

Biderivations and Commutative Post-Lie Algebra Structures on the Lie Algebra $\mathcal{W}(a, b)$

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Abstract. For $a, b \in \mathbb{C}$, the Lie algebra $\mathcal{W}(a, b)$ is the semidirect product of the Witt algebra and a module of the intermediate series. In this paper, all biderivations of $\mathcal{W}(a, b)$ are determined. Surprisingly, these Lie algebras have symmetric (and skew-symmetric) non-inner biderivations. As an application, commutative post-Lie algebra structures on $\mathcal{W}(a, b)$ are obtained.

1. Introduction

Derivations and generalized derivations (including biderivations) have become more and more powerful tools in the structure study of rings and algebras. Besides their own interests, they have wide applications to other related problems. Recently, there are many efforts on this, see [3, 6, 9, 10, 16, 20–22]. In his remarkable paper [3], Brešar showed that all biderivations on commutative prime rings are inner biderivations, and determined the biderivations of semiprime rings. The notion of biderivations was introduced to Lie algebras in [21]. Later super-biderivations on some super-algebras were introduced in [7, 22]. For the last few years many authors computed only skew-symmetric biderivations of some Lie (super)algebras due to their close relation to commuting maps, see [6, 7, 10, 20–22]. Non-skew-symmetric biderivations should not be ignored. Actually non-skew-symmetric biderivations can be used to study post-Lie algebras structures on Lie algebras. This is addressed only quite recently. For example, all biderivations of finite-dimensional complex simple Lie algebras, all biderivations of some W-algebra, all biderivations of the twisted Heisenberg-Virasoro algebra, all biderivations of Block algebras, and all the super-biderivations of classical simple Lie superalgebras were given in [11, 15–17, 23] respectively.

The present paper is to find efficient ways to determine all biderivations of the Lie algebras $\mathcal{W}(a, b)$ for all $a, b \in \mathbb{C}$, to recover and generalize results in the papers [10, 15, 17]. The Lie algebras $\mathcal{W}(a, b)$ is a class of interesting ones including many important Lie algebras as special cases. Let us first recall the Lie algebras $\mathcal{W}(a, b)$.

Received October 29, 2017; Accepted March 30, 2018.

Communicated by Ching Hung Lam.

2010 *Mathematics Subject Classification*. 17B05, 17B40, 17B68.

Key words and phrases. biderivation, Lie algebra $\mathcal{W}(a, b)$, post-Lie algebra.

Xiaomin Tang is supported by NSF of China (grant no. 11771069) and NSF of Heilongjiang Province (Grant A2015007).

Throughout the paper, we denote by \mathbb{C} and \mathbb{Z} the sets of complex numbers and integers, respectively. All vector spaces and algebras are over \mathbb{C} . For $a, b \in \mathbb{C}$, the Lie algebra $\mathcal{W}(a, b) = \text{span}_{\mathbb{C}}\{L_m, I_m \mid m \in \mathbb{Z}\}$ has the following brackets:

$$[L_m, L_n] = (m - n)L_{m+n}, \quad [L_m, I_n] = -(n + a + bm)I_{m+n}, \quad [I_m, I_n] = 0$$

for all $m, n \in \mathbb{Z}$. Note that $\mathcal{W}(a, b)$ contains a subalgebra $\mathcal{W} = \text{span}_{\mathbb{C}}\{L_m \mid m \in \mathbb{Z}\}$ isomorphic to the well-known Witt algebra, and that the space $\text{span}_{\mathbb{C}}\{I_m \mid m \in \mathbb{Z}\}$ is a \mathcal{W} -module of the intermediate series. The algebras $\mathcal{W}(a, b)$ were considered in the mathematical physics in [13]. We know that the universal central extension of $\mathcal{W}(0, 0)$ is the so-called twisted Heisenberg-Virasoro algebra in [1], which plays an important role in the representation theory of toroidal Lie algebras in [2]. The universal central extension of $\mathcal{W}(0, -1)$ is the Lie algebra $W(2, 2)$ whose representations have been studied in [24] in terms of vertex operator algebras.

In [10], the authors determined skew-symmetric biderivations for all $\mathcal{W}(a, b)$. All biderivations of $\mathcal{W}(0, -1)$ and $\mathcal{W}(0, 0)$ were later obtained in [15, 17]. In the present paper, we shall use the methods in [11] to determine all biderivations of $\mathcal{W}(a, b)$ for all $a, b \in \mathbb{C}$.

The paper is organized as follows. In Section 2, we give general results on biderivations and some lemmas which will be used to our proof. In Section 3, we completely characterize the biderivations without the skew-symmetric condition of the Lie algebra $\mathcal{W}(a, b)$ for all cases of a, b . In Section 4, by using the biderivations we characterize the forms of the commutative post-Lie algebra structures on $\mathcal{W}(a, b)$.

2. General results on biderivations and some lemmas

Let L be a Lie algebra. Recall that a linear map $\phi: L \rightarrow L$ is called a derivation if $\phi([x, y]) = [\phi(x), y] + [x, \phi(y)]$ for all $x, y \in L$. For any $x \in L$, we have the inner derivation

$$\text{ad } x: L \rightarrow L, \quad y \mapsto \text{ad } x(y) = [x, y], \quad \forall y \in L.$$

Denote by $\text{Der}(L)$ and by $\text{Inn}(L)$ the space of all derivations and the space of all inner derivations of L respectively. Now let us recall the definition of a biderivation of a Lie algebra as follows.

Definition 2.1. [21] A bilinear map $f: L \times L \rightarrow L$ is called a biderivation of L if it is a derivation with respect to both components. Namely, for all $x, y, z \in L$,

$$\begin{aligned} f([x, y], z) &= [x, f(y, z)] + [f(x, z), y], \\ f(x, [y, z]) &= [f(x, y), z] + [y, f(x, z)]. \end{aligned}$$

For any $\lambda \in \mathbb{C}$, the bilinear map $f: L \times L \rightarrow L$ given by $f(x, y) = \lambda[x, y]$ for all $x, y \in L$ is a biderivation of L . Such biderivations are said to be inner. Denote by $\text{Bid}(L)$ the set of all biderivations of L which is clearly a vector space. An $f \in \text{Bid}(L)$ is called symmetric if $f(x, y) = f(y, x)$ for all $x, y \in L$, and is called skew-symmetric if $f(x, y) = -f(y, x)$ for all $x, y \in L$. Denote by $\text{Bid}_+(L)$ and $\text{Bid}_-(L)$ the subspaces of all symmetric biderivations and all skew-symmetric biderivations on L respectively. If $f \in \text{Bid}(L)$, then it is easy to see that the bilinear map $f^{\text{op}}: L \times L \rightarrow L$ given by $f^{\text{op}}(x, y) = f(y, x)$ for all $x, y \in L$ is also a biderivation of L . Let $f_- = \frac{1}{2}(f - f^{\text{op}})$ and $f_+ = \frac{1}{2}(f + f^{\text{op}})$. It follows that $f_- \in \text{Bid}_-(L)$ and $f_+ \in \text{Bid}_+(L)$ if $f \in \text{Bid}(L)$. In view of $f = f_- + f_+$, the following result established in [11] is very useful for our later arguments.

Lemma 2.2. *Let L be any Lie algebra. Then $\text{Bid}(L) = \text{Bid}_-(L) \oplus \text{Bid}_+(L)$.*

In view of Lemma 2.2, to determine $\text{Bid}(L)$ we only need to determine $\text{Bid}_-(L)$ and $\text{Bid}_+(L)$. The following lemmas are easy to verify by direct computations.

Lemma 2.3. *Suppose that L and \tilde{L} are two Lie algebras and $\sigma: L \rightarrow \tilde{L}$ is an isomorphism of Lie algebras. For any bilinear map $f: L \times L \rightarrow L$, let the bilinear map $f^\sigma: \tilde{L} \times \tilde{L} \rightarrow \tilde{L}$ be determined by*

$$f^\sigma(\sigma(x), \sigma(y)) = \sigma(f(x, y)) \quad \text{for all } x, y \in L.$$

Then f is a biderivation of L if and only if f^σ is a biderivation of \tilde{L} .

Lemma 2.4. *Let $k \in \mathbb{Z}$, $a, b \in \mathbb{C}$. Then the linear map $\sigma: \mathcal{W}(a, b) \rightarrow \mathcal{W}(a + k, b)$ given by $\sigma(L_m) = L_m$, $\sigma(I_m) = I_{m-k}$ is an isomorphism of Lie algebras.*

Lemma 2.5. [15] *Suppose that $k_i^{(n)}, h_i^{(m)} \in \mathbb{C}$ satisfy*

$$(i - m)k_i^{(n)} = (2n - m - i)h_{m-n+i}^{(m)} \quad \text{for all } m, n, i \in \mathbb{Z}.$$

Then there exists $\lambda \in \mathbb{C}$ such that $k_i^{(m)} = h_i^{(m)} = \delta_{m,i}\lambda$.

