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Time-asymptotic Dynamics of Hermitian Riccati Differential Equations

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Abstract. The matrix Riccati differential equation (RDE) raises in a wide variety of applications for science and applied mathematics. We are particularly interested in the Hermitian Riccati Differential Equation (HRDE). Radon's lemma gives a solution representation to HRDE. Although solutions of HRDE may show the finite escape time phenomenon, we can investigate the time asymptotic dynamical behavior of HRDE by its extended solutions. In this paper, we adapt the Hamiltonian Jordan canonical form to characterize the time asymptotic phenomena of the extended solutions for HRDE in four elementary cases. The extended solutions of HRDE exhibit the dynamics of heteroclinic, homoclinic and periodic orbits in the elementary cases under some conditions.

1. Introduction

The matrix Riccati differential equation (RDE) is the quadratic differential equation

$$\dot{W} = M_{21}(t) + M_{22}(t)W - WM_{11}(t) - WM_{12}(t)W$$

where $W(t)$ and $M_{11}(t)$, $M_{12}(t)$, $M_{21}(t)$, $M_{22}(t)$ are matrices of dimensions $m \times n$, $n \times n$, $n \times m$, $m \times n$ and $m \times m$, respectively. The RDE plays an important role in a wide variety of applications for science and applied mathematics. We are particularly interested in the Hermitian Riccati Differential Equation (HRDE) which arises in optimal controls [5,12–14] and in two-point boundary value problems [2, 3, 6, 7]. It has the form

$$\dot{W} = -WSW + WA + A^H W + D \quad \text{with } W(0) = W_0 \tag{1.1}$$

where A, S and D are $n \times n$ complex-valued constant matrices with $S^H = S$ and $D^H = D$. There is an important relationship between a linear system of differential equations and HRDE. We can use it to obtain a solution representation formula for HRDE explicitly. This relation has been known at least since the work of Radon [16, 17]. Suppose that the solution $W(t)$ of HRDE (1.1) exists for $t \in (\widehat{t}_0, \widehat{t}_1)$, $0 \in (\widehat{t}_0, \widehat{t}_1)$ and $W(0) = W_0$. We first introduce the following important properties.

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Theorem 1.1 (Radon's lemma in [1]). *Let $A, S, D \in \mathbb{C}^{n\times n}$ with $S^H = S$ and $D^H = D$, then the following statements hold.*

(i) *Let $W(t)$ be a solution of HRDE (1.1) in $(\widehat{t}_0, \widehat{t}_1)$. If $Q(t)$ is a solution of the IVP $\dot{Q}(t) = (SW(t) - A)Q(t)$ with initial value $Q(0) = I_n$ and $P(t) := W(t)Q(t)$, then $Y(t) \equiv [Q(t)^\top, P(t)^\top]^\top$ is the solution of the linear IVP*

$$\dot{Y}(t) = \widetilde{\mathcal{H}}Y(t), \quad Y(0) = [I_n, W_0^\top]^\top, \tag{1.2}$$

where

$$\widetilde{\mathcal{H}} = \begin{bmatrix} -A & S \\ D & A^H \end{bmatrix} \tag{1.3}$$

is a $2n \times 2n$ complex Hamiltonian matrix.

(ii) *Let $Y(t) \equiv [Q(t)^\top, P(t)^\top]^\top$ be the solution of (1.2). If $Q(t)$ is invertible for $t \in (\widehat{t}_0, \widehat{t}_1) \subset \mathbb{R}$, then*

$$W(t) \equiv P(t)Q(t)^{-1} \tag{1.4}$$

is a solution of HRDE (1.1).

It is clear from (1.4) that the nonsingularity of $Q(t)$ determines whether the solution $W(t)$ of HRDE exists. Define the set

$$\mathcal{J}_w \equiv \{t \in \mathbb{R} \mid Q(t) \text{ is invertible}\}.$$

Since

$$Q(t) = [I_n, 0_{n\times n}] \begin{bmatrix} Q(t) \\ P(t) \end{bmatrix} = [I_n, 0_{n\times n}] e^{\widetilde{\mathcal{H}}t} \begin{bmatrix} I_n \\ W_0 \end{bmatrix},$$

the function $\det(Q(t))$ is analytic and not the zero function. This implies all zeros of $\det(Q(t))$ are isolated. It follows $\mathcal{J}_w$ is the set that $\mathbb{R}$ subtracts some isolated points. $\mathcal{J}_w$ can be written as a union of open intervals, say

$$\mathcal{J}_w = \bigcup_{k\in\mathbb{Z}} (\widehat{t}_k, \widehat{t}_{k+1}),$$

which is an unbounded set in $\mathbb{R}$. Since $P(t)$ and $Q(t)$ are analytic functions on $\mathbb{R}$, $W(t) = P(t)Q(t)^{-1}$ is a meromorphic function. The singularities of $W(t)$ are poles. This means that the solutions $W(t)$ of HRDE (1.1) may show the finite escape time phenomenon, i.e., the solutions may blow up on a finite interval. However, it is noted that the solution representation for $W(t)$ in (1.4) holds not only for $t \in (\widehat{t}_0, \widehat{t}_1)$ but also for $t \in \mathcal{J}_w$. Hence, we can define the extended solution of HRDE (1.1) by

$$W(t) = P(t)Q(t)^{-1}, \quad t \in \mathcal{J}_w. \tag{1.5}$$

Moreover, Radon's lemma also leads to a geometric version which gives connection between the solution for HRDE (1.1) and the flow defined on the Grassmann manifold. The embedding of trajectories of HRDE into trajectories of a flow on the Grassmann manifold is stated in Appendix A. This flow on the Grassmann manifold is analytic and exists for all $t \in \mathbb{R}$. Therefore, the investigation of time asymptotic phenomena for the extended solutions (1.5) of HRDE (1.1) is meaningful.

The main results (main theorem) of this paper is to give a characterization of dynamical behavior for the extended solutions of HRDE (1.1) and to study the time asymptotic estimates. The time asymptotic behavior of the solutions for RDE was also studied in [4,8,9,18] and the literature cited therein. The matrix Riccati equations is closely related, via compactification of the phase space, to the differential equations on the Grassmann manifold and the Lagrange-Grassmann manifold. In [18], the author characterized the nonwandering set and its stable/unstable manifolds of the extended Riccati differential equations. Rather than the topological structure of the invariant sets and stabilities of HRDE, we shall focus on the convergent rates of the solutions. In this paper, the explicit representations of the solutions $W(t) = P(t)Q^{-1}(t)$ are obtained by using matrix analysis, and then the time convergent rates to the asymptotic solutions can be estimated.

Based on the special structure, a canonical form of a Hamiltonian matrix under symplectic similarity transformations has been widely studied in [15]. Instead of the Jordan canonical form, we shall use the Hamiltonian Jordan canonical form $\mathfrak{J}$ of $\widetilde{\mathcal{H}}$ to investigate the time asymptotic behavior for the extended solution of HRDE (1.1). The structure of $e^{\mathfrak{J}t}$ for the general cases are complicated. Therefore, four elementary cases are studied in this paper. All general cases can be generated by using direct sums of these four elementary cases. However, combinations of some of the elementary cases need more sophisticated analysis and is the future work. We obtain the following results:

1. If $\widetilde{\mathcal{H}}$ has only eigenvalues λ and $-\overline{\lambda}$ with nonzero real part and each of the two eigenvalues has only one Jordan block, the trajectory of the extended solution $W(t)$ for HRDE (1.1) is a hetroclinic orbit;

2. If $\widetilde{\mathcal{H}}$ has one pure imaginary eigenvalue which has one Jordan block of size $2n$, then the trajectory of the extended solution $W(t)$ is a homoclinic orbit;

3. If $\widetilde{\mathcal{H}}$ has one pure imaginary eigenvalue but it has two Jordan blocks with multiplicities $2n_1 + 1$ and $2n_2 + 1$ $(n_1 + n_2 + 1 = n)$, respectively, then the trajectory of the extended solution $W(t)$ also forms a homoclinic orbit;

4. If $\widetilde{\mathcal{H}}$ has two distinct pure imaginary eigenvalues $i\delta$ and $i\gamma$ ($\delta \neq \gamma$) with partial multiplicities $2n_1 + 1$ and $2n_2 + 1$, respectively, then the extended solution $W(t)$ converges to a limit cycle which is a periodic solution with period $2\pi/(\delta - \gamma)$ and its convergent rate to the limit cycle is $\mathcal{O}(t^{-1})$.

The paper is organized as follows. In Section 2, the main theorem is presented. In Section 3.1, there are preliminaries for the proof of the main theorem. We prove the main theorem in Section 3.2. A simple combination of elementary cases is given in Section 4. It is also noticed that the notations used in this paper basically follow the rules:

- Capital letters denote matrices;
- Lowercase letters denote vectors or scalars;
- Greek letters are used for auxiliary variables;
- Hats are used for variables transformed by a matrix P_k;
- Tildes denote matrices that have been extended in some way.

2. Main theorem

A canonical form of a Hamiltonian matrix under symplectic similarity transformations has been investigated in [15]. Let S be the symplectic matrix such that

$$\mathfrak{J} = S^{-1}\widetilde{\mathcal{H}}S \tag{2.1}$$

where $\mathfrak{J}$ is the Hamiltonian Jordan canonical form of the Hamiltonian matrix $\widetilde{\mathcal{H}}$ in (1.3). Let $N_k \equiv \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$ be the $k \times k$ nilpotent matrix, let $N_k(\lambda) = \lambda I_k + N_k$ be the Jordan block of the eigenvalue λ with size k, and let e_n denote the n-th unit vector.

Throughout this paper, we assume that the Hamiltonian Jordan canonical form

$$\mathfrak{J} = \left[\begin{array}{c|c} R & D \\ \hline G & -R^H \end{array}\right] \in \mathbb{C}^{2n \times 2n} \tag{2.2}$$

for $\widetilde{\mathcal{H}}$ is one of the following four elementary cases:

Case 1: $R = N_n(\lambda)$, $D = G = 0$ and $\mathrm{Re}(\lambda) > 0$;

Case 2: $R = N_n(i\alpha)$, $D = \beta e_n e_n^H$, $G = 0$ and $\alpha \in \mathbb{R}$, $\beta \in \{-1, 1\}$;

Case 3: $n = n_1 + n_2 + 1$, $\eta \in \mathbb{R}$, $\beta \in \{-1, 1\}$, $G = 0$ and

$$R = \begin{bmatrix} N_{n_1}(i\eta) & 0 & -\frac{\sqrt{2}}{2}e_{n_1} \\ 0 & N_{n_2}(i\eta) & -\frac{\sqrt{2}}{2}e_{n_2} \\ 0 & 0 & i\eta \end{bmatrix}, \quad D = \frac{\sqrt{2}}{2}i\beta \begin{bmatrix} 0 & 0 & e_{n_1} \\ 0 & 0 & -e_{n_2} \\ -e_{n_1}^H & e_{n_2}^H & 0 \end{bmatrix};$$

Case 4: $n = n_1 + n_2 + 1$, $\beta \in \{-1, 1\}$, $\gamma, \delta \in \mathbb{R}$ with $\gamma \neq \delta$ and

$$R = \begin{bmatrix} N_{n_1}(i\gamma) & 0 & -\frac{\sqrt{2}}{2}e_{n_1} \\ 0 & N_{n_2}(i\delta) & -\frac{\sqrt{2}}{2}e_{n_2} \\ 0 & 0 & \frac{i}{2}(\gamma + \delta) \end{bmatrix}, \quad G = \beta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}(\gamma - \delta) \end{bmatrix},$$

$$D = \frac{\sqrt{2}}{2}i\beta \begin{bmatrix} 0 & 0 & e_{n_1} \\ 0 & 0 & -e_{n_2} \\ -e_{n_1}^H & e_{n_2}^H & -i\frac{\sqrt{2}}{2}(\gamma - \delta) \end{bmatrix}.$$

Due to (2.1), the solution for IVP (1.2) is

$$\begin{bmatrix} Q(t) \\ P(t) \end{bmatrix} = Y(t) = e^{\widetilde{\mathcal{H}}t} \begin{bmatrix} I_n \\ W_0 \end{bmatrix} = Se^{\mathfrak{J}t}S^{-1} \begin{bmatrix} I_n \\ W_0 \end{bmatrix}. \tag{2.3}$$

Now, we are ready to state our main results.

Theorem 2.1 (Main theorem)**.** *Suppose that $\widetilde{\mathcal{H}}$ is symplectically similar to one of the four Hamiltonian Jordan canonical forms $\mathfrak{J}$ mentioned above. Let $Y(t) = [Q(t)^\top, P(t)^\top]^\top$ and $W(t) = P(t)Q(t)^{-1}$ for $t \in \mathcal{J}_w$ be the solution of IVP* (1.2) *and be the extended solution of HRDE* (1.1)*, respectively. Define*

$$[W_1^\top, W_2^\top]^\top \equiv S^{-1}[I_n, W_0^\top]^\top. \tag{2.4}$$

Then the following assertions hold.

