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Waring-Goldbach Problem: Two Squares and Three Biquadrates

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Abstract. Assume that ψ is a function of positive variable t, monotonically increasing to infinity and $0 < \psi(t) \ll \log t/(\log\log t)$. Let $\mathcal{R}_3(n)$ denote the number of representations of the integer n as sums of two squares and three biquadrates of primes and we write $\mathcal{E}_3(N)$ for the number of integers n satisfying $n \leq N$, $n \equiv 5, 53, 101 \pmod{120}$ and

$$\left|\mathcal{R}_3(n) - \frac{\Gamma^2(1/2)\Gamma^3(1/4)}{\Gamma(7/4)} \frac{\mathfrak{S}_3(n) n^{3/4}}{\log^5 n}\right| \geq \frac{n^{3/4}}{\psi(n)\log^5 n},$$

where $0 < \mathfrak{S}_3(n) \ll 1$ is the singular series. In this paper, we prove

$$\mathcal{E}_3(N) \ll N^{23/48+\varepsilon}\psi^2(N)$$

for any $\varepsilon > 0$. This result constitutes a refinement upon that of Friedlander and Wooley [2].

1. Introduction

The celebrated Waring problem involving two squares still remains one of the most elegant problems in additive number theory. Here we outline several pieces of research about it.

Let $v(n)$ denote the number of representations of n as sums of two squares and three nonnegative cubes. In 1972, Linnik [10] proved that $v(n) \gg_\varepsilon n^{2/3-\varepsilon}$ for all large integers n and any $\varepsilon > 0$. In 1981, Hooley [5] improved upon the work of Linnik by obtaining the expected asymptotic formula for $v(n)$. In 2000, he [6] also obtained the asymptotic formula for the number of representations of n as sums of three squares and a k-th power.

Let $R_s(n)$ denote the number of representations of natural number n as sums of two squares and s biquadrates. The expected asymptotic formula for $R_s(n)$ can be established for $s \geq 5$, see Hooley [4]. But for $s \leq 4$, all techniques fail to obtain the expected asymptotic formula for $R_s(n)$. Let $E_s(N)$ be the number of integers $n \leq N$ such that the expected asymptotic formula for $R_s(n)$ fails to be valid. In 2014, Friedlander and Wooley [2] showed

$$E_3(N) \ll N^{1/2+\varepsilon} \quad \text{and} \quad E_4(N) \ll N^{1/4+\varepsilon}.$$

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Later on, Zhao [13] strengthened these results by showing

$$E_3(N) \ll N^{3/8+\varepsilon} \quad \text{and} \quad E_4(N) \ll N^{1/8+\varepsilon}.$$

Let

$$\Omega = \{n \in \mathbb{N} : n \equiv 5, 53, 101 \pmod{120}\}.$$

The purpose of this paper is to investigate the cognate problem concerning the representation of integers n in Ω such that

$$n = p_1^2 + p_2^2 + p_3^4 + p_4^4 + p_5^4, \tag{1.1}$$

where p_i are prime numbers. The congruence condition is necessary here, since we have $p^2 \equiv 1$ or $49 \pmod{120}$ and $p^4 \equiv 1 \pmod{120}$ for primes $p > 5$. Denote by $\mathcal{R}_3(n)$ the number of representations of natural number $n \in \Omega$ as the form (1.1). By applying a pruning process into the Hardy-Littlewood method, we obtain the following result, which constitutes an improvement upon that of Friedlander and Wooley [2].

Theorem 1.1. *For a function ψ of a positive variable t, monotonically increasing to infinity and $0 < \psi(t) \ll \log t/(\log\log t)$, let $\mathcal{E}_3(N)$ be the number of integers $n \in \Omega$ and $n \leq N$ such that*

$$\left| \mathcal{R}_3(n) - \frac{\Gamma^2(1/2)\Gamma^3(1/4)}{\Gamma(7/4)} \frac{\mathfrak{S}_3(n) n^{3/4}}{\log^5 n} \right| \geq \frac{n^{3/4}}{\psi(n)\log^5 n},$$

where

$$\mathfrak{S}_3(n) = \sum_{q=1}^{\infty} \frac{1}{\varphi^5(q)} \sum_{a(q)*} S_2(q,a)^2 S_4(q,a)^3 e_q(-an) \quad \textit{and} \quad S_k(q,a) = \sum_{r(q)*} e\left(\frac{ar^k}{q}\right).$$