3. Biderivations of $\mathcal{W}(a, b)$

In this section, we shall determine all biderivations of $\mathcal{W}(a, b)$. Thanks to Lemma 2.4, we may assume that $0 \leq a < 1$. First we define three classes of biderivations for various $\mathcal{W}(a, b)$. The verifications are straightforward.

Definition 3.1. Let $\Omega = (\mu_k)_{k \in \mathbb{Z}}$ be a sequence which contains only finitely many nonzero entries.

- The biderivation $\Psi_\Omega: \mathcal{W}(a, 0) \times \mathcal{W}(a, 0) \rightarrow \mathcal{W}(a, 0)$ is given by

$$\Psi_\Omega(L_m, L_n) = \sum_{k \in \mathbb{Z}} \mu_k I_{m+n+k},$$

$$\Psi_\Omega(L_m, I_n) = \Psi_\Omega(I_n, L_m) = \Psi_\Omega(I_m, I_n) = 0 \quad \text{for all } m, n \in \mathbb{Z}.$$

- The biderivation $\Upsilon_\Omega^a: \mathcal{W}(a, 1) \times \mathcal{W}(a, 1) \rightarrow \mathcal{W}(a, 1)$ is determined by

$$\Upsilon_\Omega^a(L_m, L_n) = \sum_{k \in \mathbb{Z}} (m + n + k + a) \mu_k I_{m+n+k},$$

$$\Upsilon_\Omega^a(L_m, I_n) = \Upsilon_\Omega^a(I_n, L_m) = \Upsilon_\Omega^a(I_m, I_n) = 0 \quad \text{for all } m, n \in \mathbb{Z}.$$

- The biderivation $\Theta_\mu^a: \mathcal{W}(a, -1) \times \mathcal{W}(a, -1) \rightarrow \mathcal{W}(a, -1)$ for $a \in \mathbb{Z}$ is determined by

$$\Theta_\mu^a(L_m, L_n) = (m - n) \mu I_{m+n-a},$$

$$\Theta_\mu^a(L_m, I_n) = \Theta_\mu^a(I_n, L_m) = \Theta_\mu^a(I_m, I_n) = 0 \quad \text{for all } m, n \in \mathbb{Z}.$$

Note that $\Psi_\Omega, \Upsilon_\Omega^a$ are symmetric and Θ_μ^a is skew-symmetric, and they are non-inner if they are nonzero. We know that the spaces $\text{Bid}(\mathcal{W}(0, -1)), \text{Bid}(\mathcal{W}(0, 0))$ and $\text{Bid}_-(\mathcal{W}(a, b))$ were determined in [15], [17] and [10], respectively:

(i) If $f \in \text{Bid}(\mathcal{W}(0, 0))$, then there exist $\lambda \in \mathbb{C}$ and a sequence $\Omega = (\mu_k)_{k \in \mathbb{Z}}$ which contains only finitely many nonzero entries such that

$$f(x, y) = \lambda[x, y] + \Psi_\Omega(x, y) \quad \text{for all } x, y \in \mathcal{W}(0, 0);$$

(ii) If $f \in \text{Bid}(\mathcal{W}(0, -1))$, then there exist $\lambda, \mu \in \mathbb{C}$ such that

$$f(x, y) = \lambda[x, y] + \Theta_\mu^0(x, y) \quad \text{for all } x, y \in \mathcal{W}(0, -1);$$

(iii) If $f \in \text{Bid}_-(\mathcal{W}(a, b))$, then there exist $\lambda, \mu \in \mathbb{C}$ such that

$$f(x, y) = \begin{cases} \lambda[x, y] + \Theta_\mu^a(x, y) & \text{if } a \in \mathbb{Z}, b = -1, \\ \lambda[x, y] & \text{otherwise} \end{cases}$$

for all $x, y \in \mathcal{W}(a, b)$.

Now we present our main result in this section.

Theorem 3.2. *Any biderivation f of $\mathcal{W}(a, b)$ is of the form*

$$f(x, y) = \begin{cases} \lambda[x, y] + \Psi_\Omega(x, y) & \text{if } b = 0, \\ \lambda[x, y] + \Upsilon_\Omega^a(x, y) & \text{if } b = 1, \\ \lambda[x, y] + \Theta_\mu^a(x, y) & \text{if } a \in \mathbb{Z}, b = -1, \\ \lambda[x, y] & \text{otherwise} \end{cases}$$

for some $\lambda, \mu \in \mathbb{C}$ and a sequence $\Omega = (\mu_k)_{k \in \mathbb{Z}}$ which contains only finitely many nonzero entries.

The proof of Theorem 3.2 will be completed later. We first recall and establish several auxiliary results.

Lemma 3.3. [8] *The derivation of $\mathcal{W}(a, b)$ is determined by the following:*

$$\text{Der}(\mathcal{W}(a, b)) = \begin{cases} \text{Inn}(\mathcal{W}(a, b)) \oplus \mathbb{C}D_1 \oplus \mathbb{C}D_2^{0,0} \oplus \mathbb{C}D_3 & (a, b) = (0, 0), \\ \text{Inn}(\mathcal{W}(a, b)) \oplus \mathbb{C}D_1 \oplus \mathbb{C}D_2^{0,1} & (a, b) = (0, 1), \\ \text{Inn}(\mathcal{W}(a, b)) \oplus \mathbb{C}D_1 \oplus \mathbb{C}D_2^{0,2} & (a, b) = (0, 2), \\ \text{Inn}(\mathcal{W}(a, b)) \oplus \mathbb{C}D_1 & \text{otherwise,} \end{cases}$$

where the derivations $D_1, D_2^{0,0}, D_2^{0,1}, D_2^{0,2}, D_3$ are defined as follows for all $m \in \mathbb{Z}$,

$$\begin{aligned} D_1(L_m) &= 0, & D_1(I_m) &= I_m, \\ D_2^{0,0}(L_m) &= (m - 1)I_m, & D_2^{0,0}(I_m) &= 0, \\ D_2^{0,1}(L_m) &= (m^2 - m)I_m, & D_2^{0,1}(I_m) &= 0, \\ D_2^{0,2}(L_m) &= m^3I_m, & D_2^{0,2}(I_m) &= 0, \\ D_3(L_m) &= mI_m, & D_3(I_m) &= 0. \end{aligned}$$

Lemma 3.4. *Suppose that f is a biderivation of $\mathcal{W}(a, b)$ with $(a, b) \neq (0, 0)$. Then there are linear maps $\phi^{a,b}$ and $\psi^{a,b}$ from $\mathcal{W}(a, b)$ into itself such that*

$$\begin{aligned} f(x, y) &= \rho_1^{a,b}(x)D_1(y) + \rho_2^{a,b}(x)D_2^{a,b}(y) + [\phi^{a,b}(x), y] \\ &= \theta_1^{a,b}(y)D_1(x) + \theta_2^{a,b}(y)D_2^{a,b}(x) + [x, \psi^{a,b}(y)] \end{aligned}$$

for all $x, y \in \mathcal{W}(2, 2)$, where $\rho_1^{a,b}, \rho_2^{a,b}$ and $\theta_1^{a,b}, \theta_2^{a,b}$ are linear complex-valued functions on $\mathcal{W}(a, b)$, and $D_1, D_2^{a,b}$ are given by Lemma 3.3, note that $D_2^{a,b} = 0$ when $(a, b) \notin \{(0, 1), (0, 2)\}$.

Proof. It is easy to see that, for the biderivation f of $\mathcal{W}(a, b)$ and a fixed element $x \in \mathcal{W}(a, b)$, the linear map $\phi_x(y) = f(x, y)$ is a derivation of $\mathcal{W}(a, b)$. Notice that $(a, b) \neq (0, 0)$, by Lemma 3.3, there are complex-valued functions $\rho_1^{a,b}, \rho_2^{a,b}$ on $\mathcal{W}(a, b)$ and a linear map $\phi^{a,b}$ from $\mathcal{W}(a, b)$ into itself such that $\phi_x = \rho_1^{a,b}(x)D_1 + \rho_2^{a,b}(x)D_2^{a,b} + \text{ad } \phi^{a,b}(x)$, where we provide that $D_2^{a,b} = 0$ when $(a, b) \notin \{(0, 1), (0, 2)\}$. Namely, $f(x, y) = \rho_1^{a,b}(x)D_1(y) + \rho_2^{a,b}(x)D_2^{a,b}(y) + [\phi^{a,b}(x), y]$. Because f is bilinear, the maps $\rho_1^{a,b}, \rho_2^{a,b}$ are linear. Similarly, the map $\psi_z(y) = f(y, z)$ is a derivation of $\mathcal{W}(a, b)$, and there are linear complex-valued functions $\theta_1^{a,b}, \theta_2^{a,b}$ on $\mathcal{W}(a, b)$ and a linear map $\psi^{a,b}$ from $\mathcal{W}(a, b)$ into itself such that

$$\begin{aligned} f(x, y) &= \theta_1^{a,b}(y)D_1(x) + \theta_2^{a,b}(y)D_2^{a,b}(x) + \text{ad}(-\psi^{a,b}(y))(x) \\ &= \theta_1^{a,b}(y)D_1(x) + \theta_2^{a,b}(y)D_2^{a,b}(x) + [x, \psi^{a,b}(y)]. \end{aligned}$$