(i) *Suppose that the symplectic matrix S in* (2.1) *is partitioned as*

$$S = \left[\begin{array}{c|c} U_1 & V_1 \\ \hline U_2 & V_2 \end{array}\right], \tag{2.5}$$

where $U_1, U_2, V_1, V_2 \in \mathbb{C}^{n\times n}$.

Case 1*: Assume* $\mathrm{Re}(\lambda) > 0$ *and W_1, U_1 are nonsingular. Then*

$$W(t) = U_2U_1^{-1} + \mathcal{O}(e^{-2\,\mathrm{Re}(\lambda)t}t^{2(n-1)})$$

as $t \to \infty$. On the other hand, if W_2 and V_1 are nonsingular, then

$$W(t) = V_2 V_1^{-1} + \mathcal{O}(e^{2\operatorname{Re}(\lambda)t} t^{2(n-1)})$$

as $t \to -\infty$. Therefore, the trajectory of the extended solution $W(t)$ is a hetroclinic orbit that starts from the equilibrium $V_2V_1^{-1}$ to $U_2U_1^{-1}$ which are hermitian.

Case 2: Assume W_2 and U_1 are nonsingular. Then

$$W(t) = U_2 U_1^{-1} + \mathcal{O}(t^{-1})$$

as $t \to \pm\infty$. In this case, the trajectory of the solution $W(t)$ is a homoclinic orbit.

(ii) *Suppose that the symplectic matrix S in (2.1) is further partitioned as*

$$S = \left[\begin{array}{cc|cc} U_1 & u_1 & V_1 & v_1 \\ \hline U_2 & u_2 & V_2 & v_2 \end{array}\right] \in \mathbb{C}^{2n\times 2n}, \tag{2.6}$$

where $U_1, U_2, V_1, V_2 \in \mathbb{C}^{n\times(n_1+n_2)}$, $u_1, u_2, v_1, v_2 \in \mathbb{C}^n$ and $n = n_1 + n_2 + 1$.

Case 3: Assume W_2 is nonsingular. Then there are constants $f_u, f_v, g_u, g_v \in \mathbb{C}$ with $\widetilde{\mathbb{U}}_i = [U_i \mid f_u u_i + f_v v_i] \in \mathbb{C}^{n\times n}$ and $\zeta_i = g_u u_i + g_v v_i \in \mathbb{C}^n$, $i = 1, 2$ such that

$$W(t) = \left(\widetilde{\mathbb{U}}_2 + \frac{(\zeta_2 - \widetilde{\mathbb{U}}_2\widetilde{\mathbb{U}}_1^{-1}\zeta_1)e_n^H}{1 + e_n^H\widetilde{\mathbb{U}}_1^{-1}\zeta_1}\right)\widetilde{\mathbb{U}}_1^{-1} + \mathcal{O}(t^{-1})$$

as $t \to \pm\infty$, whenever $\widetilde{\mathbb{U}}_1$ is nonsingular and $1 + e_n^H\widetilde{\mathbb{U}}_1^{-1}\zeta_1 \neq 0$. The trajectory of the solution $W(t)$ is a homoclinic orbit.

Case 4: Assume W_2 is nonsingular. Then there are constants $f_u, f_v, g_u, g_v \in \mathbb{C}$ with $\widetilde{\mathbb{U}}_i = [U_i \mid f_u u_i + f_v v_i] \in \mathbb{C}^{n\times n}$ and $\zeta_i = g_u u_i + g_v v_i \in \mathbb{C}^n$, $i = 1, 2$ such that

$$W(t) = \widetilde{\mathbb{U}}_2\widetilde{\mathbb{U}}_1^{-1} + \frac{e^{i\theta t}}{1 + e^{i\theta t}e_n^H\widetilde{\mathbb{U}}_1^{-1}\zeta_1}(\zeta_2 - \widetilde{\mathbb{U}}_2\widetilde{\mathbb{U}}_1^{-1}\zeta_1)e_n^H\widetilde{\mathbb{U}}_1^{-1} + \mathcal{O}(t^{-1})$$

as $t \to \pm\infty$, provided that $\widetilde{\mathbb{U}}_1$ is nonsingular and $1 + e_n^H\widetilde{\mathbb{U}}_1^{-1}\zeta_1 \neq 0$. Here $\theta = \delta - \gamma$. In this case, the trajectory of the extended solution $W(t)$ approaches the limit cycle

$$W^\infty(t) = \widetilde{\mathbb{U}}_2\widetilde{\mathbb{U}}_1^{-1} + \frac{e^{i\theta t}}{1 + e^{i\theta t}e_n^H\widetilde{\mathbb{U}}_1^{-1}\zeta_1}(\zeta_2 - \widetilde{\mathbb{U}}_2\widetilde{\mathbb{U}}_1^{-1}\zeta_1)e_n^H\widetilde{\mathbb{U}}_1^{-1}$$

with period $2\pi/\theta$.

3. Proof of main theorem

3.1. Preliminaries

We first introduce some notations. For $1 \leq i, j \leq k$,

$$(P_k)_{ij} = \begin{cases} (-1)^i & \text{for } j = k+1-i, \\ 0 & \text{otherwise,} \end{cases}$$

$$\Phi_k(t) \triangleq e^{N_k t} = \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{k-1}}{(k-1)!} \\ & 1 & t & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2!} \\ & & & 1 & t \\ & & & & 1 \end{bmatrix}, \quad \phi_k(t) \triangleq \begin{bmatrix} \frac{t^k}{k!} \\ \vdots \\ \frac{t^2}{2!} \\ t \end{bmatrix}, \quad \psi_k(t) \triangleq \begin{bmatrix} t \\ \frac{t^2}{2!} \\ \vdots \\ \frac{t^k}{k!} \end{bmatrix}, \tag{3.1}$$

$$\Gamma_{k_1}^{k_2}(t) \triangleq \begin{bmatrix} \frac{t^{k_1}}{k_1!} & \frac{t^{k_1+1}}{(k_1+1)!} & \cdots & \frac{t^{k_2}}{k_2!} \\ \frac{t^{k_1-1}}{(k_1-1)!} & \frac{t^{k_1}}{k_1!} & \cdots & \frac{t^{k_2-1}}{(k_2-1)!} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{t^{2k_1-k_2}}{(2k_1-k_2)!} & \cdots & \cdots & \frac{t^{k_1}}{k_1!} \end{bmatrix} \quad \text{with } k_1 < k_2 \leq 2k_1,$$

$$\widehat{\Phi}_k(t) \triangleq P_k^{-1}\Phi_k(t)P_k, \quad \widehat{\Gamma}_k^{2k-1}(t) \triangleq \Gamma_k^{2k-1}(t)P_k, \quad \widehat{\psi}_k^H(t) \triangleq \psi_k^H(t)P_k.$$

It is also noticed that the time-dependency of the above notations is omitted where no confusion can arise.

Lemma 3.1. *We have*

(i) $e^{N_k(\lambda)t} = e^{\lambda t}e^{N_k t} = e^{\lambda t}\Phi_k$;

(ii) $P_k^{-1} = P_k^H$, $P_k^{-1}N_kP_k = -N_k^H$;

(iii) $\widehat{\Phi}_k = P_k^{-1}\Phi_kP_k = e^{-N_k^H t} = \Phi_k^{-H}$.

Proof. The proof is straightforward by direct calculation. □

The following lemmas show the expression for e^{At} when A is one of the elementary cases.

Lemma 3.2. *Let A denote the Hamiltonian matrix* $\left[\begin{array}{c|c} N_k(i\alpha) & \beta e_k e_k^H \\ \hline 0 & -N_k(i\alpha)^H \end{array}\right] \in \mathbb{C}^{2k\times 2k}$, *where $\beta \in \{-1, 1\}$ and $\alpha \in \mathbb{R}$. Then for every $t \in \mathbb{R}$, e^{At} is of the form*

$$e^{At} = e^{i\alpha t}\left[\begin{array}{c|c} \Phi_k & -\beta\widehat{\Gamma}_k^{2k-1} \\ \hline 0 & \Phi_k^{-H} \end{array}\right].$$

Proof. Let $\Theta = I_k \oplus (-\beta P_k)$ where $\oplus$ means the direct sum. We have $N_{2k}(i\alpha) = \Theta A \Theta^{-1}$. Then by Lemma 3.1, we have

$$
\begin{aligned}
e^{At} &= e^{i\alpha t}\Theta^{-1}e^{N_{2k}t}\Theta \\
&= e^{i\alpha t}\left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & -\beta P_k^{-1}\end{array}\right]
\left[\begin{array}{cccc|cccc}
1 & t & \cdots & \frac{t^{k-1}}{(k-1)!} & \frac{t^k}{k!} & \cdots & \cdots & \frac{t^{2k-1}}{(2k-1)!} \\
 & 1 & \ddots & \vdots & \frac{t^{k-1}}{(k-1)!} & \frac{t^k}{k!} & \ddots & \frac{t^{2k-2}}{(2k-2)!} \\
 & & \ddots & t & \vdots & \ddots & \ddots & \vdots \\
 & & & 1 & t & \cdots & \cdots & \frac{t^k}{k!} \\ \hline
 & & & & 1 & t & \cdots & \frac{t^{k-1}}{(k-1)!} \\
 & & & & & 1 & \ddots & \vdots \\
 & & & & & & \ddots & t \\
 & & & & & & & 1
\end{array}\right]
\left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & -\beta P_k\end{array}\right] \\
&= e^{i\alpha t}\left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & -\beta P_k^{-1}\end{array}\right]
\left[\begin{array}{c|c} \Phi_k & \Gamma_k^{2k-1} \\ \hline 0 & \Phi_k\end{array}\right]
\left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & -\beta P_k\end{array}\right] \\
&= e^{i\alpha t}\left[\begin{array}{c|c} \Phi_k & -\beta\widehat{\Gamma}_k^{2k-1} \\ \hline 0 & \Phi_k^{-H}\end{array}\right].
\end{aligned}
$$

$\square$

For simplicity of expression for e^{At} in the next lemma, we introduce the following notations

$$
\begin{aligned}
\Phi_{m,n}(t) &\triangleq e^{i\gamma t}\Phi_m \oplus e^{i\delta t}\Phi_n, \\
\widehat{\Phi}_{m,n}(t) &\triangleq e^{i\gamma t}\Phi_m^{-H} \oplus e^{i\delta t}\Phi_n^{-H} = e^{i\gamma t}P_m^{-1}\Phi_m P_m \oplus e^{i\delta t}P_n^{-1}\Phi_n P_n, \\
\phi^1_{m,n}(t) &\triangleq -\frac{\sqrt{2}}{2}\begin{bmatrix} e^{i\gamma t}\phi_m \\ e^{i\delta t}\phi_n\end{bmatrix}, \quad \phi^2_{m,n}(t) \triangleq \frac{\sqrt{2}}{2}i\beta\begin{bmatrix} e^{i\gamma t}\phi_m \\ -e^{i\delta t}\phi_n\end{bmatrix}, \\
\widehat{\psi^1}^H_{m,n}(t) &\triangleq \frac{\sqrt{2}}{2}i\beta\begin{bmatrix} e^{i\gamma t}\widehat{\psi}_m^H & -e^{i\delta t}\widehat{\psi}_n^H\end{bmatrix} = \frac{\sqrt{2}}{2}i\beta\begin{bmatrix} e^{i\gamma t}\psi_m^H P_m & -e^{i\delta t}\psi_n^H P_n\end{bmatrix}, \\
\widehat{\psi^2}^H_{m,n}(t) &\triangleq -\frac{\sqrt{2}}{2}\begin{bmatrix} e^{i\gamma t}\widehat{\psi}_m^H & e^{i\delta t}\widehat{\psi}_n^H\end{bmatrix} = -\frac{\sqrt{2}}{2}\begin{bmatrix} e^{i\gamma t}\psi_m^H P_m & e^{i\delta t}\psi_n^H P_n\end{bmatrix}, \\
\widehat{\Gamma}^{2m,2n}_{m+1,n+1}(t) &\triangleq i\beta\big[-e^{i\gamma t}\widehat{\Gamma}^{2m}_{m+1} \oplus e^{i\delta t}\widehat{\Gamma}^{2n}_{n+1}\big] = i\beta\big[-e^{i\gamma t}\Gamma^{2m}_{m+1}P_m \oplus e^{i\delta t}\Gamma^{2n}_{n+1}P_n\big], \\
\begin{bmatrix}\omega_{11}(t) & \omega_{12}(t) \\ \omega_{21}(t) & \omega_{22}(t)\end{bmatrix} &\triangleq \frac{1}{2}\begin{bmatrix} e^{i\gamma t}+e^{i\delta t} & -i\beta(e^{i\gamma t}-e^{i\delta t}) \\ i\beta(e^{i\gamma t}-e^{i\delta t}) & e^{i\gamma t}+e^{i\delta t}\end{bmatrix}.
\end{aligned} \tag{3.2}
$$

Similarly, the time variable of the notations introduced in (3.2) will be omitted wherever it is not necessary to specify it.