Then for any $\varepsilon > 0$, we have

$$\mathcal{E}_3(N) \ll N^{23/48+\varepsilon} \psi^2(N).$$

2. Notations and some preliminary lemmas

In this paper, $\varepsilon \in (0, 10^{-100})$ and the value of ε may change from line to line. Let N denote a sufficiently large positive integer in terms of ε. The constants in O-term and $\ll$ symbol depend at most on ε. The letter p, with or without subscript, is reserved for a prime number. As usual, $\varphi(n)$ denotes Euler's function. We use $e(\alpha)$ to denote $e^{2\pi i\alpha}$ and $e_q(\alpha) = e(\alpha/q)$. We denote by $\sum_{x(q)*}$ a sum with x running over a reduced system of residues modulo q. For a set $\mathcal{F}$, $|\mathcal{F}|$ denotes the cardinality of $\mathcal{F}$.

Lemma 2.1. *Let*

$$g_k(\alpha) = \sum_{2 \le p \le N^{1/k}} e(\alpha p^k).$$

Then for $\alpha = a/q + \lambda$, $(a, q) = 1$, $q \le Q$ and $|\lambda| \le Q/(qN)$, we have

$$g_2(\alpha) \ll Q^{1/2} N^{11/40+\varepsilon} + V_2(\alpha),$$

where

$$V_2(\alpha) = \frac{N^{1/2} \log^c N}{q^{1/2-\varepsilon}(1 + N|\lambda|)^{1/2}} \tag{2.1}$$

and $c > 0$ denotes some absolute constant.

Proof. It follows from [9, Theorem 2]. □

Lemma 2.2. *Let*

$$S_k(q, a) = \sum_{r(q)*} e\left(\frac{ar^k}{q}\right).$$

Then for $(q, a) = 1$, we have

(i) $|S_k(q, a)| \ll q^{1/2+\varepsilon}$;

(ii) $|S_k(p, a)| \le ((k, p-1) - 1)p^{1/2} + 1$;

(iii) $S_k(p^l, a) = 0$ *for* $l \ge \gamma(p)$, *where*

$$\gamma(p) = \begin{cases} \theta + 2 & \text{if } p^\theta \parallel k,\ p \ne 2 \text{ or } p = 2,\ \theta = 0, \\ \theta + 3 & \text{if } p^\theta \parallel k,\ p = 2,\ \theta > 0. \end{cases}$$

Proof. For (i), see [7, Lemma 8.5]. For (ii), see [11, Lemma 4.3]. For (iii), see [7, Lemma 8.3]. □

Lemma 2.3. *Let $2 \le k_1 \le k_2 \le \cdots \le k_s$ be natural numbers such that*

$$\sum_{i=j+1}^{s} \frac{1}{k_i} \le \frac{1}{k_j}, \quad 1 \le j \le s-1.$$

Then we have

$$\int_0^1 \left| \prod_{i=1}^{s} g_{k_i}(\alpha) \right|^2 d\alpha \le N^{1/k_1 + \cdots + 1/k_s + \varepsilon}.$$

Proof. By considering the number of solutions of the underlying equation, Lemma 2.3 follows from [1, Lemma 1]. □

Lemma 2.4. *Let $\mathcal{F}(N)$ denote a subset of integers in the interval $(N/2, N]$ and $Z = |\mathcal{F}(N)|$. Let $\xi\colon \mathbb{Z} \to \mathbb{C}$ be a function with $|\xi(n)| \le 1$ for all $n \in \mathbb{Z}$, and set*

$$K(\alpha) = \sum_{n\in\mathcal{F}(N)} \xi(n)e(-n\alpha).$$

Then we have

(i) $\displaystyle\int_0^1 |g_2(\alpha)^2K(\alpha)^2|\,d\alpha \ll Z^2N^{\varepsilon} + ZN^{1/2}$;

(ii) $\displaystyle\int_0^1 |g_2(\alpha)|^{11/6}|g_4(\alpha)^3K(\alpha)|\,d\alpha \ll N^{2/3+\varepsilon}Z + N^{11/12+\varepsilon}Z^{1/2}$.

