u(t)) = J(u_0), \quad 0 \le t < T_0. \tag{3.4}$$

Definition 3.3 (Maximal existence time)**.** Let $u(t)$ be a weak solution to the problem (1.1). We define the maximal existence time $T_{\max}$ of $u(t)$ as follows:

(i) If $u(t)$ exists for $0 \le t < \infty$, then $T_{\max} = \infty$.

(ii) If there exists $t_0 > 0$ such that $u(t)$ exists for $0 \le t < t_0$, but does not exist at t_0, then $T_{\max} = t_0$.

We then introduce the sets

$$\mathcal{G} = \left\{u_0 \in W_0^{1,m(\cdot)}(\Omega) : u(t) \text{ exists globally, i.e., } T_{\max} = \infty\right\},$$
$$\mathcal{G}_0 = \left\{u_0 \in \mathcal{G} : u(t) \to 0 \text{ in } W_0^{1,m(\cdot)}(\Omega) \text{ as } t \to \infty\right\},$$
$$\mathcal{B} = \left\{u_0 \in W_0^{1,m(\cdot)}(\Omega) : u(t) \text{ blows up in finite time, i.e., } T_{\max} < \infty\right\}.$$

We are now in a position to state our main results. Our first theorem shows that the solution $u(t)$ to (1.1) exists globally and its potential energy $J(u(t))$ decays when it begins in the stable set $\mathcal{W}$.

Theorem 3.4. *Let* (1.2)–(1.4) *hold. If* $u_0 \in \mathcal{W}$, *then* $u_0 \in \mathcal{G}_0$. *Moreover, the energy functional* $J(u(t))$ *satisfies the following decay estimates*

$$J(u(t)) \le \begin{cases} \left(Ct + (J(u_0) + \|u_0\|_{H_0^1}^2)^{\frac{2-m^+}{2}}\right)^{\frac{2}{2-m^+}} & \text{if } m^+ > 2, \\ (J(u_0) + \|u_0\|_{H_0^1}^2)e^{-Ct} & \text{if } m^+ = 2 \end{cases}$$

for some positive constant $C := C(m, p, u_0, d)$.

We next prove the instability of $u(t)$ that starts from the unstable sets $\mathcal{U}$.

Theorem 3.5. *Let* (1.2)–(1.4) *hold. If* $u_0 \in \mathcal{U}$, *then* $u_0 \in \mathcal{B}$. *Moreover, we can estimate the upper bound for* $T_{\max}$ *as follows*

$$T_{\max} \le \frac{4(p^- - 1)\|u_0\|_{H_0^1}^2}{p^-(p^- - 2)^2(d - J(u_0))}.$$

Remark 3.6. Theorems 3.4 and 3.5 give us a sharp result on the existence and nonexistence of global solution to (1.1) when $J(u_0) < d$.

Let us introduce the set

$$\mathcal{S} = \left\{\phi \in W_0^{1,m(\cdot)}(\Omega) : \phi \text{ is a stationary solution of (1.1)}\right\}.$$

The next theorem shows the asymptotic behavior of the global solution of (1.1).

Theorem 3.7. *Let* (1.2)–(1.4) *hold and* $u(t)$ *be a global weak solution of the problem* (1.1). *Then there exists a sequence* $\{t_n\}$ *with* $t_n \to \infty$ *and a function* $\phi \in \mathcal{S}$ *such that* $u(t_n) \to \phi$ *strongly in* $W_0^{1,m(\cdot)}(\Omega)$ *as* $n \to \infty$.

Finally, we give a characterization on the data u_0 with arbitrary high energy $J(u_0)$ that leads to blow-up in finite time phenomena.

Theorem 3.8. *Let* (1.2)–(1.4) *hold. Then if* $J(u_0) > d$ *and*

$$\|u_0\|^2_{H^1_0} > \frac{2(1+\lambda_1)}{m^-\lambda_1}\left(\frac{m^+p^-}{p^- - m^+}J(u_0) + \frac{m^+ - 2}{2}|\Omega|\right), \tag{3.5}$$

then $u_0 \in \mathcal{N}_- \cap \mathcal{B}$. *Here* λ_1 *is the first eigenvalue of the problem*

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

4. Proof of Theorem 3.4

We begin this section by the boundedness of the stable set $\mathcal{W}$.

Lemma 4.1. $\mathcal{W}$ *is a bounded subset of* $W_0^{1,m(\cdot)}(\Omega)$. *More precisely, one has*

$$\|\nabla u\|_{m(\cdot)} \le \max\left\{M^{\frac{1}{m^-}}, M^{\frac{1}{m^+}}\right\}, \quad \forall\, u \in \mathcal{W},$$

where $M = \frac{p^-m^+d}{p^- - m^+} > 0$.