Lemma 3.3. *Let A denote the Hamiltonian matrix* $\left[\begin{array}{c|c} R & D \\ \hline G & -R^H \end{array}\right]$, *where*

$$R = \begin{bmatrix} N_m(i\gamma) & 0 & -\frac{\sqrt{2}}{2}e_m \\ 0 & N_n(i\delta) & -\frac{\sqrt{2}}{2}e_n \\ 0 & 0 & \frac{i}{2}(\gamma+\delta) \end{bmatrix}, \quad G = \beta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}(\gamma-\delta) \end{bmatrix},$$

$$D = \frac{\sqrt{2}}{2} i\beta \begin{bmatrix} 0 & 0 & e_m \\ 0 & 0 & -e_n \\ -e_m^H & e_n^H & -i\frac{\sqrt{2}}{2}(\gamma-\delta) \end{bmatrix},$$

$\beta \in \{-1, 1\}$ *and* $\gamma, \delta \in \mathbb{R}$. *Therefore, for every* $t \in \mathbb{R}$, e^{At} *is of the form*

$$e^{At} = \left[\begin{array}{c|c} \mathbf{B} & \mathbf{D} \\ \hline \mathbf{G} & \mathbf{E} \end{array}\right] = \left[\begin{array}{c|c} \begin{bmatrix} \Phi_{m,n} & \phi^1_{m,n} \\ 0 & \omega_{11} \end{bmatrix} & \begin{bmatrix} \widehat{\Gamma}^{2m,2n}_{m+1,n+1} & \phi^2_{m,n} \\ \widehat{\psi^1}^H_{m,n} & \omega_{12} \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 \\ 0 & \omega_{21} \end{bmatrix} & \begin{bmatrix} \widehat{\Phi}_{m,n} & 0 \\ \widehat{\psi^2}^H_{m,n} & \omega_{22} \end{bmatrix} \end{array}\right].$$

Proof. In order to transform A to a block diagonal matrix, we introduce the matrices Θ_1 and Θ_2 with

$$\Theta_1 = \left[\begin{array}{c|c} I_{m+1} \oplus (-i\beta P_m) & 0 \\ \hline 0 & I_{n+1} \oplus (i\beta P_n) \end{array}\right], \quad \Theta_2 = \left[\begin{array}{ccc|ccc} I_m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2}i\beta \\ 0 & 0 & 0 & I_m & 0 & 0 \\ \hline 0 & I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 & -\frac{\sqrt{2}}{2}i\beta \\ 0 & 0 & 0 & 0 & I_n & 0 \end{array}\right].$$

Θ_2 is a combination of a permutation matrix and a 2×2 unitary matrix. The mechanism of Θ_2 is to let A be similar to an upper block matrix. That is

$$\Theta_2 A \Theta_2^{-1} = \left[\begin{array}{ccc|ccc} N_m(i\gamma) & e_m & 0 & 0 & 0 & 0 \\ 0 & i\gamma & i\beta e_m^H & 0 & 0 & 0 \\ 0 & 0 & -N_m^H(i\gamma) & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & N_n(i\delta) & e_n & 0 \\ 0 & 0 & 0 & 0 & i\delta & -i\beta e_n^H \\ 0 & 0 & 0 & 0 & 0 & -N_n^H(i\delta) \end{array}\right].$$

The mechanism of Θ_1 is to transform $-N_m^H(i\gamma)$ and $-N_n^H(i\delta)$ to $N_m(i\gamma)$ and $N_n(i\delta)$, respectively, in $\Theta_2 A\Theta_2^{-1}$. Then we have

$$\Theta_1\Theta_2 A(\Theta_1\Theta_2)^{-1} = \left[\begin{array}{c|c} N_{2m+1}(i\gamma) & 0 \\ \hline 0 & N_{2n+1}(i\delta) \end{array}\right],$$

and

$$\begin{aligned} e^{At} &= (\Theta_1\Theta_2)^{-1}\left[\begin{array}{c|c} e^{N_{2m+1}(i\gamma)t} & 0 \\ \hline 0 & e^{N_{2n+1}(i\delta)t} \end{array}\right]\Theta_1\Theta_2 \\ &= \left[\begin{array}{ccc|ccc} e^{i\gamma t}\Phi_m & 0 & -\frac{\sqrt{2}}{2}e^{i\gamma t}\phi_m & (-i\beta)e^{i\gamma t}\widehat{\Gamma}_{m+1}^{2m} & 0 & \frac{\sqrt{2}}{2}(i\beta)e^{i\gamma t}\phi_m \\ 0 & e^{i\delta t}\Phi_n & -\frac{\sqrt{2}}{2}e^{i\delta t}\phi_n & 0 & (i\beta)e^{i\delta t}\widehat{\Gamma}_{n+1}^{2n} & -\frac{\sqrt{2}}{2}(i\beta)e^{i\delta t}\phi_n \\ 0 & 0 & \frac{1}{2}(e^{i\gamma t}+e^{i\delta t}) & \frac{\sqrt{2}}{2}(i\beta)e^{i\gamma t}\widehat{\psi}_m^H & -\frac{\sqrt{2}}{2}(i\beta)e^{i\delta t}\widehat{\psi}_n^H & -\frac{(i\beta)}{2}(e^{i\gamma t}-e^{i\delta t}) \\ \hline 0 & 0 & 0 & e^{i\gamma t}\Phi_m^{-H} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{i\delta t}\Phi_n^{-H} & 0 \\ 0 & 0 & \frac{(i\beta)}{2}(e^{i\gamma t}-e^{i\delta t}) & -\frac{\sqrt{2}}{2}e^{i\gamma t}\widehat{\psi}_m^H & -\frac{\sqrt{2}}{2}e^{i\delta t}\widehat{\psi}_n^H & \frac{1}{2}(e^{i\gamma t}+e^{i\delta t}) \end{array}\right] \\ &= \left[\begin{array}{c|c} \mathbf{B} & \mathbf{D} \\ \hline \mathbf{G} & \mathbf{E} \end{array}\right]. \end{aligned}$$

□

For integers k, l, k_1 and k_2 that satisfies $0 \le k$, $0 \le l$, $k \ne l$ and $0 < k_1 < k_2 \le 2k_1$, we denote

$$\Xi_{k,l} \equiv \Xi_{k,l}(t) = \begin{cases} \operatorname{diag}(t^k, t^{k+1}, \ldots, t^l) & \text{if } k < l, \\ \operatorname{diag}(t^k, t^{k-1}, \ldots, t^l) & \text{if } k > l, \end{cases}$$

(3.3)
$$\digamma_{k_1}^{k_2} = \begin{bmatrix} \frac{1}{k_1!} & \frac{1}{(k_1+1)!} & \cdots & \frac{1}{k_2!} \\ \frac{1}{(k_1-1)!} & \frac{1}{k_1!} & \ddots & \frac{1}{(k_2-1)!} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{1}{(2k_1-k_2)!} & \cdots & \cdots & \frac{1}{k_1!} \end{bmatrix}.$$

To give a useful expression for the inverse of the matrix $\Gamma_{k_1}^{k_2}$, we prove that the matrix $\digamma_{k_1}^{k_2}$ is invertible in the following lemma.

Lemma 3.4. *Let k_1, k_2 be positive integers that satisfy $0 < k_1 < k_2 \le 2k_1$ and $k = k_2 - k_1$. Then*

$$\det(\digamma_{k_1}^{k_2}) = \frac{k!(k-1)!\cdots 1!}{k_2!(k_2-1)!\cdots k_1!}.$$

Hence, $\digamma_{k_1}^{k_2}$ is invertible.

Proof. To prove this lemma, we need the so-called Pascal's law:

$$\sigma_r^n - \sigma_r^{n-1} = r\sigma_{r-1}^{n-1} \quad \text{for } n, r \in \mathbb{N} \text{ and } n \geq r$$

where $\sigma_r^n = n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}$. Let $D = \mathrm{diag}(k_2!, (k_2-1)!, \ldots, k_1!)$. Set

$$\widetilde{F}_{k_1}^{k_2} \equiv D F_{k_1}^{k_2} = \begin{bmatrix} \sigma_k^{k_2} & \sigma_{k-1}^{k_2} & \cdots & \sigma_0^{k_2} \\ \sigma_k^{k_2-1} & \sigma_{k-1}^{k_2-1} & \cdots & \sigma_0^{k_2-1} \\ \vdots & \vdots & & \vdots \\ \sigma_k^{k_1} & \sigma_{k-1}^{k_1} & \cdots & \sigma_0^{k_1} \end{bmatrix} \in \mathbb{R}^{(k+1)\times(k+1)}.$$

Let e_j denote the jth column vector of the matrix I_{k+1} and $E_j \triangleq I_{k+1} - e_j e_{j+1}^H$. Then by Pascal's law, we have

$$\begin{aligned} E_k \cdots E_2 E_1 \widetilde{F}_{k_1}^{k_2} &= \left[\begin{array}{ccc|c} k\sigma_{k-1}^{k_2-1} & \cdots & 1\sigma_0^{k_2-1} & 0 \\ \vdots & & \vdots & \vdots \\ k\sigma_{k-1}^{k_1} & \cdots & 1\sigma_0^{k_1} & 0 \\ \hline \sigma_k^{k_1} & \cdots & \sigma_1^{k_1} & 1 \end{array}\right] \\ &= \left[\begin{array}{c|c} \widetilde{F}_{k_1}^{k_2-1} & 0 \\ \hline * & 1 \end{array}\right] \mathrm{diag}(k, (k-1), \ldots, 1, 1). \end{aligned}$$

Due to $\det(E_j) = 1$, it is easy to get $\det(\widetilde{F}_{k_1}^{k_2}) = k! \det(\widetilde{F}_{k_1}^{k_2-1})$. Then we can get

$$\det(\widetilde{F}_{k_1}^{k_2}) = k!(k-1)!\cdots 1! \det(\widetilde{F}_{k_1}^{k_1}) = k!(k-1)!\cdots 1!.$$

Hence, we have

$$\det(F_{k_1}^{k_2}) = \frac{\det(\widetilde{F}_{k_1}^{k_2})}{\det(D)} = \frac{k!(k-1)!\cdots 1!}{k_2!(k_2-1)!\cdots k_1!}.$$ □

Then the matrix $\Gamma_{k_1}^{k_2}$ defined in (3.1) and its inverse can be written in terms of $\Xi_{k,l}$ and $F_{k_1}^{k_2}$ in (3.3) as

$$\begin{aligned} \Gamma_{k_1}^{k_2} &= t^{2k_1-k_2} \Xi_{k_2-k_1,0} F_{k_1}^{k_2} \Xi_{0,k_2-k_1}, \\ (\Gamma_{k_1}^{k_2})^{-1} &= t^{-2k_1+k_2} (\Xi_{0,k_2-k_1})^{-1} (F_{k_1}^{k_2})^{-1} (\Xi_{k_2-k_1,0})^{-1} \end{aligned} \tag{3.4}$$

by the invertibility of $F_{k_1}^{k_2}$ in Lemma 3.4. The matrices $(\Xi_{0,k-1})^{-1}$ and $(\Xi_{k-1,0})^{-1}$ shall be used to help eliminate the t powers of Φ_k and ϕ_k as done in the proofs of Lemmas 3.5 and 3.6.