Proof. By [12, (2.4)] and the bound $|\xi(n)| \le 1$, we have

$$\begin{aligned}\int_0^1 |g_2(\alpha)^2K(\alpha)^2|\,d\alpha &= \sum_{p_1,p_2\le N^{1/2}} \sum_{\substack{m,n\in\mathcal{F}(N)\\ p_1^2-p_2^2=n-m}} \xi(m)\overline{\xi(n)} \\ &\ll \sum_{p_1,p_2\le N^{1/2}} \sum_{\substack{m,n\in\mathcal{F}(N)\\ p_1^2-p_2^2=n-m}} 1 \\ &\ll Z^2N^{\varepsilon} + ZN^{1/2}.\end{aligned}$$

By Hölder's inequality and (i), we have

$$\begin{aligned}&\int_0^1 |g_2(\alpha)|^{11/6}|g_4(\alpha)^3K(\alpha)|\,d\alpha \\ &\ll \left(\int_0^1 |g_2(\alpha)^2g_4(\alpha)^4|\,d\alpha\right)^{5/12} \left(\int_0^1 |g_4(\alpha)|^{16}\,d\alpha\right)^{1/12} \left(\int_0^1 |g_2(\alpha)^2K(\alpha)^2|\,d\alpha\right)^{1/2} \\ &\ll N^{5/12+\varepsilon}N^{1/4+\varepsilon}(Z^2N^{\varepsilon} + ZN^{1/2})^{1/2} \\ &\ll N^{2/3+\varepsilon}Z + N^{11/12+\varepsilon}Z^{1/2},\end{aligned}$$

where Hua's inequality and Lemma 2.3 are used. This completes the proof. □

In order to apply the Hardy-Littlewood method, we first define the Farey dissection. For this purpose, we set

$$A = 10^{100}(1+c), \quad Q_0 = \log^A N, \quad Q_1 = N^{1/4} \quad \text{and} \quad Q_2 = N^{3/4},$$

where c is defined by (2.1). For $(a,q) = 1$, $0 \le a < q$, we put

$$\mathfrak{M}_0(q,a) = \left(\frac{a}{q} - \frac{Q_0^A}{N}, \frac{a}{q} + \frac{Q_0^A}{N}\right], \qquad \mathfrak{M}(q,a) = \left(\frac{a}{q} - \frac{1}{qQ_2}, \frac{a}{q} + \frac{1}{qQ_2}\right],$$

$$\mathfrak{M}_0 = \bigcup_{q\le Q_0^A} \bigcup_{\substack{a=1\\(a,q)=1}}^{q} \mathfrak{M}_0(q,a), \qquad \mathfrak{M} = \bigcup_{q\le Q_1} \bigcup_{\substack{a=1\\(a,q)=1}}^{q} \mathfrak{M}(q,a),$$

$$\mathfrak{I} = \left(-\frac{1}{Q_2}, 1 - \frac{1}{Q_2}\right], \qquad \mathfrak{m}_1 = \mathfrak{I}\setminus\mathfrak{M}, \quad \mathfrak{m}_2 = \mathfrak{M}\setminus\mathfrak{M}_0, \quad \mathfrak{m} = \mathfrak{m}_1\cup\mathfrak{m}_2.$$

Then we have the Farey dissection

$$\mathfrak{I} = \mathfrak{M}_0 \cup \mathfrak{m}. \tag{2.2}$$

Lemma 2.5. *For $\alpha \in \mathfrak{m}_1$, we have*

$$|g_2(\alpha)| \ll N^{7/16+\varepsilon}.$$

Proof. It follows from [3, Theorem 1]. □

Lemma 2.6. *For $(a,q)=1$, let $\mathfrak{N}_0(q,a) = \left(\frac{a}{q} - \frac{1}{qN^{7/8}}, \frac{a}{q} + \frac{1}{qN^{7/8}}\right]$. Then we have*