The proof is completed. □

Lemma 3.5. *Let f be a biderivation of $\mathcal{W}(a, b)$ with $(a, b) \neq (0, 0)$, and $\phi^{a,b}, \psi^{a,b}, \rho_i^{a,b}, \theta_i^{a,b}, i = 1, 2$ be given as in Lemma 3.4. Then the following equations hold.*

$$(3.1) \quad \begin{aligned} f(L_m, L_n) &= \rho_2^{a,b}(L_m)D_2^{a,b}(L_n) + [\phi^{a,b}(L_m), L_n] \\ &= \theta_2^{a,b}(L_n)D_2^{a,b}(L_m) + [L_m, \psi^{a,b}(L_n)], \end{aligned}$$

$$(3.2) \quad \begin{aligned} f(L_m, I_n) &= \rho_1^{a,b}(L_m)I_n + [\phi^{a,b}(L_m), I_n] \\ &= \theta_2^{a,b}(I_n)D_2^{a,b}(L_m) + [L_m, \psi^{a,b}(I_n)], \end{aligned}$$

$$(3.3) \quad \begin{aligned} f(I_n, L_m) &= \rho_2^{a,b}(I_n)D_2^{a,b}(L_m) + [\phi^{a,b}(I_n), L_m] \\ &= \theta_1^{a,b}(L_m)I_n + [I_n, \psi^{a,b}(L_m)], \end{aligned}$$

$$(3.4) \quad \begin{aligned} f(I_m, I_n) &= \rho_1^{a,b}(I_m)I_n + [\phi^{a,b}(I_m), I_n] \\ &= \theta_1^{a,b}(I_n)I_m + [I_m, \psi^{a,b}(I_n)]. \end{aligned}$$

Proof. It will follow by Lemmas 3.3 and 3.4. □

Let f be a biderivation of $\mathcal{W}(a, b)$. In view of Lemma 3.4, we can assume that

$$(3.5) \quad \phi^{a,b}(L_n) = \sum_{i \in \mathbb{Z}} a_i^{a,b}(n)L_i + \sum_{i \in \mathbb{Z}} b_i^{a,b}(n)I_i,$$

$$(3.6) \quad \psi^{a,b}(L_n) = \sum_{i \in \mathbb{Z}} c_i^{a,b}(n)L_i + \sum_{i \in \mathbb{Z}} d_i^{a,b}(n)I_i,$$

$$(3.7) \quad \phi^{a,b}(I_n) = \sum_{i \in \mathbb{Z}} p_i^{a,b}(n)L_i + \sum_{i \in \mathbb{Z}} q_i^{a,b}(n)I_i,$$

$$(3.8) \quad \psi^{a,b}(I_n) = \sum_{i \in \mathbb{Z}} s_i^{a,b}(n)L_i + \sum_{i \in \mathbb{Z}} r_i^{a,b}(n)I_i,$$

where $a_i^{a,b}(n), b_i^{a,b}(n), c_i^{a,b}(n), d_i^{a,b}(n), p_i^{a,b}(n), q_i^{a,b}(n), r_i^{a,b}(n), s_i^{a,b}(n) \in \mathbb{C}$.

Note that [10] described the sets $\text{Bid}_-(\mathcal{W}(a, b))$. Therefore, by Lemma 2.2 we only need to determine the sets $\text{Bid}_+(\mathcal{W}(a, b))$. That is, we have to study the symmetric biderivations of $\mathcal{W}(a, b)$. It will be divided into several cases based on the values of a, b . For $f \in \text{Bid}(\mathcal{W}(a, b))$, the notions given in (3.5)–(3.8) will be applied always.

3.1. The case for $(a, b) = (0, 1)$

By the definition, $\mathcal{W}(0, 1)$ has the following Lie brackets

$$[L_m, L_n] = (m - n)L_{m+n}, \quad [L_m, I_n] = -(m + n)I_{m+n}, \quad [I_m, I_n] = 0.$$

In this case, we shall prove the following result.

Proposition 3.6. *Let $f \in \text{Bid}_+(\mathcal{W}(0, 1))$. Then there is a sequence $\Omega = (\mu_k)_{k \in \mathbb{Z}}$ which contains only finitely many nonzero entries such that $f(x, y) = \Upsilon_\Omega^0(x, y)$ for all $x, y \in \mathcal{W}(0, 1)$.*

Proof. The proof will be completed by verifying the following three claims.

Claim 3.7. There is a sequence $\Omega = (\mu_k)_{k \in \mathbb{Z}}$ which contains only finitely many nonzero entries such that

$$f(L_m, L_n) = \sum_{k \in \mathbb{Z}} (m + n + k) \mu_k I_{m+n+k} \quad \text{for all } m, n \in \mathbb{Z}.$$

By Lemma 3.4, the linear maps $\phi^{0,1}, \psi^{0,1}: \mathcal{W}(0, 1) \rightarrow \mathcal{W}(0, 1)$ satisfy (3.1). This, together with (3.5) and (3.6), yields that $f(L_m, L_n)$ is equal to

$$\begin{aligned} (3.9) \quad & \rho_2^{0,1}(L_m)(n^2 - n)I_n + \sum_{i \in \mathbb{Z}} (i - n)a_i^{0,1}(m)L_{n+i} + \sum_{i \in \mathbb{Z}} (i + n)b_i^{0,1}(m)I_{n+i} \\ & = \theta_2^{0,1}(L_n)(m^2 - m)I_m + \sum_{j \in \mathbb{Z}} (m - j)c_j^{0,1}(n)L_{m+j} - \sum_{j \in \mathbb{Z}} (m + j)d_j^{0,1}(n)I_{m+j} \\ & = \theta_2^{0,1}(L_n)(m^2 - m)I_m + \sum_{i \in \mathbb{Z}} (2m - n - i)c_{n-m+i}^{0,1}(n)L_{n+i} - \sum_{i \in \mathbb{Z}} (i + n)d_{n-m+i}^{0,1}(n)I_{n+i}. \end{aligned}$$

Because f is symmetric which implies $f(L_m, L_n) = f(L_n, L_m)$, comparing both sides of the above equations, we obtain

$$(3.10) \quad (i - n)a_i^{0,1}(m) = (n - i)c_i^{0,1}(m) = (2m - n - i)c_{n-m+i}^{0,1}(n) \quad \text{for all } m, n, i \in \mathbb{Z}$$

and for any $m \neq n$ with $i \neq 0, m - n$,

$$(3.11) \quad (i + n)b_i^{0,1}(m) = -(i + n)d_i^{0,1}(m) = -(i + n)d_{n-m+i}^{0,1}(n),$$

$$(3.12) \quad \rho_2^{0,1}(L_m)(n^2 - n) + nb_0^{0,1}(m) = -nd_{n-m}^{0,1}(n),$$

$$(3.13) \quad \theta_2^{0,1}(L_n)(m^2 - m) - md_0^{0,1}(n) = mb_{m-n}^{0,1}(m).$$

By Lemma 2.5 with (3.10) and (3.11), one has $a_i^{0,1}(m) = c_i^{0,1}(m) = 0$ and

$$(3.14) \quad b_i^{0,1}(m) = -d_i^{0,1}(m), \quad d_j^{0,1}(m) = d_{n-m+j}^{0,1}(n) \quad \text{for all } i \neq 0, j \neq 0, -n, m - n.$$

Denote $J_1 = \{-2, 1, 3\}$, $J_2 = \{-3, 3, 0\}$ and $J_3 = \{-3, 1, 4\}$. In view of (3.14), we have

$$(3.15) \quad \begin{cases} d_{-2-m}^{0,1}(-2) = d_{3-m}^{0,1}(3) & \text{if } m \notin J_1, \\ d_{3-m}^{0,1}(3) = d_{-3-m}^{0,1}(-3) & \text{if } m \notin J_2, \\ d_{-3-m}^{(-3)} = d_{4-m}^{(4)} & \text{if } m \notin J_3. \end{cases}$$

Take $n = -2, 3, -3$ and 4 in (3.12), respectively, we obtain

$$(3.16) \quad 6\rho_2^{0,1}(L_m) - 2b_0^{0,1}(m) = 2d_{-2-m}^{0,1}(-2) \quad \text{if } m \neq -2,$$

$$(3.17) \quad 6\rho_2^{0,1}(L_m) + 3b_0^{0,1}(m) = -3d_{3-m}^{0,1}(3) \quad \text{if } m \neq 3,$$

$$(3.18) \quad 12\rho_2^{0,1}(L_m) - 3b_0^{0,1}(m) = 3d_{-3-m}^{0,1}(-3) \quad \text{if } m \neq -3,$$

$$(3.19) \quad 12\rho_2^{0,1}(L_m) + 4b_0^{0,1}(m) = -4d_{4-m}^{0,1}(4) \quad \text{if } m \neq 4.$$