Lemma 3.5. *Let $n \in \mathbb{N}$. Then*

(i) $(\widehat{\Gamma}_n^{2n-1})^{-1}\Phi_n = \mathcal{O}(t^{-1})$, $\widehat{\Phi}_n(\widehat{\Gamma}_n^{2n-1})^{-1} = \mathcal{O}(t^{-1})$ *and* $\widehat{\Phi}_n(\widehat{\Gamma}_n^{2n-1})^{-1}\Phi_n = \mathcal{O}(t^{-1})$;

(ii) $\widehat{\Phi}_n(\Phi_n C \pm \widehat{\Gamma}_n^{2n-1})^{-1} = \mathcal{O}(t^{-1})$

as $t \to \pm\infty$, *where* $C \in \mathbb{C}^{n\times n}$ *is a constant matrix.*

Proof. (i) Using (3.4) and Lemma 3.1(iii), we have

$$\begin{aligned}
(\widehat{\Gamma}_n^{2n-1})^{-1}\Phi_n &= t^{-1}P_n^{-1}(\Xi_{0,n-1})^{-1}(F_n^{2n-1})^{-1}[(\Xi_{n-1,0})^{-1}\Phi_n] = \mathcal{O}(t^{-1}),\\
\widehat{\Phi}_n(\widehat{\Gamma}_n^{2n-1})^{-1} &= (P_n^{-1}\Phi_n P_n)P_n^{-1}(\Gamma_n^{2n-1})^{-1}\\
&= t^{-1}P_n^{-1}[\Phi_n(\Xi_{0,n-1})^{-1}](F_n^{2n-1})^{-1}(\Xi_{n-1,0})^{-1} = \mathcal{O}(t^{-1}),\\
\widehat{\Phi}_n(\widehat{\Gamma}_n^{2n-1})^{-1}\Phi_n &= (P_n^{-1}\Phi_n P_n)P_n^{-1}(\Gamma_n^{2n-1})^{-1}\Phi_n\\
&= t^{-1}P_n^{-1}[\Phi_n(\Xi_{0,n-1})^{-1}](F_n^{2n-1})^{-1}[(\Xi_{n-1,0})^{-1}\Phi_n] = \mathcal{O}(t^{-1})
\end{aligned}$$

due to

$$(\Xi_{n-1,0})^{-1}\Phi_n = \mathcal{O}(1), \quad \Phi_n(\Xi_{0,n-1})^{-1} = \mathcal{O}(1).$$

For the proof of assertion (ii),

$$\begin{aligned}
\widehat{\Phi}_n(\Phi_n C - \widehat{\Gamma}_n^{2n-1})^{-1} &= \widehat{\Phi}_n[\widehat{\Gamma}_n^{2n-1}((\widehat{\Gamma}_n^{2n-1})^{-1}\Phi_n C - I_n)]^{-1}\\
&= -\widehat{\Phi}_n\left[I_n + \sum_{k=1}^{\infty}((\widehat{\Gamma}_n^{2n-1})^{-1}\Phi_n C)^k\right](\widehat{\Gamma}_n^{2n-1})^{-1}\\
&= -\widehat{\Phi}_n(\widehat{\Gamma}_n^{2n-1})^{-1} - \sum_{k=1}^{\infty}\widehat{\Phi}_n((\widehat{\Gamma}_n^{2n-1})^{-1}\Phi_n C)^k(\widehat{\Gamma}_n^{2n-1})^{-1}\\
&= \mathcal{O}(t^{-1}).
\end{aligned}$$

Similarly, $\widehat{\Phi}_n(\Phi_n W + \widehat{\Gamma}_n^{2n-1})^{-1} = \mathcal{O}(t^{-1})$. □

Lemma 3.6. *Let* $n_1, n_2 \in \mathbb{N}$. *Then*

$$\begin{aligned}
(\Upsilon + \widehat{\Gamma}_{n_1+1,n_2+1}^{2n_1,2n_2})^{-1} &= \mathcal{O}(t^{-2}), & (\Upsilon + \widehat{\Gamma}_{n_1+1,n_2+1}^{2n_1,2n_2})^{-1}\phi_{n_1,n_2}^{j} &= \mathcal{O}(t^{-1}),\\
(\Upsilon + \widehat{\Gamma}_{n_1+1,n_2+1}^{2n_1,2n_2})^{-1}\Phi_{n_1,n_2} &= \mathcal{O}(t^{-2}), & \widehat{\psi}_{n_1,n_2}^{j^H}(\Upsilon + \widehat{\Gamma}_{n_1+1,n_2+1}^{2n_1,2n_2})^{-1} &= \mathcal{O}(t^{-1}),\\
\widehat{\Phi}_{n_1,n_2}(\Upsilon + \widehat{\Gamma}_{n_1+1,n_2+1}^{2n_1,2n_2})^{-1} &= \mathcal{O}(t^{-2}), & \widehat{\psi}_{n_1,n_2}^{j^H}(\Upsilon + \widehat{\Gamma}_{n_1+1,n_2+1}^{2n_1,2n_2})^{-1}\Phi_{n_1,n_2} &= \mathcal{O}(t^{-1}),\\
\widehat{\Phi}_{n_1,n_2}(\Upsilon + \widehat{\Gamma}_{n_1+1,n_2+1}^{2n_1,2n_2})^{-1}\Phi_{n_1,n_2} &= \mathcal{O}(t^{-2}), & \widehat{\Phi}_{n_1,n_2}(\Upsilon + \widehat{\Gamma}_{n_1+1,n_2+1}^{2n_1,2n_2})^{-1}\phi_{n_1,n_2}^{j} &= \mathcal{O}(t^{-1})
\end{aligned}$$

as $t \to \pm\infty$, *where* $j = 1, 2$, $\widehat{\Gamma}_{n_1+1,n_2+1}^{2n_1,2n_2}$, Φ_{n_1,n_2}, $\widehat{\Phi}_{n_1,n_2}$, ϕ_{n_1,n_2}^{j} *and* $\widehat{\psi}_{n_1,n_2}^{j^H}$ *are defined in* (3.2) *and* $\Upsilon = \Phi_{n_1,n_2}W + \phi_{n_1,n_2}^{1}\omega^H$, $W \in \mathbb{C}^{(n_1+n_2)\times(n_1+n_2)}$ *and* $\omega \in \mathbb{C}^{n_1+n_2}$. *Moreover, we also have*

$$\widehat{\psi}_{n_1,n_2}^{j^H}(\Upsilon + \widehat{\Gamma}_{n_1+1,n_2+1}^{2n_1,2n_2})^{-1}\phi_{n_1,n_2}^{k} = \widehat{\psi}_{n_1,n_2}^{j^H}(\widehat{\Gamma}_{n_1+1,n_2+1}^{2n_1,2n_2})^{-1}\phi_{n_1,n_2}^{k} + \mathcal{O}(t^{-1})$$

as $t \to \pm\infty$, *where* $j, k \in \{1, 2\}$.

Proof. Using the notations in (3.2) and the expression (3.4), we can obtain

$$
\begin{aligned}
(\widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1} &= \mathcal{O}(t^{-2}),\\
(\widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1}\Phi_{n_1,n_2} &= \mathcal{O}(t^{-2}),\\
(\widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1}\phi^j_{n_1,n_2} &= \mathcal{O}(t^{-1}), \quad j=1,2
\end{aligned}
\tag{3.5}
$$

due to

$$(\Xi_{n_i-1,0})^{-1}\Phi_{n_i} = \mathcal{O}(1),\ \ i=1,2, \quad (\Xi_{n_i-1,0})^{-1}\phi_{n_i} = \mathcal{O}(t),\ \ i=1,2.$$

These imply

$$(\widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1}\Upsilon = \mathcal{O}(t^{-1})$$

and

$$
\begin{aligned}
(\Upsilon + \widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1} &= [(\widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})((\widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1}\Upsilon + I)]^{-1}\\
&= [I + (\widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1}\Upsilon]^{-1}(\widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1}\\
&= \left[I + \sum_{k=1}^{\infty}(-1)^k((\widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1}\Upsilon)^k\right](\widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1}\\
&= \mathcal{O}(t^{-2}).
\end{aligned}
$$

We can also obtain

$$
\begin{aligned}
\widehat{\psi}^{j\,H}_{n_1,n_2}(\widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1} &= \mathcal{O}(t^{-1}),\\
\widehat{\psi}^{j\,H}_{n_1,n_2}(\widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1}\Phi_{n_1,n_2} &= \mathcal{O}(t^{-1}),\\
\widehat{\psi}^{j\,H}_{n_1,n_2}(\widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1}\phi^j_{n_1,n_2} &= \mathcal{O}(1),\\
\widehat{\Phi}_{n_1,n_2}(\widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1} &= \mathcal{O}(t^{-2}),\\
\widehat{\Phi}_{n_1,n_2}(\widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1}\Phi_{n_1,n_2} &= \mathcal{O}(t^{-2}),\\
\widehat{\Phi}_{n_1,n_2}(\widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1}\phi^j_{n_1,n_2} &= \mathcal{O}(t^{-1}).
\end{aligned}
\tag{3.6}
$$

The rest of the lemma can be proved by using (3.5) and (3.6). □

Lemma 3.7. *Given $n \in \mathbb{N}$. Let $\kappa_n = \widehat{\psi}^H_n(\widehat{\Gamma}^{2n}_{n+1})^{-1}\phi_n$, where $\widehat{\psi}^H_n$, ϕ_n and $\widehat{\Gamma}^{2n}_{n+1}$ are defined in* (3.1)*. Then*

$$
\kappa_n = \begin{cases} 2 & \text{if } n \text{ is odd},\\ 0 & \text{if } n \text{ is even}. \end{cases}
\tag{3.7}
$$

Proof. By the definitions of $\widehat{\psi}^H_n$, ϕ_n and $\widehat{\Gamma}^{2n}_{n+1}$ in (3.1), it follows from (3.4) that $\kappa_n = \mathbf{x}^H_n(\varGamma^{2n}_{n+1})^{-1}\mathbf{y}_n$, where $\mathbf{x}_n = \left[1, \frac{1}{2!}, \ldots, \frac{1}{n!}\right]^H$, $\mathbf{y}_n = \left[\frac{1}{n!}, \frac{1}{(n-1)!}, \ldots, 1\right]^H$ and $\varGamma^{2n}_{n+1}$ is given in (3.3). It is obvious that

$$\varGamma^{2n}_{n} = \begin{bmatrix} \mathbf{y}_n & \varGamma^{2n}_{n+1} \\ 1 & \mathbf{x}^H_n \end{bmatrix}.$$

From Lemma 3.4, we have that $\digamma_{n+1}^{2n}$ is invertible. It is well-defined to set

$$E = \left[\begin{array}{c|c} 1 & 0 \\ \hline -(\digamma_{n+1}^{2n})^{-1}\mathbf{y}_n & I \end{array}\right].$$

Then we have

$$\digamma_n^{2n} E = \left[\begin{array}{c|c} 0 & \digamma_{n+1}^{2n} \\ \hline 1 - \mathbf{x}_n^H(\digamma_{n+1}^{2n})^{-1}\mathbf{y}_n & \mathbf{x}_n^H \end{array}\right].$$

From Lemma 3.4, we can obtain that

$$\begin{aligned} \frac{n!(n-1)!\cdots 1!}{(2n)!(2n-1)!\cdots n!} &= \det(\digamma_n^{2n}) = \det(\digamma_n^{2n}E) \\ &= (-1)^{n+2}(1-\kappa_n)\det(\digamma_{n+1}^{2n}) \\ &= (-1)^n(1-\kappa_n)\frac{(n-1)!(n-2)!\cdots 1!}{(2n)!(2n-1)!\cdots(n+1)!}. \end{aligned}$$

This implies that $(-1)^n(1-\kappa_n) = 1$ and then (3.7) is proved. □

3.2. Proof of main theorem

In this subsection, we shall prove each case of Theorem 2.1.