Proof. Let $u \in \mathcal{W}$ then by definition of $\mathcal{W}$ we have $J(u) < d$ and $I(u) \ge 0$. Taking this into account and notice that (2.2), we have the following estimate

$$J(u) \ge \left(\frac{1}{m^+} - \frac{1}{p^-}\right)\int_\Omega |\nabla u|^{m(x)}\,\mathrm{d}x,$$

which implies

$$\rho_{m(\cdot)}(\nabla u) = \int_\Omega |\nabla u|^{m(x)}\,\mathrm{d}x \le \frac{p^-m^+}{p^- - m^+}J(u) \le \frac{p^-m^+}{p^- - m^+}d = M. \tag{4.1}$$

Hence, we obtain

$$\|\nabla u\|_{m(\cdot)} \le \max\left\{\rho_{m(\cdot)}^{\frac{1}{m^-}}(\nabla u), \rho_{m(\cdot)}^{\frac{1}{m^+}}(\nabla u)\right\} \le \max\left\{M^{\frac{1}{m^-}}, M^{\frac{1}{m^+}}\right\}.$$

The proof is complete. □

The next lemma plays a crucial role in the proof of decay properties of the energy functional.

Lemma 4.2. *Let* $u(t)$ *be a weak solution to* (1.1) *on* $[0, T_{\max})$ *associated with the initial data* $u_0 \in \mathcal{W}$. *Then* $u(t) \in \mathcal{W}$ *for all* $t \in [0, T_{\max})$ *and*

$$I(u(t)) \ge \left[1 - \left(\frac{J(u_0)}{d}\right)^{\frac{p^- - m^+}{p^+}}\right]\int_\Omega |\nabla u(t)|^{m(x)}\,\mathrm{d}x, \quad 0 \le t < T_{\max}.$$

Proof. We first prove that $u(t) \in \mathcal{W}$ on $[0, T_{\max})$. Indeed, assume that $u(t)$ exists in $\mathcal{W}$ at the time $t = t_0 \in (0, T_{\max})$ then we have

$$u(t) \in \mathcal{W}, \ \ 0 \le t < t_0 \quad \text{and} \quad u(t_0) \in \partial\mathcal{W}.$$

This implies that $J(u(t_0)) = d$ or $I(u(t_0)) = 0$ and $u(t_0) \neq 0$. By (3.4) we have $J(u(t_0)) \le J(u_0) < d$, so it must be $I(u(t_0)) = 0$ and $u(t_0) \neq 0$. However this again leads to the following contradiction

$$d = \inf_{u \in \mathcal{N}} J(u) \le J(u(t_0)) < d.$$

Hence we get $u(t) \in \mathcal{W}$ on $[0, T_{\max})$.

By Lemma 2.3 there exists a constant $s^* = s^*(u(t)) > 1$ such that $I(s^* u(t)) = 0$. Then

$$\begin{aligned} 0 = I(s^* u(t)) &\le (s^*)^{m^+} \int_\Omega |\nabla u(t)|^{m(x)} \,\mathrm{d}x - (s^*)^{p^-} \int_\Omega |u(t)|^{p(x)} \,\mathrm{d}x \\ &= \left((s^*)^{m^+} - (s^*)^{p^-}\right) \int_\Omega |\nabla u(t)|^{m(x)} \,\mathrm{d}x + (s^*)^{p^-} I(u(t)), \end{aligned}$$

which implies

$$I(u(t)) \ge \left[1 - (s^*)^{m^+ - p^-}\right] \int_\Omega |\nabla u(t)|^{m(x)} \,\mathrm{d}x. \tag{4.2}$$

We next find a precisely lower bound for s^*. By variational characterization of d and the definition of J, we have

$$\begin{aligned} d &\le J(s^* u(t)) = J(s^* u(t)) - \frac{1}{p^-} I(s^* u(t)) \\ &= \int_\Omega (s^*)^{m(x)} \left(\frac{1}{m(x)} - \frac{1}{p^-}\right) |\nabla u(t)|^{m(x)} \,\mathrm{d}x + \int_\Omega (s^*)^{p(x)} \left(\frac{1}{p^-} - \frac{1}{p(x)}\right) |u(t)|^{p(x)} \,\mathrm{d}x \\ &\le (s^*)^{p^+} \left[\int_\Omega \left(\frac{1}{m(x)} - \frac{1}{p^-}\right) |\nabla u(t)|^{m(x)} \,\mathrm{d}x + \int_\Omega \left(\frac{1}{p^-} - \frac{1}{p(x)}\right) |u(t)|^{p(x)} \,\mathrm{d}x\right] \\ &= (s^*)^{p^+} \left[J(u(t)) - \frac{1}{p^-} I(u(t))\right]. \end{aligned}$$

Combining this fact, energy identity (3.4) and $u(t) \in \mathcal{W}$ we obtain

$$J(u_0) \ge J(u(t)) \ge J(u(t)) - \frac{1}{p^-} I(u(t)) \ge \frac{d}{(s^*)^{p^+}},$$

which implies

$$s^* \ge \left(\frac{d}{J(u_0)}\right)^{\frac{1}{p^+}} > 1. \tag{4.3}$$

The proof follows from (4.2) and (4.3). $\square$

With the aid of Lemmas 4.1 and 4.2, we are now ready to prove the Theorem 3.4.