(i) $$\sum_{q \le Q_0} \sum_{\substack{a=-q \\ (a,q)=1}}^{2q} \int_{\mathfrak{N}_0(q,a)} |V_2(\alpha)|^{1/6}\, d\alpha \ll N^{-\frac{77}{96}+\varepsilon};$$

(ii) $$\sum_{q \le Q_0} \sum_{\substack{a=-q \\ (a,q)=1}}^{2q} \int_{\mathfrak{N}_0(q,a)} |V_2(\alpha)|^{2}\, d\alpha \ll Q_0^3,$$

where $V_2(\alpha)$ is defined by (2.1)*.*

Proof. By (2.1), we have

$$\begin{aligned}
&\sum_{q \le Q_0} \sum_{\substack{a=-q \\ (a,q)=1}}^{2q} \int_{\mathfrak{N}_0(q,a)} |V_2(\alpha)|^{1/6}\, d\alpha \\
&\ll \sum_{q \le Q_0} \sum_{\substack{a=-q \\ (a,q)=1}}^{2q} q^{\varepsilon} \int_{|\lambda| \le 1/(qN^{7/8})} \frac{N^{1/12}\log^{c/6} N}{(q+qN|\lambda|)^{1/12}}\, d\lambda \\
&\ll N^{-11/12+\varepsilon} \sum_{q \le Q_0} \sum_{\substack{a=-q \\ (a,q)=1}}^{2q} q^{-1/12+\varepsilon} \int_{|u| \le N^{1/8}/q} \frac{1}{(1+u)^{1/12}}\, du \\
&\ll N^{-11/12+\varepsilon} Q_0^{23/12+\varepsilon} \int_0^{N^{1/8}} \frac{1}{(1+u)^{1/12}}\, du \ll N^{-77/96+\varepsilon}.
\end{aligned}$$

Now, (i) is proved, and (ii) can be proved by similar arguments. □

Lemma 2.7. *Let*

$$v_k(\lambda) = \sum_{2 < n \le N} \frac{e(n\lambda)}{n^{1-1/k}\log n}.$$

Then for $\alpha = a/q + \lambda \in \mathfrak{M}_0$, we have

$$g_k(\alpha) = \frac{S_k(q,a)}{\varphi(q)} v_k(\lambda) + O(N^{1/k}\exp(-\log^{1/3} N)).$$

Proof. See [7, Lemma 7.15]. □

Lemma 2.8. *Let* $\Omega = \{n \in \mathbb{N} : n \equiv 5, 53, 101 \pmod{120}\}$, *and let*

$$A_3(q,n) = \frac{1}{\varphi^5(q)} \sum_{a(q)*} S_2(q,a)^2 S_4(q,a)^3 e_q(-an) \quad \textit{and} \quad \mathfrak{S}_3(n) = \sum_{q=1}^{\infty} A_3(q,n).$$

Then the series $\mathfrak{S}_3(n)$ *is convergent and* $\mathfrak{S}_3(n) > 0$ *for* $n \in \Omega$.

Proof. The convergence of $\mathfrak{S}_3(n)$ follows from Lemma 2.2(i). By Lemma 2.2(iii) and the fact that $A_3(q,n)$ is multiplicative in q, we get

$$\mathfrak{S}_3(n) = (1 + A_3(2,n) + A_3(4,n) + A_3(8,n)) \prod_{p>2} (1 + A_3(p,n)). \tag{2.3}$$

When $p > 22$, we conclude from Lemma 2.2(ii) that

$$|A_3(p,n)| \le \frac{(p^{1/2}+1)^2(3p^{1/2}+1)^3}{(p-1)^4} \le \frac{100}{p^{3/2}}.$$

So we get

$$\prod_{p>22} (1 + A_3(p,n)) \ge \prod_{p>22} \left(1 - \frac{100}{p^{3/2}}\right) > c > 0. \tag{2.4}$$

Let $L(q,n)$ denote the number of solutions to the congruence

$$x_1^2 + x_2^2 + x_3^4 + x_4^4 + x_5^4 \equiv n \pmod{q}, \quad 1 \le x_i \le q,\ (x_i, q) = 1.$$