Proof of Case 1. Since $\mathfrak{J} = N_n(\lambda) \oplus (-N_n(\lambda)^H)$, then from (2.3), (2.4) and (2.5),

$$\begin{aligned} \begin{bmatrix} Q(t) \\ P(t) \end{bmatrix} &= Se^{\mathfrak{J}t}S^{-1}\begin{bmatrix} I_n \\ W_0 \end{bmatrix} = S\left[\begin{array}{c|c} e^{N_n(\lambda)t} & 0 \\ \hline 0 & e^{-N_n(\lambda)^H t} \end{array}\right]\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \\ &= \left[\begin{array}{c|c} U_1 & V_1 \\ \hline U_2 & V_2 \end{array}\right]\begin{bmatrix} e^{\lambda t}e^{N_n t}W_1 \\ e^{-\bar{\lambda}t}(e^{-N_n t})^H W_2 \end{bmatrix}. \end{aligned}$$

Recall the notations

$$\Phi_n = \Phi_n(t) \equiv e^{N_n t}, \quad \Phi_n^{-H} \equiv (\Phi_n^{-1})^H = (e^{-N_n t})^H.$$

Then

$$\begin{aligned} Q(t) &= e^{\lambda t}U_1\Phi_n W_1 + e^{-\bar{\lambda}t}V_1\Phi_n^{-H}W_2, \\ P(t) &= e^{\lambda t}U_2\Phi_n W_1 + e^{-\bar{\lambda}t}V_2\Phi_n^{-H}W_2. \end{aligned}$$

Due to $\mathrm{Re}(\lambda) > 0$, W_1 is invertible and

$$\|\Phi_n\|, \|\Phi_n^{-1}\| = \mathcal{O}(|t|^{n-1}),$$

we have for $t > 0$,

$$\|e^{-(\lambda+\overline{\lambda})t} V_1 \Phi_n^{-H} W_2 W_1^{-1} \Phi_n^{-1}\| = \mathcal{O}(e^{-2\operatorname{Re}(\lambda)t} t^{2(n-1)}).$$

Then by using Sherman-Morrison-Woodbury formula (see Appendix B), we have

$$[U_1 + e^{-(\lambda+\overline{\lambda})t} V_1 \Phi_n^{-H} W_2 W_1^{-1} \Phi_n^{-1}]^{-1} = U_1^{-1} + \mathcal{O}(e^{-2\operatorname{Re}(\lambda)t} t^{2(n-1)})$$

for t sufficiently large if U_1 is invertible. Then

$$\begin{aligned} W(t) &= P(t)Q(t)^{-1} \\ &= [U_2 + e^{-(\lambda+\overline{\lambda})t} V_2 \Phi_n^{-H} W_2 W_1^{-1} \Phi_n^{-1}][U_1 + e^{-(\lambda+\overline{\lambda})t} V_1 \Phi_n^{-H} W_2 W_1^{-1} \Phi_n^{-1}]^{-1} \\ &= [U_2 + \mathcal{O}(e^{-2\operatorname{Re}(\lambda)t} t^{2(n-1)})][U_1^{-1} + \mathcal{O}(e^{-2\operatorname{Re}(\lambda)t} t^{2(n-1)})] \\ &= U_2 U_1^{-1} + \mathcal{O}(e^{-2\operatorname{Re}(\lambda)t} t^{2(n-1)}) \end{aligned}$$

as $t \to \infty$. Similarly, if V_1 and W_2 are invertible, we have

$$\begin{aligned} W(t) &= P(t)Q(t)^{-1} \\ &= [V_2 + e^{(\lambda+\overline{\lambda})t} U_2 \Phi_n W_1 W_2^{-1} \Phi_n^{H}][V_1 + e^{(\lambda+\overline{\lambda})t} U_1 \Phi_n W_1 W_2^{-1} \Phi_n^{H}]^{-1} \\ &= V_2 V_1^{-1} + \mathcal{O}(e^{2\operatorname{Re}(\lambda)t} t^{2(n-1)}) \end{aligned}$$

as $t \to -\infty$.

Proof of Case 2. For $\mathfrak{J} = \left[\begin{array}{c|c} N_n(i\alpha) & \beta e_n e_n^H \\ \hline 0 & -N_n(i\alpha)^H \end{array}\right] \in \mathbb{C}^{2n\times 2n}$, we have, by Lemma 3.2,

$$e^{\mathfrak{J}t} = e^{i\alpha t} \left[\begin{array}{c|c} \Phi_n & -\beta\widehat{\Gamma}_n^{2n-1} \\ \hline 0 & \Phi_n^{-H} \end{array}\right].$$

Since W_2 is invertible, the solution of IVP (1.2) is

$$\begin{aligned} \begin{bmatrix} Q(t) \\ P(t) \end{bmatrix} &= Y(t) = S e^{\mathfrak{J}t} S^{-1} \begin{bmatrix} I_n \\ W_0 \end{bmatrix} \\ &= e^{i\alpha t} \left[\begin{array}{c|c} U_1 & V_1 \\ \hline U_2 & V_2 \end{array}\right] \left[\begin{array}{c|c} \Phi_n & -\beta\widehat{\Gamma}_n^{2n-1} \\ \hline 0 & \Phi_n^{-H} \end{array}\right] \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \\ &= e^{i\alpha t} \left[\begin{array}{c|c} U_1 & V_1 \\ \hline U_2 & V_2 \end{array}\right] \left[\begin{array}{c} \Phi_n W_1 W_2^{-1} - \beta\widehat{\Gamma}_n^{2n-1} \\ \hline \Phi_n^{-H} \end{array}\right] W_2. \end{aligned}$$

Define

$$\Omega(t) \equiv (\Phi_n W_1 W_2^{-1} - \beta\widehat{\Gamma}_n^{2n-1})^{-1}.$$

$\widehat{\Gamma}_n^{2n-1}$ is invertible for $t \neq 0$ and $\|(\widehat{\Gamma}_n^{2n-1})^{-1}\Phi_n\| = \mathcal{O}(|t|^{-1})$ by Lemma 3.5. Hence $\Omega(t)$ is well-defined for $|t|$ large. Then

$$\begin{aligned} Q(t) &= e^{i\alpha t}[U_1\Omega^{-1}(t)W_2 + V_1\Phi_n^{-H}W_2] \\ &= e^{i\alpha t}[U_1 + V_1\Phi_n^{-H}\Omega(t)]\Omega^{-1}(t)W_2, \\ P(t) &= e^{i\alpha t}[U_2\Omega^{-1}(t)W_2 + V_2\Phi_n^{-H}W_2] \\ &= e^{i\alpha t}[U_2 + V_2\Phi_n^{-H}\Omega(t)]\Omega^{-1}(t)W_2 \end{aligned}$$

as $|t|$ is large. Applying the time asymptotic estimate $\Phi_n^{-H}\Omega(t) = \widehat{\Phi}_n\Omega(t) = \mathcal{O}(t^{-1})$ by Lemma 3.5, we can obtain

$$\begin{aligned} W(t) = P(t)Q(t)^{-1} &= [U_2 + V_2\Phi_n^{-H}\Omega(t)][U_1 + V_1\Phi_n^{-H}\Omega(t)]^{-1} \\ &= U_2U_1^{-1} + \mathcal{O}(t^{-1}) \end{aligned}$$

as $t \to \pm\infty$.

Case 3 is a special case of Case 4 with $\eta = \delta = \gamma$.

Proof of Case 4. $\widetilde{H}$ is symplectically similar to the canonical form $\mathfrak{J}$ in (2.2) for Case 4. By Lemma 3.3, we have

$$e^{\mathfrak{J}t} = \left[\begin{array}{c|c} \begin{bmatrix} \Phi_{n_1,n_2} & \phi^1_{n_1,n_2} \\ 0 & \omega_{11} \end{bmatrix} & \begin{bmatrix} \widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1} & \phi^2_{n_1,n_2} \\ \widehat{\psi^1}^H_{n_1,n_2} & \omega_{12} \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 \\ 0 & \omega_{21} \end{bmatrix} & \begin{bmatrix} \widehat{\Phi}_{n_1,n_2} & 0 \\ \widehat{\psi^2}^H_{n_1,n_2} & \omega_{22} \end{bmatrix} \end{array}\right] \triangleq \left[\begin{array}{c|c} \mathbf{B} & \mathbf{D} \\ \hline \mathbf{G} & \mathbf{E} \end{array}\right]$$

where Φ_{n_1,n_2}, $\widehat{\Phi}_{n_1,n_2}$, $\phi^i_{n_1,n_2}$, $\widehat{\psi^i}_{n_1,n_2}$, $\widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1}$ and ω_{ij} are defined in (3.2). From (2.3), (2.4) and (2.6), the solution for IVP (1.2) is

$$\begin{aligned} \begin{bmatrix} Q(t) \\ P(t) \end{bmatrix} &= Y(t) = Se^{\mathfrak{J}t}S^{-1}\begin{bmatrix} I_n \\ W_0 \end{bmatrix} \\ &= \left[\begin{array}{cc|cc} U_1 & u_1 & V_1 & v_1 \\ \hline U_2 & u_2 & V_2 & v_2 \end{array}\right] \left[\begin{array}{c|c} \mathbf{B} & \mathbf{D} \\ \hline \mathbf{G} & \mathbf{E} \end{array}\right] \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}. \end{aligned}$$

By the assumption that W_2 is invertible, we have from above

$$\begin{bmatrix} Q(t) \\ P(t) \end{bmatrix} W_2^{-1} = \left[\begin{array}{cc|cc} U_1 & u_1 & V_1 & v_1 \\ \hline U_2 & u_2 & V_2 & v_2 \end{array}\right] \begin{bmatrix} \mathbf{B}W_1W_2^{-1} + \mathbf{D} \\ \mathbf{G}W_1W_2^{-1} + \mathbf{E} \end{bmatrix}. \tag{3.8}$$

Set

$$W \equiv W_1W_2^{-1} = \left[\begin{array}{c|c} \mathbf{W}_{1,1} & \mathbf{w}_{1,2} \\ \hline \mathbf{w}_{2,1} & \mathbf{w}_{2,2} \end{array}\right]$$

where $\mathbf{W}_{1,1} \in \mathbb{C}^{(n_1+n_2)\times(n_1+n_2)}$, $\mathbf{w}_{1,2} \in \mathbb{C}^{(n_1+n_2)\times 1}$, $\mathbf{w}_{2,1} \in \mathbb{C}^{1\times(n_1+n_2)}$ and $\mathbf{w}_{2,2} \in \mathbb{C}$. By direct computation,

$$\begin{aligned}
\mathbf{B}W_1W_2^{-1} + \mathbf{D} &= \mathbf{B}W + \mathbf{D} \\
&= \left[\begin{array}{c|c} \Phi_{n_1,n_2} & \phi^1_{n_1,n_2} \\ \hline 0 & \omega_{11} \end{array}\right] \left[\begin{array}{c|c} \mathbf{W}_{1,1} & \mathbf{w}_{1,2} \\ \hline \mathbf{w}_{2,1} & \mathbf{w}_{2,2} \end{array}\right] + \mathbf{D} \\
&= \left[\begin{array}{c|c} \Upsilon_{n_1,n_2} & \Phi_{n_1,n_2}\mathbf{w}_{1,2} + \phi^1_{n_1,n_2}\mathbf{w}_{2,2} \\ \hline \omega_{11}\mathbf{w}_{2,1} & \omega_{11}\mathbf{w}_{2,2} \end{array}\right] + \mathbf{D} \\
&= \left[\begin{array}{c|c} \Upsilon_{n_1,n_2} + \widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1} & p_{n_1,n_2} \\ \hline \widehat{\psi^1}^H_{n_1,n_2} + \omega_{11}\mathbf{w}_{2,1} & \omega_{11}\mathbf{w}_{2,2} + \omega_{12} \end{array}\right]
\end{aligned} \tag{3.9}$$

where

$$\Upsilon_{n_1,n_2} \triangleq \Phi_{n_1,n_2}\mathbf{W}_{1,1} + \phi^1_{n_1,n_2}\mathbf{w}_{2,1}, \quad p_{n_1,n_2} \triangleq \Phi_{n_1,n_2}\mathbf{w}_{1,2} + \phi^1_{n_1,n_2}\mathbf{w}_{2,2} + \phi^2_{n_1,n_2}.$$

Similarly,

$$\begin{aligned}
\mathbf{G}W_1W_2^{-1} + \mathbf{E} &= \mathbf{G}W + \mathbf{E} \\
&= \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \omega_{21} \end{array}\right] \left[\begin{array}{c|c} \mathbf{W}_{1,1} & \mathbf{w}_{1,2} \\ \hline \mathbf{w}_{2,1} & \mathbf{w}_{2,2} \end{array}\right] + \mathbf{E} \\
&= \left[\begin{array}{c|c} \widehat{\Phi}_{n_1,n_2} & 0 \\ \hline \widehat{\psi^2}^H_{n_1,n_2} + \omega_{21}\mathbf{w}_{2,1} & \omega_{21}\mathbf{w}_{2,2} + \omega_{22} \end{array}\right].
\end{aligned} \tag{3.10}$$

Plugging (3.9) and (3.10) into (3.8), we have

$$\begin{aligned}
\begin{bmatrix} Q(t) \\ P(t) \end{bmatrix} W_2^{-1} &= \left[\begin{array}{cc|cc} U_1 & u_1 & V_1 & v_1 \\ \hline U_2 & u_2 & V_2 & v_2 \end{array}\right] \left[\begin{array}{c} \begin{bmatrix} \Upsilon_{n_1,n_2} + \widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1} & p_{n_1,n_2} \\ \widehat{\psi^1}^H_{n_1,n_2} + \omega_{11}\mathbf{w}_{2,1} & \omega_{11}\mathbf{w}_{2,2} + \omega_{12} \end{bmatrix} \\ \hline \begin{bmatrix} \widehat{\Phi}_{n_1,n_2} & 0 \\ \widehat{\psi^2}^H_{n_1,n_2} + \omega_{21}\mathbf{w}_{2,1} & \omega_{21}\mathbf{w}_{2,2} + \omega_{22} \end{bmatrix} \end{array}\right] \\
&= \left[\begin{array}{c|c} U_1 & V_1 \\ \hline U_2 & V_2 \end{array}\right] \left[\begin{array}{c|c} \Upsilon_{n_1,n_2} + \widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1} & p_{n_1,n_2} \\ \hline \widehat{\Phi}_{n_1,n_2} & 0 \end{array}\right] \\
&\quad + \left[\begin{array}{c|c} u_1 & v_1 \\ \hline u_2 & v_2 \end{array}\right] \left[\begin{array}{c|c} \widehat{\psi^1}^H_{n_1,n_2} + \omega_{11}\mathbf{w}_{2,1} & \omega_{11}\mathbf{w}_{2,2} + \omega_{12} \\ \hline \widehat{\psi^2}^H_{n_1,n_2} + \omega_{21}\mathbf{w}_{2,1} & \omega_{21}\mathbf{w}_{2,2} + \omega_{22} \end{array}\right].
\end{aligned}$$

