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Toeplitz Operator for Dirichlet Space Through Sobolev Multiplier Algebra

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Abstract. This paper is mainly concerned with the Toeplitz operator T_ϕ over the Dirichlet space $\mathcal{D}$ with the symbol ϕ in the Sobolev multiplier algebra $M(W^{1,2}(\mathbb{D}))$, thereby extending several known ones in a very different manner.

1. Introduction

1.1. Sobolev and Dirichelt spaces

From now on, let

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \quad \text{and} \quad \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$$

be the open unit disk and the closed unit circle in the complex plane $\mathbb{C}$ respectively.

On the one hand, the squared Sobolev space $W^{1,2}(\mathbb{D})$ comprises all locally integrable functions f on $\mathbb{D}$ with

$$\|f\|_{1,2}^2 = \left| \int_{\mathbb{D}} f \, dA \right|^2 + \int_{\mathbb{D}} \left| \frac{\partial f}{\partial z} \right|^2 dA + \int_{\mathbb{D}} \left| \frac{\partial f}{\partial \overline{z}} \right|^2 dA < \infty,$$

where dA is the normalized area measure on $\mathbb{D}$ and $\left(\frac{\partial f}{\partial z}, \frac{\partial f}{\partial \overline{z}}\right)$ is a pair of the generalized derivatives. As is well-known, $W^{1,2}(\mathbb{D})$ is a Hilbert space with inner product

$$\begin{aligned} \langle f, g \rangle_{1,2} &= \left(\int_{\mathbb{D}} f \, dA \right) \overline{\left(\int_{\mathbb{D}} g \, dA \right)} + \left\langle \frac{\partial f}{\partial z}, \frac{\partial g}{\partial z} \right\rangle_{L^2(\mathbb{D})} + \left\langle \frac{\partial f}{\partial \overline{z}}, \frac{\partial g}{\partial \overline{z}} \right\rangle_{L^2(\mathbb{D})} \\ &= \left(\int_{\mathbb{D}} f \, dA \right) \overline{\left(\int_{\mathbb{D}} g \, dA \right)} + \int_{\mathbb{D}} \left(\frac{\partial f}{\partial z} \right) \overline{\left(\frac{\partial g}{\partial z} \right)} dA + \int_{\mathbb{D}} \left(\frac{\partial f}{\partial \overline{z}} \right) \overline{\left(\frac{\partial g}{\partial \overline{z}} \right)} dA. \end{aligned}$$

Associated with $W^{1,2}(\mathbb{D})$ is the Sobolev multiplier algebra:

$$M(W^{1,2}(\mathbb{D})) = \left\{ \phi \in W^{1,2}(\mathbb{D}) : M_\phi f = \phi f \in W^{1,2}(\mathbb{D}), f \in W^{1,2}(\mathbb{D}) \right\}. \tag{1.1}$$

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Interestingly, if $L^\infty(\mathbb{D})$ is the Lebesgue space of essentially bounded functions on $\mathbb{D}$, then the algebra $M(W^{1,2}(\mathbb{D}))$ in (1.1) contains the bounded Sobolev space

$$W^{1,\infty}(\mathbb{D}) = \left\{ f \in W^{1,2}(\mathbb{D}) : f, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial \overline{z}} \in L^\infty(\mathbb{D}) \right\}.$$

Here, it is perhaps appropriate to mention that any function in $W^{1,\infty}(\mathbb{D})$ enjoys such a nice property: $f \in W^{1,\infty}(\mathbb{D})$ if and only if there exist a constant $C > 0$ and a continuous function $\widetilde{f}$ on $\mathbb{D}$ such that $\widetilde{f} = f$ a.e. on $\mathbb{D}$ and

$$|\widetilde{f}(z) - \widetilde{f}(w)| \leq C|z - w|, \quad z, w \in \mathbb{D};$$

see [1, Theorem 5.4], [10, p. 279, Theorem 4] or [7, Proposition 3].