Then by [7, Lemma 8.6], we have

$$1 + A_3(2,n) + A_3(4,n) + A_3(8,n) = \frac{L(8,n)}{2^7}, \tag{2.5}$$

$$1 + A_3(p,n) = \frac{pL(p,n)}{(p-1)^5}. \tag{2.6}$$

For $n \equiv 5, 53, 101 \pmod{120}$, it is easy to verify that

$$L(8,n) > 0 \quad \text{and} \quad L(p,n) > 0 \quad \text{for } 2 < p \le 19. \tag{2.7}$$

Now the conclusion $\mathfrak{S}_3(n) > 0$ follows from (2.3)–(2.7). □

3. Mean value estimates

Let

$$I_j = \int_{\mathfrak{m}_j} |g_2(\alpha)^2 g_4(\alpha)^3 K(\alpha)|\, d\alpha, \quad j = 1, 2,$$

where $K(\alpha)$ is defined as in Lemma 2.4.

Proposition 3.1. *We have*

$$I_1 \ll N^{71/96+\varepsilon} Z + N^{95/96+\varepsilon} Z^{1/2},$$

where Z is defined as in Lemma 2.4.

Proof. By Lemma 2.4(ii) and Lemma 2.5, we have

$$\begin{aligned} I_1 &\ll \sup_{\alpha\in\mathfrak{m}_1} |g_2(\alpha)|^{1/6} \int_0^1 |g_2(\alpha)|^{11/6} |g_4(\alpha)^3 K(\alpha)|\, d\alpha \\ &\ll N^{7/96+\varepsilon}(N^{2/3+\varepsilon} Z + N^{11/12+\varepsilon} Z^{1/2}) \\ &\ll N^{71/96+\varepsilon} Z + N^{95/96+\varepsilon} Z^{1/2}. \end{aligned}$$

□

Proposition 3.2. *We have*

$$I_2 \ll N^{3/4} Q_0^{-A/4} Z + N^{95/96+\varepsilon} Z^{1/2}.$$

Proof. For $\alpha \in \mathfrak{m}_2$, it follows from Lemma 2.1 with $Q = N^{1/4}$ that

$$|g_2(\alpha)| \ll V_2(\alpha) + N^{2/5+\varepsilon}, \tag{3.1}$$

where $V_2(\alpha)$ is defined by (2.1). By (3.1) and Lemma 2.4(ii), we get

$$\begin{aligned} I_2 &\ll \int_{\mathfrak{m}_2} |V_2(\alpha)|^{1/6} |g_2(\alpha)|^{11/6} |g_4(\alpha)^3 K(\alpha)|\, d\alpha \\ &\quad + N^{1/15+\varepsilon} \int_0^1 |g_2(\alpha)|^{11/6} |g_4(\alpha)^3 K(\alpha)|\, d\alpha \\ &\ll \int_{\mathfrak{m}_2} |V_2(\alpha)|^{1/6} |g_2(\alpha)|^{11/6} |g_4(\alpha)^3 K(\alpha)|\, d\alpha + N^{11/15+\varepsilon} Z + N^{59/60+\varepsilon} Z^{1/2}. \end{aligned} \tag{3.2}$$

Let

$$\mathfrak{N}_0(q,a) = \left(\frac{a}{q} - \frac{1}{qN^{7/8}}, \frac{a}{q} + \frac{1}{qN^{7/8}} \right], \quad \mathfrak{N}(q,a) = \left(\frac{a}{q} - \frac{1}{qQ_0}, \frac{a}{q} + \frac{1}{qQ_0} \right],$$