According to (3.15), it follows by (3.16) and (3.17) that $\rho_2^{0,1}(L_m) = 0$ if $m \notin J_1$; by (3.17) and (3.18) that $\rho_2^{0,1}(L_m) = 0$ if $m \notin J_2$ and by (3.18) and (3.19) that $\rho_2^{0,1}(L_m) = 0$ if $m \notin J_3$. Note that $J_1 \cap J_2 \cap J_3 = \emptyset$, we get $\rho_2^{0,1}(L_m) = 0$ for all $m \in \mathbb{Z}$. Similarly, according to (3.13) we have $\theta_2^{0,1}(L_n) = 0$ for all $n \in \mathbb{Z}$. This, together with (3.9), implies that for any $m, n, i \in \mathbb{Z}$,

$$(3.20) \quad (i + n)b_i^{0,1}(m) = -(i + n)d_i^{0,1}(m) = -(i + n)d_{n-m+i}^{0,1}(n).$$

It is not difficult to see by (3.20) that $d_{m+k}^{0,1}(m) = d_{n+k}^{0,1}(n) = -b_{m+k}^{0,1}(m)$ for all $m, n, k \in \mathbb{Z}$. Denote $d_k^{0,1}(0) = -\mu_k$, we have $d_{m+k}^{0,1}(m) = -b_{m+k}^{0,1}(m) = -\mu_k$ for all $m, k \in \mathbb{Z}$. All these with (3.9) yield that

$$\begin{aligned} f(L_m, L_n) &= \sum_{i \in \mathbb{Z}} (i + n)b_i^{0,1}(m)I_{n+i} \\ &= \sum_{k \in \mathbb{Z}} (m + k + n)b_{m+k}^{0,1}(m)I_{n+m+k} \\ &= \sum_{k \in \mathbb{Z}} (m + k + n)\mu_k I_{n+m+k}. \end{aligned}$$

Note that the sequence $\Omega \doteq (\mu_k)_{k \in \mathbb{Z}}$ which contains only finitely many nonzero entries for which the above equation makes sense. This completes the proof of the claim. In addition, the above proof also implies

$$(3.21) \quad \phi^{0,1}(L_n) = \psi^{0,1}(L_n) = \sum_{k \in \mathbb{Z}} \mu_k I_{n+k} \quad \text{for all } n \in \mathbb{Z}.$$

Claim 3.8. $f(L_m, I_n) = 0$ for all $m, n \in \mathbb{Z}$.

By Lemma 3.4, the linear maps $\phi^{0,1}, \psi^{0,1}: \mathcal{W}(0, 1) \rightarrow \mathcal{W}(0, 1)$ satisfy (3.2) and (3.3). This, together with (3.7), (3.8) and (3.21), yields that $f(L_m, I_n)$ and $f(I_n, L_m)$ are of the following forms respectively:

$$(3.22) \quad \rho_1^{0,1}(L_m)I_n = \theta_2^{0,1}(I_n)(m^2 - m)I_m + \sum_{j \in \mathbb{Z}} (m - i)s_i^{0,1}(n)L_{m+i} - \sum_{i \in \mathbb{Z}} (m + i)r_i^{0,1}(n)I_{m+i},$$

$$(3.23) \quad \theta_1^{0,1}(L_m)I_n = \rho_2^{a,b}(I_n)(m^2 - m)I_m + \sum_{j \in \mathbb{Z}} (i - m)p_i^{0,1}(n)L_{i+m} + \sum_{i \in \mathbb{Z}} (m + i)q_i^{0,1}(n)I_{m+i}.$$

For any $m \neq n$, from (3.22) we get

$$(3.24) \quad (m - i)s_i^{0,1}(n) = 0,$$

$$(3.25) \quad nr_{n-m}^{0,1}(n) = \rho_1^{0,1}(L_m),$$

$$(3.26) \quad \theta_2^{0,1}(I_n)(m^2 - m) - mr_0^{0,1}(n) = 0.$$

It is easy to see by (3.24) that $s_i^{0,1}(n) = 0$ for all $i, n \in \mathbb{Z}$. Take $m = 2, 3, 4$ and 5 in (3.26), respectively, then we have

$$(3.27) \quad 2\theta_2^{0,1}(I_n) - 2r_0^{0,1}(n) = 0 \quad \text{if } n \neq 2,$$

$$(3.28) \quad 6\theta_2^{0,1}(I_n) - 3r_0^{0,1}(n) = 0 \quad \text{if } n \neq 3,$$

$$(3.29) \quad 12\theta_2^{0,1}(I_n) - 4r_0^{0,1}(n) = 0 \quad \text{if } n \neq 4,$$

$$(3.30) \quad 20\theta_2^{0,1}(I_n) - 5r_0^{0,1}(n) = 0 \quad \text{if } n \neq 5.$$

It follows by (3.27) and (3.28) that $\theta_2^{0,1}(I_n) = 0$ if $n \notin \{2, 3\}$, and by (3.29) and (3.30) that $\theta_2^{0,1}(I_n) = 0$ if $n \notin \{4, 5\}$. In view of $\{2, 3\} \cap \{4, 5\} = \emptyset$, we obtain $\theta_2^{0,1}(I_n) = 0$ for all $n \in \mathbb{Z}$. In addition, by letting $n = 0$ in (3.25) we obtain $\rho_1^{0,1}(L_m) = 0$ for all $m \neq 0$. But we also have $\rho_1^{0,1}(L_0) = 0$ by taking $m = n = 0$ in (3.22). Now, it is already shown that $f(L_m, I_n) = 0$ for all $m, n \in \mathbb{Z}$, which proves the claim.

On the other hand, the results above together with (3.22) yield $(m + i)r_i^{0,1}(n) = 0$ for any integers m, n, i . This implies $r_i^{0,1}(n) = 0$ for all $n, i \in \mathbb{Z}$. Similarly, by (3.23) we have $p_i^{0,1}(n) = q_i^{0,1}(n) = 0$ for all $i, n \in \mathbb{Z}$. Hence, we get the following useful result:

$$(3.31) \quad \phi^{0,1}(I_n) = \psi^{0,1}(I_n) = 0 \quad \text{for all } n \in \mathbb{Z}.$$

Claim 3.9. $f(I_m, I_n) = 0$ for all $m, n \in \mathbb{Z}$.

By Lemma 3.4, the linear maps $\phi^{0,1}, \psi^{0,1}: \mathcal{W}(0, 1) \rightarrow \mathcal{W}(0, 1)$ satisfy (3.4). This, together with (3.31), yields

$$f(I_m, I_n) = \rho_1^{0,1}(I_m)I_n = \theta_1^{0,1}(I_n)I_m.$$

Let m, n run over all integers with $m \neq n$ in the above equation, then $\rho_1^{0,1}(I_m) = \theta_1^{0,1}(I_n) = 0$ and the conclusion is proved.

Finally, the proof of the proposition is completed by Claims 3.7, 3.8 and 3.9. □

3.2. The case for $(a, b) = (0, 2)$

By the definition, $\mathcal{W}(0, 2)$ has the following Lie brackets

$$[L_m, L_n] = (m - n)L_{m+n}, \quad [L_m, I_n] = -(2m + n)I_{m+n}, \quad [I_m, I_n] = 0.$$

In this case, we shall prove the following result.

Proposition 3.10. *Let $f \in \text{Bid}_+(\mathcal{W}(0, 2))$. Then $f(x, y) = 0$ for all $x, y \in \mathcal{W}(0, 2)$.*

Proof. We shall complete the proof by verifying the following three claims.

Claim 3.11. $f(L_m, L_n) = 0$ for all $m, n \in \mathbb{Z}$.