On the other hand, the Dirichlet space $\mathcal{D}$ consists of all holomorphic $W^{1,2}(\mathbb{D})$-functions f with $f(0) = 0$. It is well known that $\mathcal{D}$ is not only a reproducing Hilbert space with kernel

$$R_w(z) = \sum_{k=1}^{\infty} \frac{\overline{w}^k z^k}{k} = \log \frac{1}{1 - \overline{w}z}, \quad w, z \in \mathbb{D},$$

but also can be regarded as an orthogonal projection of $W^{1,2}(\mathbb{D})$ under

$$Pf(w) = \langle f, R_w \rangle_{1,2} = \int_{\mathbb{D}} \left(\frac{\partial f}{\partial z} \right) \overline{\left(\frac{\partial R_w(z)}{\partial z} \right)} \, dA(z), \quad f \in W^{1,2}(\mathbb{D}).$$

Interestingly, $\mathcal{D}$ can be treated as a typical member of the Dirichlet-type spaces, more precisely, for each $\alpha \in \mathbb{R}$, let $\mathcal{D}_\alpha$ be the space of holomorphic functions

$$f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n, \quad z \in \mathbb{D}$$

with

$$\|f\|_{\mathcal{D}_\alpha}^2 = \sum_{n=0}^{\infty} (n+1)^\alpha |\widehat{f}(n)|^2 < \infty,$$

and

$$\langle f, g \rangle_{\mathcal{D}_\alpha} = \sum_{n=0}^{\infty} (n+1)^\alpha \widehat{f}(n) \overline{\widehat{g}(n)}, \quad (f, g) \in \mathcal{D}_\alpha \times \mathcal{D}_\alpha.$$

As is well-known, $\mathcal{D}_0$ is the Hardy space $H^2(\mathbb{D}) =: H^2$, $\mathcal{D}_{-1}$ is the Bergman space $L_a^2(\mathbb{D}) =: L_a^2$, and the Dirichlet space $\mathcal{D}$ consists of all functions $f \in \mathcal{D}_1$ with $f(0) = 0$. Along this direction, it is worth mentioning two basic facts: the first is that a holomorphic function $f \in \mathcal{D}_\alpha$ amounts to $f' \in \mathcal{D}_{\alpha-2}$, see also [3] for details and some related facts; the second is that (cf. [7])

$$f \in W^{1,\infty}(\mathbb{D}) \text{ and } F = P[f|_{\mathbb{T}}] \quad \Longrightarrow \quad \frac{\partial F}{\partial z}, \frac{\partial F}{\partial \overline{z}} \in H^2 \quad \Longrightarrow \quad F \in \mathcal{D}_2 + \overline{\mathcal{D}_2},$$

in short,

$$P(W^{1,\infty}(\mathbb{D})|_{\mathbb{T}}) \subseteq \mathcal{D}_2 + \overline{\mathcal{D}_2}.$$

1.2. Toeplitz operators on $\mathcal{D}$ with Sobolev symbols

Given a function $\phi \in W^{1,\infty}(\mathbb{D})$, define the Toeplitz operator T_ϕ on $\mathcal{D}$ with the symbol ϕ by

$$T_\phi f = P[\phi f], \quad f \in \mathcal{D}. \tag{1.2}$$

It is not hard to see that T_ϕ decided by (1.2) is a bounded operator on $\mathcal{D}$. Now, the problem is how to extend the boundedness (and its immediately-induced properties) from the 'smallest' symbol class $W^{1,\infty}(\mathbb{D})$ to the 'biggest' symbol class $W^{1,2}(\mathbb{D})$. As a matter of fact, this problem has attached a lot of attention lately from various people; see e.g., [4, 5, 7, 9, 11–16, 18, 20], especially,

(1) Lee [11] considered the commutativity of two Toeplitz operators with harmonic symbols in $W^{1,\infty}(\mathbb{D})$.