Proof of Theorem 3.4. Let $u(t)$ be a weak solution of (1.1) starting from $u_0 \in \mathcal{W}$. As in the proof of Lemma 4.2 we have $u(t) \in \mathcal{W}$ for all $t \in [0, T_{\max})$. It follows from Lemma 4.1 that $u(t)$ is uniformly bounded in time in $W_0^{1,m(\cdot)}(\Omega)$. Thus the weak solution $u(t)$ to (1.1) exists globally, i.e., $T_{\max} = \infty$. So it remains to prove the decay property of energy functional $J(u(t))$, we define the Lyapunov functional

$$L(t) = J(u(t)) + \|u(t)\|_{H_0^1}^2.$$

Then by using the embedding $W_0^{1,m(\cdot)}(\Omega) \hookrightarrow H_0^1(\Omega)$ for $m^- \geq 2$ and (4.1), one has

$$\begin{aligned}
L(t) &\leq J(u(t)) + C\|\nabla u(t)\|_{m(\cdot)}^2 \\
&\leq J(u(t)) + C \max\left\{\rho_{m(\cdot)}^{\frac{2}{m^-}}(\nabla u(t)), \rho_{m(\cdot)}^{\frac{2}{m^+}}(\nabla u(t))\right\} \\
&= J(u(t)) + C \max\left\{\rho_{m(\cdot)}^{\frac{2}{m^-}-\frac{2}{m^+}}(\nabla u(t)), 1\right\} \rho_{m(\cdot)}^{\frac{2}{m^+}}(\nabla u(t)) \\
&\leq J(u(t)) + C \max\left\{M^{\frac{2}{m^-}-\frac{2}{m^+}}, 1\right\}\rho_{m(\cdot)}^{\frac{2}{m^+}}(\nabla u(t)) \\
&\leq J(u(t)) + CJ^{\frac{2}{m^+}}(u(t)) \\
&= \left(J^{1-\frac{2}{m^+}}(u(t)) + C\right)J^{\frac{2}{m^+}}(u(t)) \\
&\leq CJ^{\frac{2}{m^+}}(u(t)).
\end{aligned} \tag{4.4}$$

Here $C := C(m, p, u_0)$ is a constant and $J(u(t)) \leq J(u_0)$ is used for the last estimate. By Lemma 4.2 and (2.3), we obtain

$$J(u(t)) \leq CI(u(t)), \tag{4.5}$$

where $C := C(d, m, p, u_0)$ is a positive constant given by

$$C = \left(\frac{1}{m^-} - \frac{1}{p^+}\right)\left[1 - \left(\frac{J(u_0)}{d}\right)^{\frac{p^- - m^+}{p^+}}\right]^{-1} + \frac{1}{p^+}.$$

Combining (4.4) and (4.5), we find that

$$L'(t) = -\|u_t(t)\|_{H_0^1}^2 - 2I(u(t)) \leq -2C^{-1}J(u(t)) \leq -CL^{\frac{m^+}{2}}(t),$$

where $C := C(d, m, p, u_0)$. If $m^+ > 2$, then

$$J(u(t)) \leq L(t) \leq \left(Ct + L^{\frac{2-m^+}{2}}(0)\right)^{\frac{2}{2-m^+}} = \left(Ct + \left(J(u_0) + \|u_0\|_{H_0^1}^2\right)^{\frac{2-m^+}{2}}\right)^{\frac{2}{2-m^+}}.$$

If $m^+ = 2$, then

$$J(u(t)) \le L(t) \le L(0)e^{-Ct} = \big(J(u_0) + \|u_0\|_{H_0^1}^2\big)e^{-Ct}.$$

Thus $J(u(t)) \to 0$ as $t \to \infty$ in both cases. It is from this and (4.1) that $u_0 \in \mathcal{G}_0$. The proof is complete. □

5. Proof of Theorem 3.5

We begin this section by the invariance property of $\mathcal{U}$ which can be proved similarly as in the proof of Lemma 4.2. So we omit it here.

Lemma 5.1. *If $u_0 \in \mathcal{U}$ and $u(t)$ is a weak solution to* (1.1) *on $[0, T_{\max})$, then $u(t) \in \mathcal{U}$ for $0 \le t < T_{\max}$.*

With the aid of Lemma 5.1, we are now ready to prove Theorem 3.5. Fix $T \in (0, T_{\max})$ and consider the function $F(t)$ defined by

$$F(t) = \int_0^t \|u(\tau)\|_{H_0^1}^2 \,\mathrm{d}\tau + (T-t)\|u_0\|_{H_0^1}^2 + \mu(t) \quad \text{for } t \in [0, T).$$

Here $\mu(t) \in C^2[0, T)$ is a positive function given later. We then have

$$F'(t) = \|u(t)\|_{H_0^1}^2 - \|u_0\|_{H_0^1}^2 + \mu'(t), \quad \text{and} \quad F''(t) = -2I(u(t)) + \mu''(t).$$