Let

$$\Omega(t) \equiv \left[\begin{array}{c|c} (\Upsilon_{n_1,n_2} + \widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1} & -(\Upsilon_{n_1,n_2} + \widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1} p_{n_1,n_2} \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} \mathcal{O}(t^{-2}) & \mathcal{O}(t^{-1}) \\ \hline 0 & 1 \end{array}\right]$$

be the time asymptotic estimates in Lemma 3.6. Hence $\Omega(t)$ is well defined for $|t|$ large. Then we have

$$\begin{aligned} \left[\begin{array}{c} \mathbf{\Lambda}_1(t) \\ \hline \mathbf{\Lambda}_2(t) \end{array}\right] &\triangleq \begin{bmatrix} Q(t) \\ P(t) \end{bmatrix} W_2^{-1}\Omega(t) \\ &= \left[\begin{array}{c|c} U_1 & V_1 \\ \hline U_2 & V_2 \end{array}\right] \left[\begin{array}{c|c} I_{n_1+n_2} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \widehat{\Phi}_{n_1,n_2}(\Upsilon_{n_1,n_2} + \widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1} & -\widehat{\Phi}_{n_1,n_2}(\Upsilon_{n_1,n_2} + \widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1} p_{n_1,n_2} \end{array}\right] \\ &+ \left[\begin{array}{c|c} u_1 & v_1 \\ \hline u_2 & v_2 \end{array}\right] \left[\begin{array}{c|c} \begin{matrix} (\widehat{\psi^1}^H_{n_1,n_2} + \omega_{11}\mathbf{w}_{2,1}) \\ \times(\Upsilon_{n_1,n_2} + \widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1} \end{matrix} & \begin{matrix} -(\widehat{\psi^1}^H_{n_1,n_2} + \omega_{11}\mathbf{w}_{2,1}) \\ \times(\Upsilon_{n_1,n_2} + \widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1} p_{n_1,n_2} \\ +\omega_{11}\mathbf{w}_{2,2} + \omega_{12} \end{matrix} \\ \hline \begin{matrix} (\widehat{\psi^2}^H_{n_1,n_2} + \omega_{21}\mathbf{w}_{2,1}) \\ \times(\Upsilon_{n_1,n_2} + \widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1} \end{matrix} & \begin{matrix} -(\widehat{\psi^2}^H_{n_1,n_2} + \omega_{21}\mathbf{w}_{2,1}) \\ \times(\Upsilon_{n_1,n_2} + \widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1} p_{n_1,n_2} \\ +\omega_{21}\mathbf{w}_{2,2} + \omega_{22} \end{matrix} \end{array}\right]. \end{aligned}$$

From the time asymptotic estimates in Lemma 3.6, we have

$$\begin{aligned} \widehat{\Phi}_{n_1,n_2}(\Upsilon_{n_1,n_2} + \widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1} &= \mathcal{O}(t^{-2}), \\ \widehat{\Phi}_{n_1,n_2}(\Upsilon_{n_1,n_2} + \widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1} p_{n_1,n_2} &= \mathcal{O}(t^{-1}), \\ (\widehat{\psi^i}^H_{n_1,n_2} + \omega_{i1}\mathbf{w}_{2,1})(\Upsilon_{n_1,n_2} + \widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1} &= \mathcal{O}(t^{-1}), \quad i = 1, 2. \end{aligned}$$

Moreover, due to $(\widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1}\Upsilon_{n_1,n_2} = \mathcal{O}(t^{-1})$, we have by Sherman-Morrison-Woodbury formula

$$\begin{aligned} &(\Upsilon_{n_1,n_2} + \widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1} \\ &= (\widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1} - (\widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1}\Upsilon_{n_1,n_2}[I_k + (\widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1}\Upsilon_{n_1,n_2}]^{-1}(\widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1} \end{aligned}$$

for t sufficiently large. Then by the above identity, Lemma 3.6 and (3.6), it can be obtained that the time asymptotic leading terms of $-(\widehat{\psi^i}^H_{n_1,n_2} + \omega_{i1}\mathbf{w}_{2,1})(\Upsilon_{n_1,n_2} + \widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1} p_{n_1,n_2} + \omega_{i1}\mathbf{w}_{2,2} + \omega_{i2}$ are

$$-\widehat{\psi^i}^H_{n_1,n_2}(\widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1}(\phi^1_{n_1,n_2}\mathbf{w}_{2,2} + \phi^2_{n_1,n_2}) + \omega_{i1}\mathbf{w}_{2,2} + \omega_{i2}, \quad i = 1, 2,$$

respectively, which are both $\mathcal{O}(1)$. In the following, we shall use above asymptotic estimates to decompose $\mathbf{\Lambda}_i(t) = \mathcal{O}(1) + \mathcal{O}(t^{-1})$. To this end, we can obtain

$$\mathbf{\Lambda}_i(t) = \left[\begin{array}{c|cc} U_i & u_i & v_i \end{array}\right] \left[\begin{array}{c|c} I_{n_1+n_2} & \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \\ \hline 0 \cdots 0 & \begin{array}{c} -\widehat{\psi^1}^H_{n_1,n_2}(\widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1}(\phi^1_{n_1,n_2}\mathbf{w}_{2,2} + \phi^2_{n_1,n_2}) \\ +\omega_{11}\mathbf{w}_{2,2} + \omega_{12} \end{array} \\ 0 \cdots 0 & \begin{array}{c} -\widehat{\psi^2}^H_{n_1,n_2}(\widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1}(\phi^1_{n_1,n_2}\mathbf{w}_{2,2} + \phi^2_{n_1,n_2}) \\ +\omega_{21}\mathbf{w}_{2,2} + \omega_{22} \end{array} \end{array}\right] + \mathbf{M}^{\epsilon}_i(t)$$

where

$$\begin{aligned} \mathbf{M}^{\epsilon}_i(t) = {} & V_i \cdot \left[\begin{array}{c|c} \widehat{\Phi}_{n_1,n_2}(\Upsilon_{n_1,n_2} + \widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1} & -\widehat{\Phi}_{n_1,n_2}(\Upsilon_{n_1,n_2} + \widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1}p_{n_1,n_2} \end{array}\right] \\ & + \left[\begin{array}{c|c} \begin{array}{c} [u_i(\widehat{\psi^1}^H_{n_1,n_2} + \omega_{11}\mathbf{w}_{2,1}) + v_i(\widehat{\psi^2}^H_{n_1,n_2} + \omega_{21}\mathbf{w}_{2,1})] \\ \times(\Upsilon_{n_1,n_2} + \widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1} \end{array} & \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \end{array}\right] \\ & + \left[\begin{array}{c|l} \mathbf{0}_{n\times(n_1+n_2)} & \begin{array}{l} -u_i\,\omega_{11}\mathbf{w}_{2,1}(\Upsilon_{n_1,n_2} + \widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1}p_{n_1,n_2} \\ -u_i\,\widehat{\psi^1}^H_{n_1,n_2}(\Upsilon_{n_1,n_2} + \widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1}\Phi_{n_1,n_2}\mathbf{w}_{1,2} \\ -u_i\,\widehat{\psi^1}^H_{n_1,n_2}[(\Upsilon_{n_1,n_2} + \widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1} - (\widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1}] \\ \quad\times(\phi^1_{n_1,n_2}\mathbf{w}_{2,2} + \phi^2_{n_1,n_2}) \\ -v_i\,\omega_{21}\mathbf{w}_{2,1}(\Upsilon_{n_1,n_2} + \widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1}p_{n_1,n_2} \\ -v_i\,\widehat{\psi^2}^H_{n_1,n_2}(\Upsilon_{n_1,n_2} + \widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1}\Phi_{n_1,n_2}\mathbf{w}_{1,2} \\ -v_i\,\widehat{\psi^2}^H_{n_1,n_2}[(\Upsilon_{n_1,n_2} + \widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1} - (\widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1}] \\ \quad\times(\phi^1_{n_1,n_2}\mathbf{w}_{2,2} + \phi^2_{n_1,n_2}) \end{array} \end{array}\right] \\ = {} & \mathcal{O}(t^{-1}) \end{aligned}$$

for $i = 1, 2$ by Lemma 3.6. If we focus on the $\mathcal{O}(1)$ term of $\mathbf{\Lambda}_i(t)$, using the notations (3.2) and Lemma 3.7, a direct calculation yields

$$\begin{bmatrix} -\widehat{\psi^1}^H_{n_1,n_2}(\widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1}(\phi^1_{n_1,n_2}\mathbf{w}_{2,2} + \phi^2_{n_1,n_2}) + \omega_{11}\mathbf{w}_{2,2} + \omega_{12} \\ -\widehat{\psi^2}^H_{n_1,n_2}(\widehat{\Gamma}^{2n_1,2n_2}_{n_1+1,n_2+1})^{-1}(\phi^1_{n_1,n_2}\mathbf{w}_{2,2} + \phi^2_{n_1,n_2}) + \omega_{21}\mathbf{w}_{2,2} + \omega_{12} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} (-1)^{n_1}(\mathbf{w}_{2,2} - i\beta) & (-1)^{n_2}(\mathbf{w}_{2,2} + i\beta) \\ (-1)^{n_1}(i\beta\mathbf{w}_{2,2} + 1) & (-1)^{n_2}(-i\beta\mathbf{w}_{2,2} + 1) \end{bmatrix}\begin{bmatrix} e^{i\gamma t} \\ e^{i\delta t} \end{bmatrix} \equiv \begin{bmatrix} f_u & g_u \\ f_v & g_v \end{bmatrix}\begin{bmatrix} e^{i\gamma t} \\ e^{i\delta t} \end{bmatrix}.$$

Here, f_u, f_v, g_u and g_v are constants independent of t. Therefore, we can get for $i = 1, 2$,

$$\begin{aligned}
\mathbf{\Lambda}_i(t) &= \left[\begin{array}{c|c} U_i & \begin{bmatrix} u_i & v_i \end{bmatrix}\begin{bmatrix} f_u & g_u \\ f_v & g_v \end{bmatrix}\begin{bmatrix} e^{i\gamma t} \\ e^{i\delta t} \end{bmatrix} \end{array}\right] + \mathbf{M}_i^{\epsilon}(t) \\
&= \left[\begin{array}{c|c} U_i & e^{i\gamma t}(f_u u_i + f_v v_i) \end{array}\right] + \left[\begin{array}{c|c} 0 & e^{i\delta t}(g_u u_i + g_v v_i) \end{array}\right] + \mathbf{M}_i^{\epsilon}(t) \\
&= \left(\left[\begin{array}{c|c} U_i & (f_u u_i + f_v v_i) \end{array}\right] + e^{i\theta t}\zeta_i e_n^H + \widetilde{\mathbf{M}}_i^{\epsilon}(t)\right) \cdot (I_{n_1+n_2} \oplus e^{i\gamma t}) \\
&= \left(\widetilde{\mathbb{U}}_i + e^{i\theta t}\zeta_i e_n^H + \widetilde{\mathbf{M}}_i^{\epsilon}(t)\right) \cdot (I_{n_1+n_2} \oplus e^{i\gamma t})
\end{aligned}$$

where

$$\zeta_i \equiv g_u u_i + g_v v_i, \quad \widetilde{\mathbf{M}}_i^{\epsilon}(t) \equiv \mathbf{M}_i^{\epsilon}(t) \cdot (I_{n_1+n_2} \oplus e^{-i\gamma t}), \quad \widetilde{\mathbb{U}}_i \equiv \left[\begin{array}{c|c} U_i & (f_u u_i + f_v v_i) \end{array}\right], \quad \theta = \delta - \gamma.$$