(2) Chen and Nguyen [7] generalized Lee's results to the Toeplitz operators with general $W^{1,\infty}(\mathbb{D})$ symbols.

(3) Lee and Zhu [14] investigated finite sums of products of several Toeplitz operators with $W^{1,\infty}(\mathbb{D})$ symbols.

In this paper, keeping in mind the following fundamental fact

$$M(W^{1,2}(\mathbb{D})) \subseteq W^{1,2}(\mathbb{D}) \cap L^\infty(\mathbb{D}),$$

we will not just show that the main results in the just-mentioned three papers are actually true for Toeplitz operators on $\mathcal{D}$ with more general symbols, by using a method essentially different from any of the ones used in these papers, but also discover several more new results through exploiting the action of T_ϕ on $H^2(\mathbb{D})$. More precise information can be seen as below.

In §2, as the mid-process of this paper we handle the basic behaviour of T_ϕ on $\mathcal{D}$ with $\phi \in W^{1,2}(\mathbb{D})$; see Theorem 2.2 whose (i) was established in [7, Proposition 2] via a different argument, and whose (iv) shows that the boundedness of T_ϕ on $\mathcal{D}$ amounts to $P[\phi|_{\partial D}] \in L^\infty(\mathbb{D})$ and $|\partial_z P[\phi|_{\partial D}]|^2\, dA$ is a Carleson measure for $\mathcal{D}$.

In §3, as the major issue of this paper we discover two intrinsic properties of T_ϕ on $\mathcal{D}$ with $\phi \in M(W^{1,2}(\mathbb{D}))$, the first is Theorem 3.2 whose (i)–(ii) extend [12, Theorem 1.2], and whose (iii) corresponds nicely to Aleman-Vutotic's finite rank property of T_ϕ acting on $H^2(\mathbb{D})$ in [2, Theorem A]; the second is Theorem 3.6 showing that compactness of $\sum_{i=1}^n \prod_{j=1}^m T_{\phi_{ij}}$ on $\mathcal{D}$ with $\phi_{ij} \in M(W^{1,2}(\mathbb{D}))$ forces $\sum_{i=1}^n \prod_{j=1}^m \phi_{ij}|_{\mathbb{T}} = 0$.

Notation. Throughout this paper, we will use C for a general constant which may change from one line to another. We say two quantities A and B are equivalent, denoted by $A \asymp B$, if there exist constants $c, C > 0$ such that $cA \le B \le CA$.

2. T_ϕ on $\mathcal{D}$ with $\phi \in W^{1,2}(\mathbb{D})$

2.1. Nature of $W^{1,2}(\mathbb{D})$

First, a positive measure μ is called a Carleson measure for $\mathcal{D}$, denoted by $\mu \in \mathrm{CM}(\mathcal{D})$, if there is a constant $C > 0$ such that

$$\left(\int_{\mathbb{D}} |f|^2 \, d\mu\right)^{1/2} \leq C\|f\|_{\mathcal{D}}, \quad f \in \mathcal{D}.$$

The smallest C is called the square of the Carleson measure norm of μ, denoted by $\|\mu\|_{\mathrm{CM}(\mathcal{D})}$.

Secondly, we identify the functions in $W^{1,2}(\mathbb{D})$. From the Poincaré inequality [10, p. 275, Theorem 1] it follows that for any $f \in W^{1,2}(\mathbb{D})$ there exists a constant C (independent of f) enjoying

$$\left\| f - \int_{\mathbb{D}} f \, dA \right\|_{L^2(\mathbb{D})} \leq C \left(\left\| \frac{\partial f}{\partial z} \right\|_{L^2(\mathbb{D})} + \left\| \frac{\partial f}{\partial \bar{z}} \right\|_{L^2(\mathbb{D})} \right).$$