By using Cauchy–Schwarz inequality, we have, for any $\xi_1 > 0$,

$$\begin{aligned}
&\left(\int_0^t \|u(\tau)\|_{H_0^1}^2 \,\mathrm{d}\tau + \mu(t)\right)\left(\int_0^t \|u'(\tau)\|_{H_0^1}^2 \,\mathrm{d}\tau + \xi_1\right) \\
&\ge \left(\sqrt{\int_0^t \|u'(\tau)\|_{H_0^1}^2 \,\mathrm{d}s \int_0^t \|u(\tau)\|_{H_0^1}^2 \,\mathrm{d}s} + \sqrt{\xi_1 \mu(t)}\right)^2 \\
&\ge \left(\int_0^t \langle u'(\tau), u(\tau)\rangle_{H_0^1} \,\mathrm{d}\tau + \sqrt{\xi_1 \mu(t)}\right)^2 \\
&= \frac{1}{4}\left(\|u(t)\|_{H_0^1}^2 - \|u_0\|_{H_0^1}^2 + 2\sqrt{\xi_1 \mu(t)}\right)^2.
\end{aligned}$$

We choose $\mu(t)$ such that $\mu'(t) = 2\sqrt{\xi_1 \mu(t)}$, that is, $\mu(t) = \xi_1 (t + \xi_2)^2$ with $\xi_2 > 0$. Then

$$\begin{aligned}
(F'(t))^2 &= \left(\|u(t)\|_{H_0^1}^2 - \|u_0\|_{H_0^1}^2 + 2\sqrt{\xi_1 \mu(t)}\right)^2 \\
&\le 4\left(\int_0^t \|u(\tau)\|_{H_0^1}^2 \,\mathrm{d}\tau + \mu(t)\right)\left(\int_0^t \|u'(\tau)\|_{H_0^1}^2 \,\mathrm{d}\tau + \xi_1\right) \\
&\le 4F(t)\left(\int_0^t \|u'(\tau)\|_{H_0^1}^2 \,\mathrm{d}\tau + \xi_1\right).
\end{aligned} \tag{5.1}$$

On the other hand, since $u_0 \in \mathcal{U}$ it follows from Lemma 5.1 that $u(t) \in \mathcal{U}$, that is, $I(u(t)) < 0$. Then by analogous arguments in the proof of Lemma 2.4, we deduce that

$$d < J(u(t)) - \frac{1}{p^-} I(u(t)),$$

which implies that

$$-I(u(t)) > p^-(d - J(u(t))) = p^-(d - J(u_0)) + p^- \int_0^t \|u'(\tau)\|_{H_0^1}^2 \, \mathrm{d}\tau.$$

From the above inequality and (5.1), we get

$$\begin{aligned} F''(t)F(t) - \frac{p^-}{2}(F'(t))^2 &\geq F(t) \left[F''(t) - 2p^- \left(\int_0^t \|u'(\tau)\|_{H_0^1}^2 \, \mathrm{d}\tau + \xi_1 \right) \right] \\ &= F(t) \left[-2I(u(t)) - 2p^- \int_0^t \|u'(\tau)\|_{H_0^1}^2 \, \mathrm{d}\tau - 2\xi_1(p^- - 1) \right] \\ &\geq F(t) \big[2p^-(d - J(u_0)) - 2\xi_1(p^- - 1) \big]. \end{aligned}$$

Choosing $\xi_1 = \frac{p^-(d - J(u_0))}{p^- - 1} > 0$, we have

$$F''(t)F(t) - \frac{p^-}{2}(F'(t))^2 \geq 0.$$

Putting $G(t) = F^{1 - \frac{p^-}{2}}(t)$, we get

$$G'(t) = \left(1 - \frac{p^-}{2}\right) \frac{F'(t)}{F^{\frac{p^-}{2}}(t)}, \quad \text{and} \quad G''(t) = \left(1 - \frac{p^-}{2}\right) \frac{F(t)F''(t) - \frac{p^-}{2}(F'(t))^2}{F^{1 + \frac{p^-}{2}}(t)}.$$

Then we have $G''(t) \leq 0$ for all $t \in [0, T]$, that is, $G(t)$ is concave downward on $[0, T]$. By definition of concavity, we get

$$G(T) \leq G(0) + TG'(0).$$

From this and notice that $G(T) > 0$ and $G'(0) < 0$, one has

$$T \leq -\frac{G(0)}{G'(0)} = \frac{F(0)}{\left(\frac{p^-}{2} - 1\right)F'(0)} = \frac{T\|u_0\|_{H_0^1}^2 + \xi_1 \xi_2^2}{(p^- - 2)\xi_1 \xi_2}.$$

By choosing $\xi_2 > \frac{\|u_0\|_{H_0^1}^2}{(p^- - 2)\xi_1}$, we have

$$T \leq \frac{\xi_1 \xi_2^2}{(p^- - 2)\xi_1 \xi_2 - \|u_0\|_{H_0^1}^2} := \gamma(\xi_2).$$

Hence,

$$T \leq \inf_{\xi_2 > \frac{\|u_0\|^2_{H_0^1}}{(p^- -2)\xi_1}} \gamma(\xi_2) = \gamma\left(\frac{2\|u_0\|^2_{H_0^1}}{(p^- -2)\xi_1}\right) = \frac{4(p^- -1)\|u_0\|^2_{H_0^1}}{p^-(p^- -2)^2(d-J(u_0))}.$$

Letting $T \to T_{\max}$, we obtain

$$T_{\max} \leq \frac{4(p^- -1)\|u_0\|^2_{H_0^1}}{p^-(p^- -2)^2(d-J(u_0))}.$$

The proof is complete.