By Lemma 3.4, there are linear maps $\phi^{0,2}, \psi^{0,2}: \mathcal{W}(0, 2) \rightarrow \mathcal{W}(0, 2)$ satisfying (3.1). This, together with (3.5) and (3.6), yields that $f(L_m, L_n)$ is equal to

$$\begin{aligned} & \rho_2^{0,2}(L_m)n^3I_n + \sum_{i \in \mathbb{Z}}(i-n)a_i^{0,2}(m)L_{n+i} + \sum_{i \in \mathbb{Z}}(2n+i)b_i^{0,2}(m)I_{n+i} \\ &= \theta_2^{0,2}(L_n)m^3I_m + \sum_{j \in \mathbb{Z}}(m-j)c_j^{0,2}(n)L_{m+j} - \sum_{j \in \mathbb{Z}}(2m+j)d_j^{0,2}(n)I_{m+j} \\ &= \theta_2^{0,2}(L_n)m^3I_m + \sum_{i \in \mathbb{Z}}(2m-n-i)c_{n-m+i}^{0,2}(n)L_{n+i} - \sum_{i \in \mathbb{Z}}(m+n+i)d_{n-m+i}^{0,2}(n)I_{n+i}. \end{aligned}$$

Because f is symmetric which implies $f(L_m, L_n) = f(L_n, L_m)$, comparing both sides of the above equations, we obtain

$$(3.32) \quad (i-n)a_i^{0,2}(m) = (n-i)c_i^{0,2}(m) = (2m-n-i)c_{n-m+i}^{0,2}(n) \quad \text{for all } m, n, i \in \mathbb{Z}$$

and for $m \neq n$ and $i \neq 0, m-n$,

$$(3.33) \quad (2n+i)b_i^{0,2}(m) = -(2n+i)d_i^{0,2}(m) = -(m+n+i)d_{n-m+i}^{0,2}(n),$$

$$(3.34) \quad \rho_2^{0,2}(L_m)n^3 + 2nb_0^{0,2}(m) = -(m+n)d_{n-m}^{0,2}(n),$$

$$(3.35) \quad \theta_2^{0,2}(L_n)m^3 - 2md_0^{0,2}(n) = (m+n)b_{m-n}^{0,2}(m).$$

It is easy to see by Lemma 2.5 with (3.32) and (3.33) that $a_i^{0,2}(m) = c_i^{0,2}(m) = 0$ and

$$(3.36) \quad b_i^{0,2}(m) = -d_i^{0,2}(m) \quad \text{for all } i \neq 0.$$

By taking $i = -2n$ in (3.33), we get $(n-m)d_{-m-n}^{0,2}(n) = 0$ if $-2n \neq 0, m-n$. It follows that

$$(3.37) \quad d_k^{0,2}(n) = 0 \quad \text{for all } n \neq 0, k \neq 0, -2n.$$

Let $m = 0$ and $i = n \neq 0$ in (3.33), then one has $d_n^{0,2}(0) = \frac{2}{3}d_{2n}^{0,2}(n)$. Note that $2n \neq 0, -2n$, by (3.37) we have $d_{2n}^{0,2}(n) = 0$, which yields $d_n^{0,2}(0) = 0$ for all $n \neq 0$. By taking $n = 0$ and $m \neq 0$ in (3.35), notice that $b_m^{0,2}(m) = -d_m^{0,2}(m) = 0$ according to (3.36) and (3.37), we obtain $\theta_2^{0,2}(L_0)m^3 - 2md_0^{0,2}(0) = 0$. This implies $d_0^{0,2}(0) = 0$. Now, we have shown that

$$(3.38) \quad b_s^{0,2}(0) = d_s^{0,2}(0) = b_k^{0,2}(n) = d_k^{0,2}(n) = 0 \quad \text{for } k, s, n \in \mathbb{Z} \text{ with } k \neq 0, -2n.$$

Then, from (3.38) we know $b_{m-n}^{0,2}(m) = 0$ if $m-n \neq 0, -2m$. This, together with (3.35), deduces

$$\theta_2^{0,2}(L_n)m^3 - 2md_0^{0,2}(n) = 0 \quad \text{for all } n \neq m, 3m.$$

Taking $m = 1, 2, 4$ and 5 in the above equation, respectively, we obtain

$$(3.39) \quad \theta_2^{0,2}(L_n) - 2d_0^{0,2}(n) = 0 \quad \text{if } n \neq 1, 3,$$

$$(3.40) \quad 8\theta_2^{0,2}(L_n) - 4d_0^{0,2}(n) = 0 \quad \text{if } n \neq 2, 6,$$

$$(3.41) \quad 64\theta_2^{0,2}(L_n) - 8d_0^{0,2}(n) = 0 \quad \text{if } n \neq 4, 12,$$

$$(3.42) \quad 125\theta_2^{0,2}(L_n) - 10d_0^{0,2}(n) = 0 \quad \text{if } n \neq 5, 15.$$

It follows by (3.39) and (3.40) that $\theta_2^{0,2}(L_n) = 0$ if $n \notin \{1, 2, 3, 6\}$, by (3.41) and (3.42) that $\theta_2^{0,2}(L_n) = 0$ if $n \notin \{4, 5, 12, 15\}$. Note that $\{1, 2, 3, 6\} \cap \{4, 5, 12, 15\} = \emptyset$, we get $\theta_2^{0,2}(L_n) = 0$ for all $n \in \mathbb{Z}$. Similarly, according to (3.34) we have $\rho_2^{0,2}(L_m) = 0$ for all $m \in \mathbb{Z}$. This, together with (3.9), implies that (3.33) holds for any $m, n, i \in \mathbb{Z}$. By letting $i = 0$ and $n = -m \neq 0$ in (3.33), we have $d_0^{0,2}(m) = 0$ for all $m \neq 0$. This, together with (3.38), gives that $b_k^{0,2}(n) = d_k^{0,2}(n) = 0$ for all $k, n \in \mathbb{Z}$ with $k \neq -2n$ if $n \neq 0$. This completes the proof of the claim. In addition, the above proof also implies

$$(3.43) \quad \phi^{0,2}(L_n) = \psi^{0,2}(L_n) = 0 \quad \text{for all } n \in \mathbb{Z}.$$

Claim 3.12. $f(L_m, I_n) = 0$ for all $m, n \in \mathbb{Z}$.

By Lemma 3.4, the linear maps $\phi^{0,2}, \psi^{0,2}: \mathcal{W}(0, 2) \rightarrow \mathcal{W}(0, 2)$ satisfy (3.2) and (3.3). This, together with (3.43), (3.7) and (3.8), yields that $f(L_m, I_n)$ and $f(I_n, L_m)$ are of the following forms respectively:

$$(3.44) \quad \rho_1^{0,2}(L_m)I_n = \theta_2^{0,2}(I_n)m^3I_m + \sum_{i \in \mathbb{Z}}(m - i)s_i^{0,2}(n)L_{m+i} - \sum_{i \in \mathbb{Z}}(2m + i)r_i^{0,2}(n)I_{m+i},$$

$$(3.45) \quad \theta_1^{0,2}(L_m)I_n = \rho_2^{0,2}(I_n)m^3I_m + \sum_{i \in \mathbb{Z}}(i - m)p_i^{0,2}(n)L_{m+i} + \sum_{i \in \mathbb{Z}}(2m + i)q_i^{0,2}(n)I_{m+i}.$$

From (3.44) we get $(m - i)s_i^{0,2}(n) = 0$ for all $m, n, i \in \mathbb{Z}$ which implies $s_i^{0,2}(n) = 0$ for any integer $i, n \in \mathbb{Z}$, and by (3.44) we also obtain for any $m \neq n$,

$$(3.46) \quad \begin{aligned} (2m + i)r_i^{0,2}(n) &= 0, \quad i \neq 0, n - m, \\ -(m + n)r_{n-m}^{0,2}(n) &= \rho_1^{0,2}(L_m), \end{aligned}$$

$$(3.47) \quad \theta_2^{0,2}(I_n)m^3 - 2mr_0^{0,2}(n) = 0.$$

By letting $m = -n \neq n$ in (3.46), we have $\rho_1^{0,2}(L_m) = 0$ for all $m \neq 0$. In addition, by taking $m = n = 0$ in (3.44) we have $\rho_1^{0,2}(L_0) = 0$ and then $f(L_m, I_n) = \rho_1^{0,2}(L_m)I_n = 0$ for all $m, n \in \mathbb{Z}$. This completes the proof of the claim. On the other hand, in view of (3.47), in a similar way to Claim 3.11 we get $\theta_2^{0,2}(I_n) = r_0^{0,2}(n) = 0$ for all $n \in \mathbb{Z}$. These results with (3.47) yield $r_i^{0,2}(n) = 0$ for all $i, n \in \mathbb{Z}$. Similarly, by (3.45) we obtain $\rho_2^{0,2}(I_n) = p_i^{0,2}(n) = q_i^{0,2}(n) = \theta_1^{0,2}(L_n) = 0$ for any integer n . We have shown that

$$(3.48) \quad \phi^{0,2}(I_n) = \psi^{0,2}(I_n) = 0 \quad \text{for all } n \in \mathbb{Z}.$$

Claim 3.13. $f(I_m, I_n) = 0$ for all $m, n \in \mathbb{Z}$.

From (3.48), the proof can be finished by a similar way to Claim 3.9.

Finally, the proof of the proposition is completed by Claims 3.11, 3.12 and 3.13. □

3.3. The case for a, b with $a \notin \mathbb{Z}$ or $a = 0, b \notin \{0, 1, 2\}$

In this case, we shall prove the following result.