This implies

$$\begin{cases} f \in L^2(\mathbb{D}), \\ \|f\|_{L^2(\mathbb{D})} \leq C\|f\|_{1,2} \qquad \text{for a constant } C > 0, \\ \|f\|_{1,2}^2 \asymp \|f\|_{L^2(\mathbb{D})}^2 + \int_{\mathbb{D}} \left|\frac{\partial f}{\partial z}\right|^2 dA + \int_{\mathbb{D}} \left|\frac{\partial f}{\partial \bar{z}}\right|^2 dA. \end{cases}$$

Accordingly,

$$W^{1,2}(\mathbb{D}) = \left\{ f \in L^2(\mathbb{D}) : \frac{\partial f}{\partial z}, \frac{\partial f}{\partial \bar{z}} \in L^2(\mathbb{D}) \right\},$$

where the derivatives are taken in the sense of distributions. We will use $\|\cdot\|_{1,2}$ as the norm of $W^{1,2}(\mathbb{D})$ unless otherwise specified. Note that

$$\|f\|_{L^2(\mathbb{D})} \leq C\|f\|_{1,2}, \quad f \in W^{1,2}(\mathbb{D})$$

holds for a constant $C > 0$. So, by the closed graph theorem, every function $\phi \in M(W^{1,2}(\mathbb{D}))$ induces that $M_\phi \colon f \to \phi f$ exists as a bounded linear operator on $W^{1,2}(\mathbb{D})$.

From [7, p. 370] we see such a statement that if $f \in W^{1,2}(\mathbb{D})$ then $(r,\theta) \mapsto f(re^{i\theta})$ belongs to $W^{1,2}(E)$ with $E = [0,1) \times [0,2\pi)$. But nevertheless we remark here that the statement is not generally true, in fact, if

$$f(z) = |z|^\alpha \quad \text{under } 0 < \alpha < \frac{1}{2},$$

then $f \in W^{1,2}(\mathbb{D})$ (see [10, p. 246]) and however

$$(r,\theta) \mapsto f(re^{i\theta}) = r^\alpha$$

is not in $W^{1,2}(E)$ because

$$r \mapsto \frac{\partial f}{\partial r} = \alpha r^{\alpha-1}$$

is not in $L^2(0,1)$. If

$$C^1(\overline{\mathbb{D}}) = \left\{ f \in C^1(\mathbb{D}) : \frac{\partial f}{\partial z}, \frac{\partial f}{\partial \bar{z}} \text{ are continuous on the closed unit disk } \overline{\mathbb{D}} \right\},$$

then $C^1(\overline{\mathbb{D}})$ is dense in $W^{1,2}(\mathbb{D})$. By [10, p. 258, Theorem 1], we have

$$W^{1,2}(\mathbb{D}) \subseteq L^2(\mathbb{T})$$

and a constant $C > 0$ enjoying

$$\|f|_{\mathbb{T}}\|_{L^2(\mathbb{T})} \leq C\|f\|_{1,2}, \quad f \in W^{1,2}(\mathbb{D}).$$

Here and henceforth, this last $f|_{\mathbb{T}}$ is treated as the trace of f on $\mathbb{T}$, and can be calculated by

$$f|_{\mathbb{T}} = \lim_{n\to\infty} f_n|_{\mathbb{T}} \quad \text{(taken in norm)},$$

where $f_n \in C^1(\overline{\mathbb{D}})$ and f_n converges to f in $W^{1,2}(\mathbb{D})$.

Lemma 2.1. *If*

$$f \in W^{1,2}(\mathbb{D}) \quad \textit{and} \quad f_k(1) = \int_0^{2\pi} f|_{\mathbb{T}}(e^{i\theta})e^{-ik\theta}\frac{d\theta}{2\pi}, \quad k \in \mathbb{Z},$$

then

$$Pf(z) = \int_{\mathbb{T}} ze^{-i\theta}\frac{f(e^{i\theta})}{1 - ze^{-i\theta}}\frac{d\theta}{2\pi} = \sum_{k=1}^{\infty} f_k(1)z^k.$$