6. Proof of Theorem 3.7

Let $u = u(t)$ be a global solution to (1.1). By Theorems 3.4 and 3.5, without loss of generality, we may assume that $J(u(t)) \geq d$ for all $t \geq 0$. It follows from this and (3.4) that

$$\int_0^t \|u'(\tau)\|^2_{H_0^1}\,\mathrm{d}\tau \leq J(u_0) - d.$$

Letting $t \to \infty$, we get $\int_0^\infty \|u'(\tau)\|^2_{H_0^1}\,\mathrm{d}\tau < \infty$. And hence there exists a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \|u'(t_n)\|_{H_0^1} = 0. \tag{6.1}$$

Then we have

$$|I(u(t_n))| = \left|\langle u'(t_n), u(t_n)\rangle_{H_0^1}\right| \leq \|u'(t_n)\|_{H_0^1}\|u(t_n)\|_{H_0^1} \leq C\|\nabla u(t_n)\|_{m(\cdot)},$$

which, together with (2.2), implies

$$\begin{aligned} J(u_0) \geq J(u(t_n)) &\geq \left(\frac{1}{m^+} - \frac{1}{p^-}\right)\int_\Omega |\nabla u(t_n)|^{m(x)}\,\mathrm{d}x + \frac{1}{p^-}I(u(t_n)) \\ &\geq \left(\frac{1}{m^+} - \frac{1}{p^-}\right)\min\left\{\|\nabla u(t_n)\|^{m^+}_{m(\cdot)}, \|\nabla u(t_n)\|^{m^-}_{m(\cdot)}\right\} - \frac{C}{p^-}\|\nabla u(t_n)\|_{m(\cdot)}. \end{aligned}$$

This and $m^- > 1$ imply

$$\|\nabla u(t_n)\|_{m(\cdot)} \leq C, \quad \forall\, n \in \mathbb{N} \tag{6.2}$$

for some constant $C > 0$. Then since (1.3), there exists a subsequence of $\{t_n\}$, still denoted by $\{t_n\}$ such that

$$u_n := u(t_n) \rightharpoonup \phi \quad \text{weakly in } W_0^{1,m(\cdot)}(\Omega), \tag{6.3}$$

$$u_n \to \phi \quad \text{strongly in } L^{p(\cdot)}(\Omega). \tag{6.4}$$

It follows from (6.3) that

$$|\nabla u_n|^{m(x)-2}\nabla u_n \rightharpoonup \chi \quad \text{weakly in } L^{\frac{m(\cdot)}{m(\cdot)-1}}(\Omega). \tag{6.5}$$

We next show that $\chi = |\nabla\phi|^{m(x)}\nabla\phi$. Testing the equation (1.1) by $v \in W_0^{1,m(\cdot)}(\Omega)$ yields

$$\left|\int_\Omega |\nabla u_n|^{m(x)-2}\nabla u_n \cdot \nabla v \,\mathrm{d}x - \int_\Omega |u_n|^{p(x)-2}u_n v \,\mathrm{d}x\right| = \left|\langle u_n', v\rangle_{H_0^1}\right| \le \|u_n'\|_{H_0^1}\|v\|_{H_0^1}. \tag{6.6}$$

Letting $n \to \infty$ and using (6.1), (6.4) and (6.5) we have

$$\int_\Omega \chi \cdot \nabla v \,\mathrm{d}x = \int_\Omega |\phi|^{p(x)-2}\phi v \,\mathrm{d}x. \tag{6.7}$$

By taking $v = u_n$ in (6.6) and notice that (6.2), one has

$$\left|\int_\Omega |\nabla u_n|^{m(x)} \,\mathrm{d}x - \int_\Omega |u_n|^{p(x)} \,\mathrm{d}x\right| \le C\|u_n'\|_{H_0^1}. \tag{6.8}$$

We deduce from (6.1), (6.4), (6.7) and (6.8) that

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^{m(x)} \,\mathrm{d}x = \int_\Omega |\phi|^{p(x)} \,\mathrm{d}x = \int_\Omega \chi \cdot \nabla\phi \,\mathrm{d}x. \tag{6.9}$$

Since $u \mapsto K(u) := \int_\Omega \frac{1}{m(x)}|\nabla u|^{m(x)} \,\mathrm{d}x$ is a convex functional, we imply (see [23, Proposition 1.1, p. 158]) the mapping K' is monotonic and hemicontinuous, where

$$\langle K'(u), v\rangle = \int_\Omega |\nabla u|^{m(x)-2}\nabla u \cdot \nabla v \,\mathrm{d}x, \quad \forall\, u, v \in W_0^{1,m(\cdot)}(\Omega).$$

By the monotonicity of K', we have, for any $v \in W_0^{1,m(\cdot)}(\Omega)$,

$$\int_\Omega \left(|\nabla u_n|^{m(x)-2}\nabla u_n - |\nabla v|^{m(x)-2}\nabla v\right) \cdot (\nabla u_n - \nabla v) \,\mathrm{d}x \ge 0,$$

which yields that

$$\int_\Omega |\nabla u_n|^{m(x)} \,\mathrm{d}x - \int_\Omega |\nabla u_n|^{m(x)-2}\nabla u_n \cdot \nabla v \,\mathrm{d}x - \int_\Omega |\nabla v|^{m(x)-2}\nabla v \cdot (\nabla u_n - \nabla v) \,\mathrm{d}x \ge 0.$$