Proposition 3.14. *Suppose that $a \notin \mathbb{Z}$ or $a = 0, b \notin \{0, 1, 2\}$. Let $f \in \text{Bit}_+(\mathcal{W}(a, b))$. Then there is a sequence $\Omega = (\mu_k)_{k \in \mathbb{Z}}$ which contains only finitely many nonzero entries such that $f(x, y) = \Delta_\Omega(x, y)$ for all $x, y \in \mathcal{W}(a, b)$, where Δ_Ω is a bilinear map on $\mathcal{W}(a, b)$ defined by*

$$\Delta_\Omega(L_m, L_n) = \begin{cases} \sum_{k \in \mathbb{Z}} (b(m+n) + k + a)\mu_k I_{m+n+k} & a \notin \mathbb{Z}, b = 0 \text{ or } 1, \\ 0 & \text{otherwise,} \end{cases}$$

and $\Delta_\Omega(L_m, I_n) = \Delta_\Omega(I_m, L_n) = \Delta_\Omega(I_m, I_n) = 0$ for all $m, n \in \mathbb{Z}$.

Proof. The proof will be completed by verifying the following three claims.

Claim 3.15. There is a sequence $\Omega = (\mu_k)_{k \in \mathbb{Z}}$ which contains only finitely many nonzero entries such that

$$f(L_m, L_n) = \begin{cases} \sum_{k \in \mathbb{Z}} (b(m+n) + k + a)\mu_k I_{m+n+k} & a \notin \mathbb{Z}, b = 0 \text{ or } 1, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 3.4, the linear maps $\phi^{a,b}, \psi^{a,b}: \mathcal{W}(a, b) \rightarrow \mathcal{W}(a, b)$ satisfy (3.1). This, together with (3.5) and (3.6), yields that $f(L_m, L_n)$ is equal to

$$\begin{aligned} & \sum_{i \in \mathbb{Z}} (i-n)a_i^{a,b}(m)L_{n+i} + \sum_{i \in \mathbb{Z}} (i+a+bn)b_i^{a,b}(m)I_{n+i} \\ = & \sum_{j \in \mathbb{Z}} (m-j)c_j^{a,b}(n)L_{m+j} - \sum_{j \in \mathbb{Z}} (j+a+bm)d_j^{a,b}(n)I_{m+j} \\ = & \sum_{i \in \mathbb{Z}} (2m-n-i)c_{n-m+i}^{a,b}(n)L_{n+i} - \sum_{i \in \mathbb{Z}} (i+a+n+(b-1)m)d_{n-m+i}^{a,b}(n)I_{n+i}. \end{aligned}$$

Because f is symmetric which implies $f(L_m, L_n) = f(L_n, L_m)$, comparing both sides of the above equations, we obtain

$$(3.49) \quad \begin{aligned} (i-n)a_i^{a,b}(m) &= (n-i)c_i^{a,b}(m) \\ &= (2m-n-i)c_{n-m+i}^{a,b}(n), \end{aligned}$$

$$(3.50) \quad \begin{aligned} (i+a+bn)b_i^{a,b}(m) &= -(i+a+n+(b-1)m)d_{n-m+i}^{a,b}(n) \\ &= -(i+a+bn)d_i^{a,b}(m). \end{aligned}$$

It is easy to see by Lemma 2.5 with (3.49) that

$$(3.51) \quad a_i^{a,b}(m) = c_i^{a,b}(m) = 0 \quad \text{for all } m, i \in \mathbb{Z}.$$

Next, we shall finish the proof based on different cases of a, b .

Case 1: $a = 0$ and $b \neq 0, 1, 2$. By (3.50), $(i + bn)b_i^{0,b}(m) = -(i + bn)d_i^{0,b}(m)$, which implies from $b \neq 0$ that $b_i^{0,b}(m) = -d_i^{0,b}(m)$ for any integers m, i . By letting $m = 0$ and $n = -i$ in (3.50), we get $(b - 1)nb_{-n}^{0,b}(0) = 0$, which from $b \neq 1$ gives that $b_{-n}^{0,b}(0) = 0$ for any $n \neq 0$. Thus, $d_{-m+i}^{0,b}(0) = -b_{-m+i}^{0,b}(0) = 0$ if $i \neq m$. This, together with (3.50) by letting $n = 0$, yields $ib_i^{0,b}(m) = 0$ when $i \neq m$. Therefore, we get $b_i^{0,b}(m) = 0$ for all $i, m \in \mathbb{Z}$ with $i \neq 0, m$. In particular, $d_{n-m}^{0,b}(n) = -b_{n-m}^{0,b}(n) = 0$ if $n - m \neq 0, n$. This, together with (3.50) by letting $i = 0$, yields $bnb_0^{0,b}(m) = 0$ for all $n \neq 0, m$. The above discussion tells us that

$$(3.52) \quad b_i^{0,b}(m) = d_i^{0,b}(m) = 0 \quad \text{for all } i \neq m.$$

Let $i = m$ in (3.50), we get

$$(3.53) \quad (m + bn)b_m^{0,b}(m) = (n + bm)d_n^{0,b}(n) \quad \text{for all } m, n \in \mathbb{Z}.$$

Taking $n = 0$ in the above equation, we obtain $mb_m^{0,b}(m) = mbm_0^{0,b}(0)$ for all $m \in \mathbb{Z}$. Denote $b_0^{0,b}(0) = \mu$, then $b_m^{0,b}(m) = b\mu$ for all $m \in \mathbb{Z}$ with $m \neq 0$. From this, we have by (3.53) that $b(b - 1)(m - n)\mu = 0$ for all $m, n \in \mathbb{Z}$ with $m, n \neq 0$. It follows by $b \neq 0, 1$ that $\mu = 0$, and so $b_m^{0,b}(m) = 0$ for all $m \in \mathbb{Z}$. This, together with (3.52), yields $b_i^{0,b}(m) = d_i^{0,b}(m) = 0$ for all $i, m \in \mathbb{Z}$. With (3.51), one can conclude by (3.1) that

$$(3.54) \quad \phi^{0,b}(L_n) = \psi^{0,b}(L_n) = 0 \quad \text{and} \quad f(L_m, L_n) = 0.$$

This completes the proof of the claim in which $a = 0$ and $b \neq 0, 1, 2$.

Case 2: $a \notin \mathbb{Z}$. In view of (3.50), we see $(i + a)b_i^{a,b}(m) = -(i + a)d_i^{a,b}(m)$. Note that $i + a \neq 0$ since $a \notin \mathbb{Z}$, one has $b_i^{a,b}(m) = -d_i^{a,b}(m)$ for all $i \in \mathbb{Z}$. This, together with (3.50), yields

$$(i + a + bn)b_i^{a,b}(m) = (i + a + n + (b - 1)m)b_{n-m+i}^{a,b}(n).$$

Let $m = 0$ in the above equation, then we have $(i + a)b_i^{a,b}(m) = (i + a + (b - 1)m)t_{i-m}^{(0)}$. From this, by letting $k = i - m$ and $\mu_k = t_k^{(0)} \in \mathbb{C}$, we obtain $(m + k + a)b_{m+k}^{(m)} = (bm + k + a)\mu_k$, which implies

$$(3.55) \quad b_{m+k}^{(m)} = \frac{bm + k + a}{m + k + a}\mu_k = -d_{m+k}^{(m)} \quad \text{for all } m, k \in \mathbb{Z}.$$

Now, equations (3.51), (3.55) with (3.1) yield that

$$(3.56) \quad \begin{aligned} f(L_m, L_n) &= \sum_{k \in \mathbb{Z}} (m + k + a + bn) \cdot \frac{bm + k + a}{m + k + a} \mu_k I_{n+m+k} \\ &= \sum_{k \in \mathbb{Z}} \left((b(m + n) + k + a) + \frac{b(b - 1)mn}{m + k + a} \right) \mu_k I_{n+m+k}. \end{aligned}$$

Since $f(L_m, L_n) = f(L_m, L_n)$, so if there is some $\mu_k \neq 0$, then by (3.56) we get $b = 0$ or $b = 1$. Note that the sequence $\Omega \doteq (\mu_k)_{k \in \mathbb{Z}}$ which contains only finitely many nonzero entries for which the above equation makes sense. This completes the proof of the claim for $a \notin \mathbb{Z}$. In addition, the above process also implies

$$(3.57) \quad \phi^{a,b}(L_n) = -\psi^{a,b}(L_n) = \sum_{k \in \mathbb{Z}} \frac{bm + k + a}{m + k + a} \mu_k I_{n+k} \quad \text{for all } n \in \mathbb{Z}.$$

Claim 3.16. $f(L_m, I_n) = 0$ for all $m, n \in \mathbb{Z}$.