and

$$\mathfrak{N}_1(q,a) = \mathfrak{N}(q,a) \setminus \mathfrak{N}_0(q,a).$$

From Dirichlet's approximation theorem, we have

$$\begin{aligned} &\int_{\mathfrak{m}_2} |V_2(\alpha)|^{1/6} |g_2(\alpha)|^{11/6} |g_4(\alpha)^3 K(\alpha)|\, d\alpha \\ &\le \sum_{q\le Q_0} \sum_{\substack{a=-q \\ (a,q)=1}}^{2q} \int_{\mathfrak{m}_2 \cap \mathfrak{N}_1(q,a)} |V_2(\alpha)|^{1/6} |g_2(\alpha)|^{11/6} |g_4(\alpha)^3 K(\alpha)|\, d\alpha \\ &\quad + \sum_{q\le Q_0} \sum_{\substack{a=-q \\ (a,q)=1}}^{2q} \int_{\mathfrak{m}_2 \cap \mathfrak{N}_0(q,a)} |V_2(\alpha)|^{1/6} |g_2(\alpha)|^{11/6} |g_4(\alpha)^3 K(\alpha)|\, d\alpha. \end{aligned} \tag{3.3}$$

For $\alpha = a/q + \lambda \in \mathfrak{N}_1(q,a)$, it is easy to see that $q(1+N|\lambda|) \gg N^{1/8}$, hence

$$\sup_{\alpha\in\mathfrak{N}_1(q,a)} |V_2(\alpha)| \ll N^{7/16+\varepsilon}. \tag{3.4}$$

By (3.4), we obtain

$$\begin{aligned}
&\sum_{q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathfrak{m}_2\cap\mathfrak{N}_1(q,a)} |V_2(\alpha)|^{1/6}|g_2(\alpha)|^{11/6}|g_4(\alpha)^3K(\alpha)|\,d\alpha\\
&\ll \sup_{\alpha\in\mathfrak{N}_1(q,a)}|V_2(\alpha)|^{1/6}\int_0^1 |g_2(\alpha)|^{11/6}|g_4(\alpha)^3K(\alpha)|\,d\alpha\\
&\ll N^{7/96+\varepsilon}(N^{2/3+\varepsilon}Z + N^{11/12+\varepsilon}Z^{1/2})\\
&\ll N^{71/96+\varepsilon}Z + N^{95/96+\varepsilon}Z^{1/2},
\end{aligned} \tag{3.5}$$

where Lemma 2.4(ii) is used. For $\alpha \in \mathfrak{m}_2$, we have $q + qN|\lambda| \gg Q_0^A$. Then it follows from [8, Lemma 3.3] that

$$\sup_{\alpha\in\mathfrak{m}_2}|g_4(\alpha)| \ll N^{31/128+\varepsilon} + \frac{N^{1/4}\log^4 N}{q^{1/8-\varepsilon}(1+N|\lambda|)^{1/8}} \ll \frac{N^{1/4}}{Q_0^{A/9}}. \tag{3.6}$$

Moreover for $\alpha \in \mathfrak{N}_0(q,a)$, by Lemma 2.1 with $Q = N^{1/8}$, we have

$$|g_2(\alpha)| \ll V_2(\alpha) + N^{27/80+\varepsilon}. \tag{3.7}$$

From (3.6), (3.7) and Lemma 2.6(i)(ii), we get

$$\begin{aligned}
&\sum_{q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathfrak{m}_2\cap\mathfrak{N}_0(q,a)} |V_2(\alpha)|^{1/6}|g_2(\alpha)|^{11/6}|g_4(\alpha)^3K(\alpha)|\,d\alpha\\
&\ll \sum_{q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathfrak{m}_2\cap\mathfrak{N}_0(q,a)} |V_2(\alpha)^2 g_4(\alpha)^3K(\alpha)|\,d\alpha\\
&\quad + N^{99/160+\varepsilon}\sum_{q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathfrak{m}_2\cap\mathfrak{N}_0(q,a)} |V_2(\alpha)|^{1/6}|g_4(\alpha)^3K(\alpha)|\,d\alpha\\
&\ll Z\sup_{\alpha\in\mathfrak{m}_2}|g_4(\alpha)|^3\sum_{q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathfrak{N}_0(q,a)}|V_2(\alpha)^2|\,d\alpha\\
&\quad + ZN^{99/160+\varepsilon}\sup_{\alpha\in\mathfrak{m}_2}|g_4(\alpha)|^3\sum_{q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathfrak{N}_0(q,a)}|V_2(\alpha)|^{1/6}\,d\alpha\\
&\ll ZN^{3/4}Q_0^{-3A/8}Q_0^3 + ZN^{99/160+\varepsilon}N^{3/4}Q_0^{-A/2}N^{-77/96+\varepsilon}\\
&\ll N^{3/4}Q_0^{-A/4}Z,
\end{aligned} \tag{3.8}$$