By (6.3), (6.5) and (6.9), letting $n \to \infty$ in the above inequality yields

$$\int_\Omega \chi \cdot \nabla\phi \,\mathrm{d}x - \int_\Omega \chi \cdot \nabla v \,\mathrm{d}x - \int_\Omega |\nabla v|^{m(x)-2}\nabla v \cdot (\nabla\phi - \nabla v) \,\mathrm{d}x \ge 0,$$

which is equivalent to

$$\int_\Omega (\chi - |\nabla v|^{m(x)-2}\nabla v) \cdot (\nabla\phi - \nabla v) \,\mathrm{d}x \ge 0.$$

For any $w \in W_0^{1,m(\cdot)}(\Omega)$. Note that the hemicontinuous property of K'. Choosing $v = \phi \pm \lambda w$ and letting $\lambda \to 0^+$ in the above inequality, one has $\chi = |\nabla\phi|^{m(x)}\nabla\phi$. This and (6.7) imply that $\phi \in \mathcal{S}$. It follows from this and (6.9) that

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^{m(x)} \,\mathrm{d}x = \int_\Omega |\phi|^{p(x)} \,\mathrm{d}x = \int_\Omega |\nabla\phi|^{m(x)} \,\mathrm{d}x. \tag{6.10}$$

Combining (6.3) and (6.10), we get $u_n \to \phi$ strongly in $W_0^{1,m(\cdot)}(\Omega)$. The proof is complete.

7. Proof of Theorem 3.8

In order to prove this theorem, we need to use the following lemma which gives sufficient conditions for the existence and nonexistence of global solution to (1.1) in terms of the variational values λ_k and Λ_k.

Lemma 7.1. (see [38, Theorem 2.1]) *Let* (1.2)–(1.4) *hold and* $J(u_0) > d$. *If* $u_0 \in \mathcal{N}_+$ *and* $\|u_0\|_{H_0^1} \le \lambda_{J(u_0)}$, *then* $u_0 \in \mathcal{G}_0$. *If* $u_0 \in \mathcal{N}_-$ *and* $\|u_0\|_{H_0^1} \ge \Lambda_{J(u_0)}$, *then* $u_0 \in \mathcal{B}$.

Let any $u \in W_0^{1,m(\cdot)}(\Omega) \setminus \{0\}$, we have

$$\begin{aligned} \int_\Omega |\nabla u|^{m(x)}\,\mathrm{d}x &= \int_{\Omega_1} |\nabla u|^{m(x)}\,\mathrm{d}x + \int_{\Omega_2} |\nabla u|^{m(x)}\,\mathrm{d}x \\ &\ge \int_{\Omega_1} |\nabla u|^{m^+}\,\mathrm{d}x + \int_{\Omega_2} |\nabla u|^{m^-}\,\mathrm{d}x, \end{aligned} \tag{7.1}$$

where $\Omega_1 = \{x \in \Omega : |\nabla u| \le 1\}$ and $\Omega_2 = \{x \in \Omega : |\nabla u| > 1\}$. By virtue of Hölder inequality and Young inequality, one has

$$\int_{\Omega_1} |\nabla u|^{m^+}\,\mathrm{d}x \ge \frac{m^+}{2}\int_{\Omega_1} |\nabla u|^2\,\mathrm{d}x - \frac{m^+-2}{2}|\Omega_1| \ge \frac{m^-}{2}\int_{\Omega_1} |\nabla u|^2\,\mathrm{d}x - \frac{m^+-2}{2}|\Omega_1|, \tag{7.2}$$

and

$$\int_{\Omega_2} |\nabla u|^{m^-}\,\mathrm{d}x \ge \frac{m^-}{2}\int_{\Omega_2} |\nabla u|^2\,\mathrm{d}x - \frac{m^--2}{2}|\Omega_2| \ge \frac{m^-}{2}\int_{\Omega_2} |\nabla u|^2\,\mathrm{d}x - \frac{m^+-2}{2}|\Omega_2|. \tag{7.3}$$

Combining (7.1)–(7.3) we get

$$\int_\Omega |\nabla u|^{m(x)}\,\mathrm{d}x \ge \frac{m^-}{2}\|\nabla u\|_2^2 - \frac{m^+-2}{2}|\Omega|. \tag{7.4}$$

On the other hand, it follows from Poincaré inequality that $\|\nabla u\|_2^2 \ge \lambda_1 \|u\|_2^2$. Thus, one has

$$\|\nabla u\|_2^2 \ge \frac{\lambda_1}{1+\lambda_1}\left(\|u\|_2^2 + \|\nabla u\|_2^2\right) = \frac{\lambda_1}{1+\lambda_1}\|u\|_{H_0^1}^2.$$

This, together with (2.2) and (7.4), implies that

$$J(u) \ge \left(\frac{1}{m^+} - \frac{1}{p^-}\right)\left(\frac{m^-\lambda_1}{2(1+\lambda_1)}\|u\|_{H_0^1}^2 - \frac{m^+-2}{2}|\Omega|\right) + \frac{1}{p^-}I(u). \tag{7.5}$$