By Lemma 3.4, the linear maps $\phi^{a,b}, \psi^{a,b}: \mathcal{W}(a, b) \rightarrow \mathcal{W}(a, b)$ satisfy (3.2) and (3.3). This, together with (3.7), (3.8), (3.57) or (3.54), yields that $f(L_m, I_n)$ and $f(I_n, L_m)$ are of the following forms respectively:

$$(3.58) \quad \rho_1^{a,b}(L_m)I_n = \sum_{j \in \mathbb{Z}} (m - i) s_i^{a,b}(n) L_{m+i} - \sum_{i \in \mathbb{Z}} (i + a + bm) r_i^{a,b}(n) I_{m+i},$$

$$(3.59) \quad \sum_{j \in \mathbb{Z}} (i - m) p_i^{a,b}(n) L_{i+m} + \sum_{i \in \mathbb{Z}} (i + a + bm) q_i^{a,b}(n) I_{m+i} = \theta_1^{a,b}(L_m)I_n.$$

From (3.58) we get $(m - i) s_i^{a,b}(n) = 0$, which deduces $s_i^{a,b}(n) = 0$ for all $i, n \in \mathbb{Z}$. For any $m \neq n$, in view of (3.58) in which $m = 0$ we get $(i + a) r_i^{a,b}(n) = 0$ for any $i \neq m$. This, together with $a \notin \mathbb{Z}$, yields $r_i^{a,b}(n) = 0$ for any $n, i \in \mathbb{Z}$. Furthermore, it also follows that $\rho_1^{a,b}(L_m) = 0$. Similarly, by (3.59) one can obtain that $p_i^{a,b}(n) = q_i^{a,b}(n) = \theta_1^{a,b}(L_m) = 0$ for all $n, i \in \mathbb{Z}$. Therefore, we have shown that

$$(3.60) \quad \phi^{a,b}(L_m) = \psi^{a,b}(I_m) = f(L_m, I_n) = 0 \quad \text{for all } m, n \in \mathbb{Z}.$$

The proof of the claim is completed.

Claim 3.17. $f(I_m, I_n) = 0$ for all $m, n \in \mathbb{Z}$.

Notice that (3.60), the proof is similar to Claim 3.9.

Finally, the proof of the proposition is completed by Claims 3.15, 3.16 and 3.17. □

Now we are ready to give the proof of our main result.

Proof of Theorem 3.2. The “if” part is easy to verify, we now prove the “only if” part.

Now we assume that f is a biderivation of $\mathcal{W}(a, b)$. We shall finish the proof according to the cases of values allowed for a, b .

Case 1: $a \in \mathbb{Z}, b = 0$.

By [17] we know that if $f \in \text{Bid}(\mathcal{W}(0, 0))$, then there exist $\lambda \in \mathbb{C}$ and a sequence $\Omega = (\mu_k)_{k \in \mathbb{Z}}$ which contains only finitely many nonzero entries such that $f = \lambda[x, y] + \Psi_\Omega(x, y)$ for all $x, y \in \mathcal{W}(0, 0)$, where Ψ_Ω is given by Definition 3.1. Let $\sigma: \mathcal{W}(0, 0) \rightarrow \mathcal{W}(a, 0)$ be a linear map determined by $\sigma(L_m) = L_m, \sigma(I_m) = I_{m-a}$. Then by Lemma 2.4, σ is an

isomorphism of Lie algebras. Let f^σ be a linear map from $\mathcal{W}(a, 0)$ into itself determined by $f^\sigma(\sigma(x), \sigma(y)) = \sigma(f(x, y))$ for all $x, y \in \mathcal{W}(a, 0)$. Thanks to Lemma 2.3, any biderivation of $\mathcal{W}(a, 0)$ must be of the form f^σ which satisfies

$$\begin{aligned} f^\sigma(L_m, L_n) &= f^\sigma(\sigma(L_m), \sigma(L_n)) = \sigma(f(L_m, L_n)) \\ &= \lambda[L_m, L_n] + \sum_{k \in \mathbb{Z}} (m + n + k)\mu_k I_{m+n+k-a} \\ &= \lambda[L_m, L_n] + \sum_{t \in \mathbb{Z}} (m + n + t + a)\mu_{t+a} I_{m+n+t} \\ &= \lambda[L_m, L_n] + \Psi_{\Omega'}(L_m, L_n), \end{aligned}$$

where $\Omega' = (\mu'_k)_{k \in \mathbb{Z}}$ with $\mu'_k = \mu_{k+a}$; and obviously $f^\sigma(L_m, I_n) = \lambda[L_m, I_n]$, $f^\sigma(I_n, L_m) = \lambda[I_n, L_m]$, $f^\sigma(I_m, I_n) = 0$. Hence, we have $f^\sigma(x, y) = \lambda[x, y] + \Psi_{\Omega'}(x, y)$ for all $x, y \in \mathcal{W}(a, 0)$.

Case 2: $a \notin \mathbb{Z}, b = 0$.

By Lemma 2.2, $\text{Bid}(\mathcal{W}(a, 0)) = \text{Bid}_-(\mathcal{W}(a, 0)) \oplus \text{Bid}_+(\mathcal{W}(a, 0))$. By [10], any $f_- \in \text{Bid}_-(\mathcal{W}(a, 0))$ has the form as $f_-(x, y) = \lambda[x, y]$ for all $x, y \in \mathcal{W}(a, 0)$, where $\lambda \in \mathbb{C}$. Let $f_+ \in \text{Bid}_+(\mathcal{W}(a, 0))$. Then by Proposition 3.14, there is a sequence $\Omega = (\mu_k)_{k \in \mathbb{Z}}$ which contains only finitely many nonzero entries such that $f_+(x, y) = \Delta_\Omega(x, y)$ for all $x, y \in \mathcal{W}(a, 0)$, where Δ_Ω is given by Proposition 3.14. Therefore, we have $f_+(L_m, I_n) = f_+(I_m, L_n) = f_+(I_m, I_n) = 0$ and

$$f_+(L_m, L_n) = \sum_{k \in \mathbb{Z}} (k + a)\mu_k I_{m+n+k} = \sum_{k \in \mathbb{Z}} \mu'_k I_{m+n+k}$$

for all $m, n \in \mathbb{Z}$, where $\mu'_k = (k + a)\mu_k$. Let $\Omega' = (\mu'_k)_{k \in \mathbb{Z}}$, we see that $f_+ = \Psi_{\Omega'}$. Hence, in this case any biderivation f of $\mathcal{W}(a, 0)$ is of the form

$$f(x, y) = f_-(x, y) + f_+(x, y) = \lambda[x, y] + \Psi_{\Omega'}(x, y) \quad \text{for all } x, y \in \mathcal{W}(a, 0).$$

Case 3: $a \in \mathbb{Z}, b = 1$.

By Proposition 3.6, we know that if $f_+ \in \text{Bid}_+(\mathcal{W}(0, 1))$ then there is a sequence $\Omega = (\mu_k)_{k \in \mathbb{Z}}$ which contains only finitely many nonzero entries such that $f_+(x, y) = \Upsilon_\Omega^0(x, y)$ for all $x, y \in \mathcal{W}(0, 1)$, where Υ_Ω^0 is given by Definition 3.1. Let $\sigma: \mathcal{W}(0, 1) \rightarrow \mathcal{W}(a, 1)$ be a linear map determined by $\sigma(L_m) = L_m$, $\sigma(I_m) = I_{m-a}$. Then by Lemma 2.4, σ is an isomorphism of Lie algebras. Let f_+^σ be a linear map from $\mathcal{W}(a, 1)$ into itself determined by $f_+^\sigma(\sigma(x), \sigma(y)) = \sigma(f_+(x, y))$ for all $x, y \in \mathcal{W}(a, 1)$. Thanks to Lemma 2.3, any symmetric biderivation of $\mathcal{W}(a, 0)$ must be of the form f_+^σ which is given by

$$\begin{aligned} f_+^\sigma(L_m, L_n) &= f_+^\sigma(\sigma(L_m), \sigma(L_n)) = \sigma(f_+(L_m, L_n)) \\ &= \sum_{k \in \mathbb{Z}} (m + n + k)\mu_k I_{m+n+k-a} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{t \in \mathbb{Z}} (m + n + t + a) \mu_{t+a} I_{m+n+t} \\
 &= \Upsilon_{\Omega'}^a(L_m, L_n),
 \end{aligned}$$

where $\Omega' = \{\mu'_k = \mu_{k+a} \mid k \in \mathbb{Z}\}$; and obviously $f_+^\sigma(L_m, I_n) = f_+^\sigma(I_n, L_m) = f_+^\sigma(I_m, I_n) = 0$. On the other hand, if $f_- \in \text{Bid}_-(\mathcal{W}(a, 1))$, then by [10] we see that $f_-(x, y) = \lambda[x, y]$ for some $\lambda \in \mathbb{C}$. Now, by Lemma 2.2 we conclude that any biderivation of $\mathcal{W}(a, 1)$ is of the form

$$f_-(x, y) + f_+^\sigma(x, y) = \lambda[x, y] + \Upsilon_{\Omega'}^a(x, y) \quad \text{for all } x, y \in \mathcal{W}(a, 1).$$

Case 4: $a \notin \mathbb{Z}, b = 1$.