Set

$$\widetilde{\mathbf{\Lambda}}_i(t) \equiv \widetilde{\mathbb{U}}_i + e^{i\theta t}\zeta_i e_n^H + \widetilde{\mathbf{M}}_i^{\epsilon}(t)$$

for $i = 1, 2$. If $\widetilde{\mathbb{U}}_1$ is invertible and $1 + e^{i\theta t} e_n^H \widetilde{\mathbb{U}}_1^{-1}\zeta_1 \neq 0$, then Sherman-Morrison-Woodbury formula implies that $\widetilde{\mathbf{\Lambda}}_1(t)$ is invertible when $|t|$ is large and

$$\begin{aligned}
\widetilde{\mathbf{\Lambda}}_1^{-1}(t) &= (\widetilde{\mathbb{U}}_1 + \widetilde{\mathbf{M}}_1^{\epsilon}(t))^{-1} - \frac{(\widetilde{\mathbb{U}}_1 + \widetilde{\mathbf{M}}_1^{\epsilon}(t))^{-1} e^{i\theta t}\zeta_1 e_n^H (\widetilde{\mathbb{U}}_1 + \widetilde{\mathbf{M}}_1^{\epsilon}(t))^{-1}}{1 + e^{i\theta t} e_n^H (\widetilde{\mathbb{U}}_1 + \widetilde{\mathbf{M}}_1^{\epsilon}(t))^{-1}\zeta_1} \\
&= \widetilde{\mathbb{U}}_1^{-1} - \frac{\widetilde{\mathbb{U}}_1^{-1} e^{i\theta t}\zeta_1 e_n^H \widetilde{\mathbb{U}}_1^{-1}}{1 + e^{i\theta t} e_n^H \widetilde{\mathbb{U}}_1^{-1}\zeta_1} + \mathcal{O}(t^{-1}).
\end{aligned}$$

Then due to $\widetilde{\mathbf{M}}_2^{\epsilon}(t) = \mathcal{O}(t^{-1})$, we can obtain

$$\begin{aligned}
W(t) &= P(t)Q(t)^{-1} = \widetilde{\mathbf{\Lambda}}_2(t)\widetilde{\mathbf{\Lambda}}_1^{-1}(t) \\
&= (\widetilde{\mathbb{U}}_2 + e^{i\theta t}\zeta_2 e_n^H + \widetilde{\mathbf{M}}_2^{\epsilon}(t)) \cdot \left(\widetilde{\mathbb{U}}_1^{-1} - \frac{\widetilde{\mathbb{U}}_1^{-1} e^{i\theta t}\zeta_1 e_n^H \widetilde{\mathbb{U}}_1^{-1}}{1 + e^{i\theta t} e_n^H \widetilde{\mathbb{U}}_1^{-1}\zeta_1} + \mathcal{O}(t^{-1})\right) \\
&= \widetilde{\mathbb{U}}_2\widetilde{\mathbb{U}}_1^{-1} - e^{i\theta t}\frac{\widetilde{\mathbb{U}}_2\widetilde{\mathbb{U}}_1^{-1}\zeta_1 e_n^H \widetilde{\mathbb{U}}_1^{-1}}{1 + e^{i\theta t} e_n^H \widetilde{\mathbb{U}}_1^{-1}\zeta_1} + e^{i\theta t}\zeta_2 e_n^H \widetilde{\mathbb{U}}_1^{-1} \\
&\quad - \frac{e^{i\theta t} e^{i\theta t}(e_n^H \widetilde{\mathbb{U}}_1^{-1}\zeta_1)\zeta_2 e_n^H \widetilde{\mathbb{U}}_1^{-1}}{1 + e^{i\theta t} e_n^H \widetilde{\mathbb{U}}_1^{-1}\zeta_1} + \mathcal{O}(t^{-1}) \\
&= \widetilde{\mathbb{U}}_2\widetilde{\mathbb{U}}_1^{-1} - e^{i\theta t}\frac{\widetilde{\mathbb{U}}_2\widetilde{\mathbb{U}}_1^{-1}\zeta_1 e_n^H \widetilde{\mathbb{U}}_1^{-1}}{1 + e^{i\theta t} e_n^H \widetilde{\mathbb{U}}_1^{-1}\zeta_1} + e^{i\theta t}\frac{\zeta_2 e_n^H \widetilde{\mathbb{U}}_1^{-1}}{1 + e^{i\theta t} e_n^H \widetilde{\mathbb{U}}_1^{-1}\zeta_1} + \mathcal{O}(t^{-1}) \\
&= \widetilde{\mathbb{U}}_2\widetilde{\mathbb{U}}_1^{-1} + \frac{e^{i\theta t}}{1 + e^{i\theta t} e_n^H \widetilde{\mathbb{U}}_1^{-1}\zeta_1}(\zeta_2 - \widetilde{\mathbb{U}}_2\widetilde{\mathbb{U}}_1^{-1}\zeta_1) e_n^H \widetilde{\mathbb{U}}_1^{-1} + \mathcal{O}(t^{-1})
\end{aligned}$$

as $|t|$ is large. The proof is completed.

Remark 3.8. (i) If we assume $S = BB^H$ and $D = C^H C$ for some $B \in \mathbb{C}^{n\times m}$ and $C \in \mathbb{C}^{\ell\times n}$, the equation of the steady state of HRDE (that is, the algebraic Riccati equation) arises from the optimal control problems [1, 11]. It follows from Theorem 5.3 in [11] that if (C, A) is detectable, then the Hamiltonian matrix $\widetilde{\mathcal{H}}$ in (1.3) has no eigenvalues on the imaginary axis; if also (A, B) is stabilizable, then the matrix U_1 given in (2.5) is invertible and by Lemma 2.4.1 in [1], $U_2U_1^{-1}$ is the positive semi-definite steady state of HRDE. Consequently, under the stabilizable and detectable assumptions, the elementary Cases 2, 3 and 4 in Theorem 2.1 are absent, and hence, HRDE has heteroclinic orbits.

(ii) In [18], some assumptions are imposed such as eigenvalues of $\widetilde{\mathcal{H}}$ are distinct and $\widetilde{\mathcal{H}}$ has no pure imaginary eigenvalue. Therefore, Case 1 in the main theorem holds. In Case 1, a totally stable equilibrium $U_2U_1^{-1}$ and a totally unstable equilibrium $V_2V_1^{-1}$ are found and the heteroclinic orbits connecting these two equilibria are established. The other equilibria of saddle type and periodic orbits are characterized in [18]. The asymptotic analysis in this paper can also be applied to these invariant sets whenever the initial points are in the stable/unstable manifolds.

4. The behavior of the combination of elementary cases

The solution of HRDE depends on the Hamiltonian matrix $\widetilde{\mathcal{H}}$. A canonical form of a Hamiltonian matrix under symplectic similarity transformations has been studied in [15]. We adopt the Hamiltonian Jordan canonical form of $\widetilde{\mathcal{H}}$ to study the asymptotic behavior of HRDE. The general Hamiltonian Jordan canonical form is stated in the following theorem.

Theorem 4.1 (Hamiltonian Jordan canonical form [15])**.** *Given a complex Hamiltonian matrix $\widetilde{\mathcal{H}}$, there exists a complex symplectic matrix S such that*

$$\mathfrak{J} \triangleq S^{-1}\widetilde{\mathcal{H}}S = \left[\begin{array}{cccc|cccc} R_r & & & & 0 & & & \\ & R_e & & & & D_e & & \\ & & R_c & & & & D_c & \\ & & & R_d & & & & D_d \\ \hline 0 & & & & -R_r^H & & & \\ & 0 & & & & -R_e^H & & \\ & & 0 & & & & -R_c^H & \\ & & & G_d & & & & -R_d^H \end{array}\right],$$

where the different blocks have the following structures.

(1) *The blocks with index r have the form*

$$R_r = \operatorname{diag}(R_1^r, \dots, R_{\mu_r}^r), \quad R_k^r = \operatorname{diag}(N_{d_{k,1}}(\lambda_k), \dots, N_{d_{k,p_k}}(\lambda_k)), \quad k = 1, \dots, \mu_r,$$

where λ_k *are distinct and* $\mathrm{Re}(\lambda_k) > 0$.

(2) *The blocks with index e have the form*

$$R_e = \mathrm{diag}(R_1^e, \ldots, R_{\mu_e}^e), \qquad R_k^e = \mathrm{diag}(N_{l_{k,1}}(i\alpha_k), \ldots, N_{l_{k,q_k}}(i\alpha_k)),$$
$$D_e = \mathrm{diag}(D_1^e, \ldots, D_{\mu_e}^e), \qquad D_k^e = \mathrm{diag}(\beta_{k,1}^e e_{l_{k,1}} e_{l_{k,1}}^H, \ldots, \beta_{k,q_k}^e e_{l_{k,q_k}} e_{l_{k,q_k}}^H),$$

where for $k = 1, \ldots, \mu_e$ *and* $j = 1, \ldots, q_k$ *we have* $\alpha_k \in \mathbb{R}$ *are distinct and* $\beta_{k,j}^e \in \{-1, 1\}$.

(3) *The blocks with index c have the form*

$$R_c = \mathrm{diag}(R_1^c, \ldots, R_{\mu_c}^c), \qquad R_k^c = \mathrm{diag}(B_{k,1}, \ldots, B_{k,r_k}),$$
$$D_c = \mathrm{diag}(D_1^c, \ldots, D_{\mu_c}^c), \qquad D_k^c = \mathrm{diag}(D_{k,1}, \ldots, D_{k,r_k}),$$

where for $k = 1, \ldots, \mu_c$ *and* $j = 1, \ldots, r_k$ *we have*

$$B_{k,j} = \begin{bmatrix} N_{m_{k,j}}(i\eta_k) & 0 & -\frac{\sqrt{2}}{2} e_{m_{k,j}} \\ 0 & N_{n_{k,j}}(i\eta_k) & -\frac{\sqrt{2}}{2} e_{n_{k,j}} \\ 0 & 0 & i\eta_k \end{bmatrix}, \quad D_{k,j} = \frac{\sqrt{2}}{2} i\beta_{k,j}^c \begin{bmatrix} 0 & 0 & e_{m_{k,j}} \\ 0 & 0 & -e_{n_{k,j}} \\ -e_{m_{k,j}}^H & e_{n_{k,j}}^H & 0 \end{bmatrix},$$

$\eta_k \in \mathbb{R}$ *are distinct and* $\beta_{k,j}^c \in \{-1, 1\}$.

(4) *The blocks with index d have the form*

$$R_d = \mathrm{diag}(R_1^d, \ldots, R_{\mu_d}^d), \quad G_d = \mathrm{diag}(G_1^d, \ldots, G_{\mu_d}^d), \quad D_d = \mathrm{diag}(D_1^d, \ldots, D_{\mu_d}^d),$$

where for $k = 1, \ldots, \mu_d$, *we have*

$$R_k^d = \begin{bmatrix} N_{s_k}(i\eta_k) & 0 & -\frac{\sqrt{2}}{2} e_{s_k} \\ 0 & N_{t_k}(i\delta_k) & -\frac{\sqrt{2}}{2} e_{t_k} \\ 0 & 0 & \frac{1}{i}(\gamma_k + \delta_k) \end{bmatrix}, \quad G_k^d = \beta_k^d \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}(\gamma_k - \delta_k) \end{bmatrix},$$

$$D_k^d = \frac{\sqrt{2}}{2} i\beta_k^d \begin{bmatrix} 0 & 0 & e_{s_k} \\ 0 & 0 & -e_{t_k} \\ -e_{s_k}^H & e_{t_k}^H & -i\frac{\sqrt{2}}{2}(\gamma_k - \delta_k) \end{bmatrix},$$

$\gamma_k \neq \delta_k$ *and* $\beta_{k,j}^d \in \{-1, 1\}$.

It is noted that $e^{\mathfrak{J}t}$ has the same block form as $\mathfrak{J}$ due to the structure of $\mathfrak{J}$. Suppose that the Hamiltonian matrix $\widetilde{\mathcal{H}}$ has Hamiltonian Jordan canonical form $\mathfrak{J}$ defined in the above theorem. Then the solution for IVP (1.2) is

$$Y(t) = \begin{bmatrix} Q(t) \\ P(t) \end{bmatrix} = S e^{\mathfrak{J}t} S^{-1} \begin{bmatrix} I \\ W_0 \end{bmatrix}.$$

In the following example, we consider a special case that $\mathfrak{J}$ is the combination of Cases 1 and 4.