Replacing u by u_0 in (7.5) and using (3.5), we obtain

$$\begin{aligned} J(u_0) &\ge \left(\frac{1}{m^+} - \frac{1}{p^-}\right)\left(\frac{m^-\lambda_1}{2(1+\lambda_1)}\|u_0\|_{H_0^1}^2 - \frac{m^+-2}{2}|\Omega|\right) + \frac{1}{p^-}I(u_0) \\ &> J(u_0) + \frac{1}{p^-}I(u_0), \end{aligned}$$

which gives $I(u_0) < 0$, i.e.,

$$u_0 \in \mathcal{N}_-. \tag{7.6}$$

For any $u \in \mathcal{N}_{J(u_0)}$, we have $I(u) = 0$ and $J(u) \le J(u_0)$. Then by using (7.5), we obtain

$$\|u\|_{H_0^1}^2 \le \frac{2(1+\lambda_1)}{m^-\lambda_1}\left(\frac{m^+p^-}{p^- - m^+}J(u_0) + \frac{m^+ - 2}{2}|\Omega|\right),$$

which, together with (3.5), implies $\|u\|_{H_0^1} \le \|u_0\|_{H_0^1}$. Taking supremum over $u \in \mathcal{N}_{J(u_0)}$, we obtain $\Lambda_{J(u_0)} \le \|u_0\|_{H_0^1}$. Then by Lemma 7.1, it follows from this and (7.6) that $u_0 \in \mathcal{N}_- \cap \mathcal{B}$. The proof is complete.

Acknowledgments

The second author (Le Cong Nhan) is supported by Ho Chi Minh City University of Technology and Education (HCMUTE), Vietnam. The third author (Le Xuan Truong) is supported by University of Economics Ho Chi Minh City (UEH), Vietnam.

References

[1] E. Acerbi and G. Mingione, *Regularity results for stationary electro-rheological fluids*, Arch. Ration. Mech. Anal. **164** (2002), no. 3, 213–259.

[2] S. N. Adimurthi and G. Vallet, *On the equation of Barenblatt–Sobolev*, Commun. Contemp. Math. **13** (2011), no. 5, 843–862.

[3] A. B. Al'shin, M. O. Korpusov and A. G. Sveshnikov, *Blow-up in nonlinear Sobolev type equations*, in: *De Gruyter Series in Nonlinear Analysis and Applications*, **15**, Walter de Gruyter, Berlin, 2011.

[4] S. Antontsev and S. Shmarev, *Anisotropic parabolic equations with variable nonlinearity*, Publ. Mat. **53** (2009), no. 2, 355–399.

[5] G. I. Barenblatt, I. P. Zheltov and I. N. Kochiva, *Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks*, J. Appl. Math. Mech. **24** (1960), no. 5, 1286–1303.

[6] T. B. Benjamin, J. L. Bona and J. J. Mahony, *Model equations for long waves in nonlinear dispersive systems*, Philos. Trans. Roy. Soc. London Ser. A **272** (1972), no. 1220, 47–78.

[7] Y. Cao, J. Yin and C. Wang, *Cauchy problems of semilinear pseudo-parabolic equations*, J. Differential Equations **246** (2009), no. 12, 4568–4590.

[8] Y. Chen, S. Levine and M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math. **66** (2006), no. 4, 1383–1406.

[9] H. Di, Y. Shang and X. Peng, *Blow-up phenomena for a pseudo-parabolic equation with variable exponents*, Appl. Math. Lett. **64** (2017), 67–73.

[10] L. Diening, F. Ettwein and M. Růžička, $C^{1,\alpha}$*-regularity for electrorheological fluids in two dimensions*, NoDEA Nonlinear Differential Equations Appl. **14** (2007), no. 1-2, 207–217.

[11] L. Diening, P. Harjulehto, P. Hästö and M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics **2017**, Springer, Heidelberg, 2011.

[12] L. Diening, P. Nägele and M. Růžička, *Monotone operator theory for unsteady problems in variable exponent spaces*, Complex Var. Elliptic Equ. **57** (2012), no. 11, 1209–1231.

[13] F. Ettwein and M. Růžička, *Existence of local strong solutions for motions of electrorheological fluids in three dimensions*, Comput. Math. Appl. **53** (2007), no. 3-4, 595–604.

[14] X. Fan, J. Shen and D. Zhao, *Sobolev embedding theorems for spaces* $W^{k,p(x)}(\Omega)$, J. Math. Anal. Appl. **262** (2001), no. 2, 749–760.

[15] E. I. Kaĭkina, P. I. Naumkin and I. A. Shishmarëv, *The Cauchy problem for a Sobolev-type equation with a power nonlinearity*, Izv. Math. **69** (2005), no. 1, 59–111.

[16] M. O. Korpusov and A. G. Sveshnikov, *Three-dimensional nonlinear evolution equations of pseudoparabolic type in problems of mathematical physics*, Zh. Vychisl. Mat. Mat. Fiz. **43** (2003), no. 12, 1835–1869.