For any $f_- \in \text{Bid}_-(\mathcal{W}(a, 1))$, by [10] one has $f_-(x, y) = \lambda[x, y]$ for some $\lambda \in \mathbb{C}$. On the other hand, if $f_+ \in \text{Bid}_+(\mathcal{W}(a, 1))$ then by Proposition 3.14, there is a sequence $\Omega = (\mu_k)_{k \in \mathbb{Z}}$ which contains only finitely many nonzero entries such that $f_+(x, y) = \Delta_\Omega(x, y)$ for all $x, y \in \mathcal{W}(a, 1)$, where Δ_Ω is given by Proposition 3.14. Therefore, we have $f_+(L_m, I_n) = f_+(I_m, L_n) = f_+(I_m, I_n) = 0$ and

$$f_+(L_m, L_n) = \sum_{k \in \mathbb{Z}} (m + n + k + a) \mu_k I_{m+n+k} = \Upsilon_\Omega^a(L_m, L_n)$$

for all $m, n \in \mathbb{Z}$. Then we see that $f_+ = \Upsilon_\Omega^a$. Thanks to Lemma 2.2, we deduce that any biderivation of $\mathcal{W}(a, 1)$ is of the form $f_-(x, y) + f_+(x, y) = \lambda[x, y] + \Upsilon_\Omega^a(x, y)$ for all $x, y \in \mathcal{W}(a, 1)$.

Case 5: $a \in \mathbb{Z}, b = -1$.

By [15], we see that if $f \in \text{Bid}(\mathcal{W}(0, -1))$, then there exist $\lambda, \mu \in \mathbb{C}$ such that $f = \lambda[x, y] + \Theta_\mu^0(x, y)$ for all $x, y \in \mathcal{W}(0, -1)$, where Θ_μ^0 is given by Definition 3.1. Let $\sigma: \mathcal{W}(0, -1) \rightarrow \mathcal{W}(a, -1)$ be a linear map determined by $\sigma(L_m) = L_m, \sigma(I_m) = I_{m-a}$. Then by Lemma 2.4, σ is an isomorphism of Lie algebras. Let f^σ be a linear map from $\mathcal{W}(a, -1)$ into itself determined by $f^\sigma(\sigma(x), \sigma(y)) = \sigma(f(x, y))$ for all $x, y \in \mathcal{W}(a, -1)$. Thanks to Lemma 2.3, any biderivation of $\mathcal{W}(a, -1)$ must be of the form f^σ which satisfies

$$\begin{aligned}
 f^\sigma(L_m, L_n) &= f^\sigma(\sigma(L_m), \sigma(L_n)) = \sigma(f(L_m, L_n)) \\
 &= \sigma(\lambda[L_m, L_n] + \Theta_\mu^0(L_m, L_n)) \\
 &= \lambda[L_m, L_n] + \sum_{k \in \mathbb{Z}} \mu_k \sigma(I_{m+n}) \\
 &= \lambda[L_m, L_n] + \sum_{k \in \mathbb{Z}} \mu_k I_{m+n-a} \\
 &= \lambda[L_m, L_n] + \Theta_\mu^a(L_m, L_n);
 \end{aligned}$$

and obviously $f^\sigma(L_m, I_n) = \lambda[L_m, I_n], f^\sigma(I_n, L_m) = \lambda[I_n, L_m], f^\sigma(I_m, I_n) = 0$. Hence, we have $f^\sigma(x, y) = \lambda[x, y] + \Theta_\mu^a(x, y)$ for all $x, y \in \mathcal{W}(a, -1)$.

Case 6: (i) $a \in \mathbb{Z}, b = 2$; (ii) $a \in \mathbb{Z}, b \notin \{-1, 0, 1, 2\}$ and (iii) $a \notin \mathbb{Z}, b \notin \{0, 1\}$.

Proposition 3.10 tells us that if $f_+ \in \text{Bid}_+(\mathcal{W}(0, 2))$ then $f_+ = 0$. In a similar way to the proof of Case 1, by Lemmas 2.4 and 2.3 we conclude that any $f_+^\sigma \in \text{Bid}_+(\mathcal{W}(a, 2))$ has to be 0 for all $a \in \mathbb{Z}$. In view of Proposition 3.14, we see that whatever $a \in \mathbb{Z}, b \notin \{-1, 0, 1, 2\}$ or $a \notin \mathbb{Z}, b \notin \{0, 1\}$, any $f_+ \in \text{Bid}_+(\mathcal{W}(a, b))$ has to be 0. This indicated that $\text{Bid}_+(\mathcal{W}(a, b))$ contains only zero biderivation in all cases (i), (ii) and (iii). On the other hand, it is easy to see by [10] that $\text{Bid}_-(\mathcal{W}(a, b))$ contains only inner biderivation in these cases. Therefore, by Lemma 2.2 we deduce that any biderivation f of $\mathcal{W}(a, b)$ must be inner in all cases (i), (ii) and (iii).

Now, we summarize Cases 1–6 and complete the proof of Theorem 3.2. □

4. Post-Lie algebra structures on $\mathcal{W}(a, b)$

Recall that the post-Lie algebras have been introduced by Valette in connection with the homology of partition posets and the study of Koszul operads in [19]. As [5] pointed out, post-Lie algebras are natural common generalization of pre-Lie algebras and LR-algebras in the geometric context of nil-affine actions of Lie groups. Recently, many authors study some post-Lie algebras and post-Lie algebra structures (see [4, 5, 12, 14, 18]). In particular, the authors of [5] study the commutative post-Lie algebra structure on Lie algebra. By using our results, we can characterize the commutative post-Lie algebra structure on $\mathcal{W}(a, b)$. Let us recall the following definition of commutative post-Lie algebra.

Definition 4.1. [15] Let $(L, [\cdot, \cdot])$ be a complex Lie algebra. A commutative post-Lie algebra structure on L is a \mathbb{C} -bilinear product $x \circ y$ on L satisfying the following identities:

$$(4.1) \quad x \circ y = y \circ x, \quad [x, y] \circ z = x \circ (y \circ z) - y \circ (x \circ z), \quad x \circ [y, z] = [x \circ y, z] + [y, x \circ z]$$

for all $x, y, z \in L$. We also say $(L, [\cdot, \cdot], \circ)$ a commutative post-Lie algebra.

A post-Lie algebra $(L, [\cdot, \cdot], \circ)$ is said to be trivial if $x \circ y = 0$ for all $x, y \in L$. The following lemma shows the connection between commutative post-Lie algebra and biderivation of a Lie algebra.

Lemma 4.2. [15] *Suppose that $(L, [\cdot, \cdot], \circ)$ is a commutative post-Lie algebra. If we define a bilinear map $f: L \times L \rightarrow L$ by $f(x, y) = x \circ y$ for all $x, y \in L$, then f is a biderivation of L .*

Theorem 4.3. *Any commutative post-Lie algebra structure on the algebra $\mathcal{W}(a, b)$ is trivial.*

Proof. Suppose that $(\mathcal{W}(a, b), [\cdot, \cdot], \circ)$ is a commutative post-Lie algebra. By Lemma 4.2 and Theorem 3.2, we know that there are $\lambda, \mu \in \mathbb{C}$ and a sequence $\Omega = (\mu_k)_{k \in \mathbb{Z}}$ which contains only finitely many nonzero entries such that

$$x \circ y = \begin{cases} \lambda[x, y] + \Psi_{\Omega}(x, y) & \text{if } b = 0, \\ \lambda[x, y] + \Upsilon_{\Omega}^a(x, y) & \text{if } b = 1, \\ \lambda[x, y] + \Theta_{\mu}^a(x, y) & \text{if } a \in \mathbb{Z}, b = -1, \\ \lambda[x, y] & \text{otherwise} \end{cases}$$

for all $x, y \in \mathcal{H}$, where Ψ_{Ω} , Υ_{Ω}^a and Θ_{μ}^a are given by Definition 3.1. Because the product \circ is commutative, we have by $L_1 \circ L_2 = L_1 \circ L_2$ that $\lambda = \mu = 0$. By (4.1), we see that

$$[L_m, L_n] \circ L_t = L_m \circ (L_n \circ L_t) - L_n \circ (L_m \circ L_t)$$

for all $m, n, t \in \mathbb{Z}$. If there is $\mu_k \in \Omega$ such that $\mu_k \neq 0$, then it is easy to find some $m, n, t \in \mathbb{Z}$ such that the left-hand side of the above equation contains at least a nonzero item, whereas the right-hand side is equal to zero, which is a contradiction. Thus, we have $\mu_i = 0$ for any $i \in \mathbb{Z}$. In other words, $x \circ y = 0$ for all $x, y \in \mathcal{W}(a, b)$. \square

Acknowledgments

I would like to thank the referee for invaluable comments and suggestions. I also thank Dr. C. Xia for sending the paper [10], and thank Prof. K. Zhao for some suggestions to revise the paper.

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