Example 4.2. Assume $S = I_{10}$ and

$$\mathfrak{J} = \left[\begin{array}{cc|cc} N_2(\lambda) & 0 & 0 & 0 \\ 0 & \begin{bmatrix} i\gamma & 0 & \frac{-\sqrt{2}}{2} \\ & i\delta & \frac{-\sqrt{2}}{2} \\ & & \frac{i}{2}(\gamma+\delta) \end{bmatrix} & 0 & \frac{\sqrt{2}}{2}(i\beta)\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & -i\frac{\sqrt{2}}{2}(\gamma-\delta) \end{bmatrix} \\ \hline 0 & 0 & -N_2(\lambda)^H & 0 \\ 0 & \beta\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{2}(\gamma-\delta) \end{bmatrix} & 0 & \begin{bmatrix} i\gamma & 0 & 0 \\ 0 & i\delta & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{i}{2}(\gamma+\delta) \end{bmatrix} \end{array}\right],$$

where $N_2(\lambda) = \left[\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right]$, $\mathrm{Re}(\lambda) > 0$, $\gamma, \delta \in \mathbb{R}$ with $\gamma \neq \delta$ and $\beta \in \{1, -1\}$. Then we have

$$e^{\widetilde{\mathcal{H}}t} = \left[\begin{array}{cc|cc} e^{N_2(\lambda)t} & 0 & 0 & 0 \\ 0 & \mathbf{B} & 0 & \mathbf{D} \\ \hline 0 & 0 & e^{-N_2(\lambda)^H t} & 0 \\ 0 & \mathbf{G} & 0 & \mathbf{E} \end{array}\right],$$

where

$$\mathbf{B} \triangleq \begin{bmatrix} \Phi_{1,1} & \phi^1_{1,1} \\ 0 & \omega_{11} \end{bmatrix}, \quad \mathbf{D} \triangleq \begin{bmatrix} \widehat{\Gamma}^{2,2}_{2,2} & \phi^2_{1,1} \\ \widehat{\psi^1}^H_{1,1} & \omega_{12} \end{bmatrix}, \quad \mathbf{G} \triangleq \begin{bmatrix} 0 & 0 \\ 0 & \omega_{21} \end{bmatrix}, \quad \mathbf{E} \triangleq \begin{bmatrix} \widehat{\Phi}_{1,1} & 0 \\ \widehat{\psi^2}^H_{1,1} & \omega_{22} \end{bmatrix}.$$

Then the solution for IVP (1.2) is represented by

$$\left[\begin{array}{c} Q(t) \\ \hline P(t) \end{array}\right] = e^{\widetilde{\mathcal{H}}t}\left[\begin{array}{c} I_5 \\ \hline W_0 \end{array}\right] = e^{\widetilde{\mathcal{H}}t}\left[\begin{array}{cc} I_2 & 0 \\ 0 & I_3 \\ \hline W_{01} & W_{02} \\ W_{03} & W_{04} \end{array}\right]$$

where W_0 is partitioned into W_{0i}, $i = 1, 2, 3, 4$, with $W_{01} \in \mathbb{C}^{2\times 2}$, $W_{02} \in \mathbb{C}^{2\times 3}$, $W_{03} \in \mathbb{C}^{3\times 2}$ and $W_{04} \in \mathbb{C}^{3\times 3}$. Hence,

$$\left[\begin{array}{c} Q(t) \\ \hline P(t) \end{array}\right] = \left[\begin{array}{cc} e^{N_2(\lambda)t} & 0_{2\times 3} \\ \mathbf{D}W_{03} & \mathbf{B} + \mathbf{D}W_{04} \\ \hline e^{-N_2(\lambda)^H t}W_{01} & e^{-N_2(\lambda)^H t}W_{02} \\ \mathbf{E}W_{03} & \mathbf{G} + \mathbf{E}W_{04} \end{array}\right].$$

If $\mathbf{B}+\mathbf{D}W_{04}$ is invertible, then $Q(t)$ is invertible. Moreover,

$$Q^{-1}(t)=\left[\begin{array}{c|c} e^{-N_2(\lambda)t} & 0_{2\times 3} \\ \hline -(\mathbf{B}+\mathbf{D}W_{04})^{-1}\mathbf{D}W_{03}e^{-N_2(\lambda)t} & (\mathbf{B}+\mathbf{D}W_{04})^{-1} \end{array}\right],$$

and the solution of HRDE (1.1) is

$$\begin{aligned} W(t) &= P(t)Q^{-1}(t) \\ &= \left[\begin{array}{c|c} e^{-N_2(\lambda)^H t}[W_{01}-W_{02}(\mathbf{B}+\mathbf{D}W_{04})^{-1}\mathbf{D}W_{03}]e^{-N_2(\lambda)t} & e^{-N_2(\lambda)^H t}W_{02}(\mathbf{B}+\mathbf{D}W_{04})^{-1} \\ \hline [\mathbf{E}-(\mathbf{G}+\mathbf{E}W_{04})(\mathbf{B}+\mathbf{D}W_{04})^{-1}\mathbf{D}]W_{03}e^{-N_2(\lambda)t} & (\mathbf{G}+\mathbf{E}W_{04})(\mathbf{B}+\mathbf{D}W_{04})^{-1} \end{array}\right]. \end{aligned}$$

Using the analysis same as in Case 4, we have $(\mathbf{G}+\mathbf{E}W_{04})(\mathbf{B}+\mathbf{D}W_{04})^{-1}$ tends to a periodic orbit if W_{04} is invertible. For t sufficiently large, the other blocks in $W(t)$ tend to zero due to $\mathrm{Re}(\lambda)>0$. The analysis for the other combinations of the elementary cases and the general symplectic matrix S can be done analogously but more complicated.

Appendix A. Embedding of trajectories of HRDE into trajectories of a flow on the Grassmann manifold

The geometric insight of Radon's lemma gives the connection between an extended solution of HRDE and an analytic flow on a Grassmann manifold. Define

$$G^n(\mathbb{C}^{2n})=\left\{\mathrm{Im}\left(\begin{bmatrix} A \\ B\end{bmatrix}\right)\ \Big|\ A,B\in\mathbb{C}^{n\times n} \text{ and } \mathrm{rank}\left(\begin{bmatrix} A \\ B\end{bmatrix}\right)=n\right\}$$

where $\mathrm{Im}([A^\top,B^\top]^\top)$ is the linear space that is spanned by the matrix $[A^\top,B^\top]^\top$. $G^n(\mathbb{C}^{2n})$ is the Grassmann manifold with an appropriate topology (see, e.g., [1]). The Grassmann manifold $G^n(\mathbb{C}^{2n})$ is compact analytic and of dimension n^2. Then $\mathbb{C}^{n\times n}$ can be embedded into $G^n(\mathbb{C}^{2n})$ through $\psi(W)=\mathrm{Im}([I,W^\top]^\top)$. Set

$$G_0^n(\mathbb{C}^{2n})=\{\mathrm{Im}([A^\top,B^\top]^\top)\in G^n(\mathbb{C}^{2n})\mid A\in\mathbb{C}^{n\times n} \text{ is invertible}\}.$$

We can obtain $G_0^n(\mathbb{C}^{2n})=\psi(\mathbb{C}^{n\times n})$, the image of $\mathbb{C}^{n\times n}$ under the map ψ. Moreover, $G_0^n(\mathbb{C}^{2n})$ is an open dense subset of $G^n(\mathbb{C}^{2n})$.

Define a flow on the Grassmann manifold $G^n(\mathbb{C}^{2n})$ by

$$S(t,S_0,t_0)=\Phi(t,t_0)(S_0)$$

where $\Phi(t,t_0)$ is a transition matrix of (1.2) and $\Phi(t,t_0)(S_0)$ denotes the image of the n-dimensional subspace S_0 under the non-singular transformation $\Phi(t,t_0)$. Let $W(t,W_0,t_0)$ denote the solution of HRDE with initial value $W(t_0)=P(t_0)Q(t_0)^{-1}=W_0$. $Q(t_0)$ is assumed to be invertible. Through the embedding ψ, Radon's lemma also leads to a geometric version:

$$(A.1)\quad \begin{aligned} \psi(W(t,W_0,t_0)) &= \operatorname{Im}\left(\begin{bmatrix} I \\ P(t)Q(t)^{-1}\end{bmatrix}\right) = \operatorname{Im}\left(\begin{bmatrix} Q(t) \\ P(t)\end{bmatrix}\right) \\ &= \operatorname{Im}\left(\Phi(t,t_0)\begin{pmatrix} Q(t_0) \\ P(t_0)\end{pmatrix}\right) = \Phi(t,t_0)\operatorname{Im}\left(\begin{bmatrix} I \\ W_0\end{bmatrix}\right) \\ \Longrightarrow\quad & \psi(W(t,W_0,t_0)) = S(t,\psi(W_0),t_0). \end{aligned}$$

Equation (A.1) holds if $Q(t)^{-1}$ exists. This is equivalent that the trajectory $S(t,\psi(W_0),t_0)$ stays in the subset $G_0^n(\mathbb{C}^{2n})$ of $G^n(\mathbb{C}^{2n})$. The embedding ψ maps the trajectories of HRDE onto the restriction of the flow $S(t,\psi(W_0),t_0)$ to $G_0^n(\mathbb{C}^{2n})$.

The transition matrix $\Phi(t,t_0)$ associated with the constant matrix $\widetilde{\mathcal{H}}$ in the system (1.2) is holomorphic in $\mathbb{C}$. Hence, the flow $S(t,\psi(W_0),t_0)$ on $G^n(\mathbb{C}^{2n})$ exists and is holomorphic for all $t\in\mathbb{C}$. This infers that the singularities of $W(t,W_0,t_0)$ are isolated and they are poles. Therefore, it is meaningful and interesting to investigate the time asymptotic behavior of HRDE (1.1) through the extended solution $W(t)$, $t\in\mathcal{J}_w$ in (1.5).

To clarify the difference between $W(t)$ and the linear flow on the Grassmann manifold, we can consider the following example.

Example A.1. Let $w(t)$ be the solution of the scalar HRDE

$$w' = 1 + w^2 \quad \text{with } w(0) = w_0,$$

and consider the Grassmann manifold

$$G^1(\mathbb{R}^2) = \left\{ \operatorname{Im}\left(\begin{bmatrix} a \\ b\end{bmatrix}\right) \;\middle|\; a,b\in\mathbb{R} \text{ and } a^2+b^2\neq 0\right\}.$$

The solution of the scalar HRDE is $w(t)=\tan(t+c_0)$ for $t\in(-\pi/2-c_0,\pi/2-c_0)$, where $\tan(c_0)=w_0$. The corresponding linear IVP, i.e., equation (1.2) in the manuscript, turns out to be

$$\begin{bmatrix}\dot q \\ \dot p\end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0\end{bmatrix}\begin{bmatrix} q \\ p\end{bmatrix}, \quad \text{with } \begin{bmatrix} q(0) \\ p(0)\end{bmatrix} = \begin{bmatrix} 1 \\ w_0\end{bmatrix}.$$

Here $\widetilde{\mathcal{H}} = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right]$ is Hamiltonian. The solution of the linear IVP is

$$\begin{aligned}\begin{bmatrix} q(t) \\ p(t)\end{bmatrix} &= e^{\widetilde{\mathcal{H}}t}\begin{bmatrix} 1 \\ w_0\end{bmatrix} = \begin{bmatrix}\cos t & -\sin t \\ \sin t & \cos t\end{bmatrix}\begin{bmatrix} 1 \\ w_0\end{bmatrix} \\ &= \begin{bmatrix}\cos t - w_0\sin t \\ \sin t + w_0\cos t\end{bmatrix} = \sqrt{1+w_0^2}\begin{bmatrix}\cos(t+c_0) \\ \sin(t+c_0)\end{bmatrix},\end{aligned}$$

where $c_0 = \arctan(w_0)$. Therefore, $\left\{\mathrm{Im}\left(\begin{bmatrix} q(t) \\ p(t) \end{bmatrix}\right) \mid t \in \mathbb{R}\right\}$ is the analytic orbit on the Grassmann manifold which can be though of an animation of rotating straight lines on the plane.

By applying Radon's lemma, we see that $w(t) = p(t)/q(t) = \tan(t + c_0)$ for $t \in (-\pi/2 - c_0, \pi/2 - c_0)$ is the solution of the HRDE. Here, $(-\pi/2 - c_0, \pi/2 - c_0)$ is the maximal interval of the solution and $w(t)$ blows up at the end points of the interval. Therefore, we extend

$$w(t) = \frac{p(t)}{q(t)}$$

for $t \neq (k + 1/2)\pi - c_0$. This is the so-called extended solution and is also named by $w(t)$. For a given t, the point $(1, w(t))$ is the intersection of the line $\mathrm{Im}\left(\begin{bmatrix} q(t) \\ p(t) \end{bmatrix}\right)$ and the vertical line $x = 1$. Therefore, the intersection does not exist whenever $q(t) = 0$, i.e., $t = (k + 1/2)\pi - c_0$. This is the reason why we see that $\begin{bmatrix} q(t) \\ p(t) \end{bmatrix}$ is analytic but $w(t)$ blows up periodically.

Appendix B. Sherman-Morrison-Woodbury formula [10]

Given a square invertible $n \times n$ matrix A, an $n \times k$ matrix U, and a $k \times n$ matrix V, let B be an $n \times n$ matrix such that $B = A + UV$. Then, assuming $(I_k + VA^{-1}U)$ is invertible, we have

$$B^{-1} = A^{-1} - A^{-1}U(I_k + VA^{-1}U)^{-1}VA^{-1}.$$

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