where the trivial bound $|K(\alpha)| \le Z$ is used. Now from (3.2), (3.3), (3.5) and (3.8), we get

$$I_2 \ll N^{3/4} Q_0^{-A/4} Z + N^{95/96+\varepsilon} Z^{1/2}. \qquad \square$$

Proposition 3.3. *For $N/2 \le n \le N$, we have*

$$\int_{\mathfrak{M}_0} g_2(\alpha)^2 g_4(\alpha)^3 e(-n\alpha)\, d\alpha = \frac{\Gamma^2(1/2)\Gamma^3(1/4)}{\Gamma(7/4)} \frac{\mathfrak{S}_3(n) n^{3/4}}{\log^5 n} + O\left(\frac{n^{3/4}\log\log n}{\log^6 n}\right).$$

Proof. For $\alpha = a/q + \lambda$, let $f_k(\alpha) = \frac{S_k(q,a)}{\varphi(q)} v_k(\lambda)$. Then it follows from Lemma 2.7 that

$$\begin{aligned}\int_{\mathfrak{M}_0} & g_2(\alpha)^2 g_4(\alpha)^3 e(-n\alpha)\, d\alpha \\ &= \int_{\mathfrak{M}_0} f_2(\alpha)^2 f_4(\alpha)^3 e(-n\alpha)\, d\alpha + O(n^{3/4}\exp(-\log^{1/4} n)).\end{aligned} \tag{3.9}$$

It is easy to see that

$$\int_{\mathfrak{M}_0} f_2(\alpha)^2 f_4(\alpha)^3 e(-n\alpha)\, d\alpha = \sum_{q \le Q_0^A} A_3(q,n) \int_{|\lambda| \le Q_0^A/N} v_2(\lambda)^2 v_4(\lambda)^3 e(-n\lambda)\, d\lambda. \tag{3.10}$$

It follows from [7, Lemma 7.16] that

$$\begin{aligned}\int_{|\lambda| \le Q_0^A/N} & v_2(\lambda)^2 v_4(\lambda)^3 e(-n\lambda)\, d\lambda \\ &= \int_0^1 v_2(\lambda)^2 v_4(\lambda)^3 e(-n\lambda)\, d\lambda + O\left(\int_{Q_0^A/N}^1 \frac{1}{\lambda^{7/4}\log^5 N}\, d\lambda\right) \\ &= \int_0^1 v_2(\lambda)^2 v_4(\lambda)^3 e(-n\lambda)\, d\lambda + O(N^{3/4} Q_0^{-3A/4}).\end{aligned} \tag{3.11}$$

Similar to [7, Lemma 7.19], we have

$$\int_0^1 v_2(\lambda)^2 v_4(\lambda)^3 e(-n\lambda)\, d\lambda = \frac{\Gamma^2(1/2)\Gamma^3(1/4)}{\Gamma(7/4)} \frac{n^{3/4}}{\log^5 n} + O\left(\frac{n^{3/4}\log\log n}{\log^6 n}\right). \tag{3.12}$$

By (3.11) and (3.12), we have

$$\int_{|\lambda| \le Q_0^A/N} v_2(\lambda)^2 v_4(\lambda)^3 e(-n\lambda)\, d\lambda = \frac{\Gamma^2(1/2)\Gamma^3(1/4)}{\Gamma(7/4)} \frac{n^{3/4}}{\log^5 n} + O\left(\frac{n^{3/4}\log\log n}{\log^6 n}\right). \tag{3.13}$$

From Lemma 2.2(i) and the inequality $\varphi(q) \gg q/\log q$, we get

$$\sum_{q \le Q_0^A} A_3(q,n) = \mathfrak{S}_3(n) + O\left(\sum_{q > Q_0^A} q^{-3/2+\varepsilon}\right) = \mathfrak{S}_3(n) + O(Q_0^{-A/2+\varepsilon}). \tag{3.14}$$