[17] ———, *Three-dimensional nonlinear evolution equations of pseudoparabolic type in problems of mathematical physics II*, Zh. Vychisl. Mat. Mat. Fiz. **44** (2004), no. 11, 2041–2048.

[18] ———, *On the "blow-up" of the solution of a strongly nonlinear equation of pseudoparabolic type with a double nonlinearity*, Math. Notes. **79** (2006), no. 5-6, 820–840.

[19] H. A. Levine, *Some nonexistence and instability theorems for solutions of formally parabolic equations of the form* $Pu_t = -Au + \mathscr{F}(u)$, Arch. Rational Mech. Anal. **51** (1973), 371–386.

[20] M. Liao, *Non-global existence of solutions to pseudo-parabolic equations with variable exponents and positive initial energy*, Comptes Rendus Mécanique **347** (2019), no. 10, 710–715.

[21] M. Liao, B. Guo and Q. Li, *Global existence and energy decay estimates for weak solutions to the pseudo-parabolic equation with variable exponents*, Math. Methods Appl. Sci. **43** (2020), no. 5, 2516–2527.

[22] M. Liao, Q. Liu and H. Ye, *Global existence and blow-up of weak solutions for a class of fractional p-Laplacian evolution equations*, Adv. Nonlinear Anal. **9** (2020), no. 1, 1569–1591.

[23] J.-L. Lions, *Quelques méthodes de résolution des Problèmes aux limites non linéaires*, Dunod, ParisGauthier-Villars, Paris, 1969.

[24] P. Luo, *Blow-up phenomena for a pseudo-parabolic equation*, Math. Methods Appl. Sci. **38** (2015), no. 12, 2636–2641.

[25] V. Padrón, *Effect of aggregation on population recovery modeled by a forward-backward pseudoparabolic equation*, Trans. Amer. Math. Soc. **356** (2004), no. 7, 2739–2756.

[26] L. E. Payne and D. H. Sattinger, *Saddle points and instability of nonlinear hyperbolic equations*, Israel J. Math. **22** (1975), no. 3-4, 273–303.

[27] M. Ptashnyk, *Degenerate quaslinear pseudoparabolic equations with memory terms and variational inequalities*, Nonlinear Anal. **66** (2007), no. 12, 2653–2675.

[28] Q. Qi, Y. Chen and Q. Wang, *Blow-up phenomena for a pseudo-parabolic system with variable exponents*, Electron. J. Qual. Theory Differ. Equ. (2017), Paper No. 36, 9 pp.

[29] C. Qu and W. Zhou, *Asymptotic analysis for a pseudo-parabolic equation with non-standard growth conditions*, Appl. Anal. **101** (2022), no. 13, 4701–4720.

[30] K. Rajagopal and M. Růžička, *Mathematical modeling of electrorheological materials*, Continuum Mech. Thermodyn. **13** (2001), 59–78.

[31] M. Růžička, *Electrorheological Fluids: Modeling and mathematical theory*, Lecture Notes in Mathematics **1748**, Springer-Verlag, Berlin, 2000.

[32] D. H. Sattinger, *On global solution of nonlinear hyperbolic equations*, Arch. Rational Mech. Anal. **30** (1968), 148–172.

[33] R. E. Showalter, *A nonlinear parabolic-Sobolev equation*, J. Math. Anal. Appl. **50** (1975), 183–190.

[34] R. E. Showalter and T. W. Ting, *Pseudoparabolic partial differential equations*, SIAM J. Math. Anal. **1** (1970), 1–26.

[35] T. W. Ting, *Parabolic and pseudo-parabolic partial differential equations*, J. Math. Soc. Japan **21** (1969), 440–453.

[36] ———, *A cooling process according to two-temperature theory of heat conduction*, J. Math. Anal. Appl. **45** (1974), 23–31.

[37] R. Xu and J. Su, *Global existence and finite time blow-up for a class of semilinear pseudo-parabolic equations*, J. Funct. Anal. **264** (2013), no. 12, 2732–2763.

[38] X. Zhu, B. Guo and M. Liao, *Global existence and blow-up of weak solutions for a pseudo-parabolic equation with high initial energy*, Appl. Math. Lett. **104** (2020), 106270, 7 pp.

[39] X. Zhu, F. Li and Y. Li, *Some sharp results about the global existence and blowup of solutions to a class of pseudo-parabolic equations*, Proc. Roy. Soc. Edinburgh Sect. A **147** (2017), no. 6, 1311–1331.

Quach Van Chuong
Faculty of Mathematics and Computer Science, University of Science, Ho Chi Minh City, Vietnam
Vietnam National University, Ho Chi Minh City, Vietnam
and
Department of Mathematics, Dong Nai University, Bien Hoa City, Dong Nai Province, Vietnam
E-mail address: `chuongqv@dnpu.edu.vn`

Le Cong Nhan
Faculty of Applied Sciences, Ho Chi Minh City University of Technology and Education, Ho Chi Minh City, Vietnam
E-mail address: `nhanlc@hcmute.edu.vn`

Le Xuan Truong
Department of Mathematics, Univerisity of Economics Ho Chi Minh City (UEH), Ho Chi Minh City, Vietnam
E-mail address: `lxuantruong@ueh.edu.vn`