Now combining (3.9), (3.10), (3.13) and (3.14), we have

$$\int_{\mathfrak{M}_0} g_2(\alpha)^2 g_4(\alpha)^3 e(-n\alpha)\, d\alpha = \frac{\Gamma^2(1/2)\Gamma^3(1/4)}{\Gamma(7/4)} \frac{\mathfrak{S}_3(n) n^{3/4}}{\log^5 n} + O\left(\frac{n^{3/4}\log\log n}{\log^6 n}\right). \qquad \square$$

4. Proof of Theorem 1.1

By the Farey dissection (2.2), we have

$$\mathcal{R}_3(n) = \int_{\mathfrak{M}_0} g_2(\alpha)^2 g_4(\alpha)^3 e(-n\alpha)\, d\alpha + \int_{\mathfrak{m}} g_2(\alpha)^2 g_4(\alpha)^3 e(-n\alpha)\, d\alpha. \tag{4.1}$$

Let ψ be a function of positive variable t, monotonically increasing to infinity and $0 < \psi(t) \ll \log t/(\log\log t)$. By Proposition 3.3 and Lemma 2.8, we may define $\mathcal{F}(N)$ to be the set of integers $n \in \Omega$, $N/2 \le n \le N$ such that

$$\left|\mathcal{R}_3(n) - \frac{\Gamma^2(1/2)\Gamma^3(1/4)}{\Gamma(7/4)} \frac{\mathfrak{S}_3(n) n^{3/4}}{\log^5 n}\right| \ge \frac{n^{3/4}}{\psi(n)\log^5 n}. \tag{4.2}$$

For $n \in \mathcal{F}(N)$, by (4.1), (4.2) and Proposition 3.3, we get

$$\left|\int_{\mathfrak{m}} g_2(\alpha)^2 g_4(\alpha)^3 e(-n\alpha)\, d\alpha\right| \ge \frac{n^{3/4}}{\psi(n)\log^5 n}. \tag{4.3}$$

For $n \in \mathcal{F}(N)$, let $\xi(n)$ be defined by the following equation

$$\left|\int_{\mathfrak{m}} g_2(\alpha)^2 g_4(\alpha)^3 e(-n\alpha)\, d\alpha\right| = \xi(n) \int_{\mathfrak{m}} g_2(\alpha)^2 g_4(\alpha)^3 e(-n\alpha)\, d\alpha. \tag{4.4}$$

Then it is easy to see that $|\xi(n)| \le 1$. Write $Z(N) = |\mathcal{F}(N)|$. From (4.3), (4.4), we have

$$\begin{aligned}
\frac{Z(N)N^{3/4}}{\psi(N)\log^5 N} &\ll \sum_{n\in\mathcal{F}(N)} \frac{n^{3/4}}{\psi(n)\log^5 n} \\
&\ll \int_{\mathfrak{m}} |g_2(\alpha)^2 g_4(\alpha)^3 K(\alpha)|\, d\alpha \\
&\ll \left(\int_{\mathfrak{m}_1} + \int_{\mathfrak{m}_2}\right) |g_2(\alpha)^2 g_4(\alpha)^3 K(\alpha)|\, d\alpha,
\end{aligned} \tag{4.5}$$

where

$$K(\alpha) = \sum_{n\in\mathcal{F}(N)} \xi(n) e(-n\alpha).$$

From (4.5), Propositions 3.1 and 3.2, we obtain

$$\frac{Z(N)N^{3/4}}{\psi(N)\log^5 N} \ll N^{3/4} Q_0^{-A/4} Z(N) + N^{95/96+\varepsilon} Z(N)^{1/2}. \tag{4.6}$$

It follows from (4.6) that

$$Z(N) \ll N^{23/48+\varepsilon}\psi^2(N). \tag{4.7}$$

Now by (4.7), we have

$$\mathcal{E}_3(N) \ll N^{1/3} + \sum_{1\le 2^j \le N^{2/3}} Z\left(\frac{N}{2^j}\right) \ll N^{23/48+\varepsilon}\psi^2(N),$$

and the proof of the Theorem 1.1 is completed.

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