Consequently, if

$$\Lambda = \left\{ f - \sum_{k\in\mathbb{Z}\setminus\{0\}} f_k(1)r^{|k|}e^{ik\theta} : f \in W^{1,2}(\mathbb{D}) \right\},$$

then

$$W^{1,2}(\mathbb{D}) = \Lambda \oplus \mathcal{D} \oplus \overline{\mathcal{D}}.$$

Proof. To verify this projection formula, for $f \in W^{1,2}(\mathbb{D})$ let ∇f be the gradient of f, i.e., $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$, then

$$|\nabla f|^2 = \left|\frac{\partial f}{\partial x}\right|^2 + \left|\frac{\partial f}{\partial y}\right|^2 = 2\left[\left|\frac{\partial f}{\partial z}\right|^2 + \left|\frac{\partial f}{\partial \bar{z}}\right|^2\right],$$

and hence for $g \in W^{1,2}(\mathbb{D})$,

$$\begin{aligned}\langle f, g\rangle_{1,2} &= \left(\int_{\mathbb{D}} f\,dA\right)\overline{\left(\int_{\mathbb{D}} g\,dA\right)} + \int_{\mathbb{D}}\left(\left(\frac{\partial f}{\partial z}\right)\overline{\left(\frac{\partial g}{\partial z}\right)} + \left(\frac{\partial f}{\partial \overline{z}}\right)\overline{\left(\frac{\partial g}{\partial \overline{z}}\right)}\right) dA \\ &= \left(\int_{\mathbb{D}} f\,dA\right)\overline{\left(\int_{\mathbb{D}} g\,dA\right)} + \frac{1}{2}\int_{\mathbb{D}} \nabla f \cdot \nabla \overline{g}\,dA.\end{aligned}$$

By approximation, we only need to prove the lemma for all functions $f \in C^1(\overline{\mathbb{D}})$. By Green's first identity, if $u, v \in C^1(\overline{\mathbb{D}})$, then

$$\int_{\mathbb{D}} (\Delta u)v + \nabla u \cdot \nabla v\,dxdy = \int_{\mathbb{T}} v\frac{\partial u}{\partial \vec{n}}\,d\theta,$$

where $\vec{n}$ is the outward unit normal vector, and

$$\frac{\partial u}{\partial \vec{n}} = \nabla u \cdot \vec{n}.$$

Consequently, an application of the last formula for $\langle f, g\rangle_{1,2}$ gives

$$\begin{aligned}Pf(z) = \langle f, R_z\rangle_{1,2} &= \frac{1}{2}\int_{\mathbb{D}} \nabla f \cdot \nabla\overline{R_z}\,dA \\ &= \frac{1}{2\pi}\left[\int_{\mathbb{D}} \nabla f \cdot \nabla \overline{R_z}\,dxdy + \int_{\mathbb{D}} (\Delta\overline{R_z}) f\,dxdy\right] \\ &= \int_{\mathbb{T}} f\frac{\partial \overline{R_z}}{\partial \vec{n}}\frac{d\theta}{2\pi},\end{aligned}$$

where we have used $dA = \frac{1}{\pi}\,dxdy$ and $\Delta\overline{R_z} = 0$. Recall that

$$R_z(w) = \log\frac{1}{1 - \overline{z}w},$$

by a simple calculation, we get

$$\left.\frac{\partial R_z(w)}{\partial \vec{n}}\right|_{w=e^{i\theta}} = \frac{\overline{z}e^{i\theta}}{1 - \overline{z}e^{i\theta}},$$

thereby finding

$$P[f](z) = \int_{\mathbb{T}} ze^{-i\theta}\frac{f(e^{i\theta})}{1 - ze^{-i\theta}}\frac{d\theta}{2\pi} = \sum_{k=1}^{\infty} f_k(1)z^k.$$

Next, observe that $P\colon W^{1,2}(\mathbb{D}) \to \mathcal{D}$ is bounded. By the above formula for Pf we have

$$\sum_{k=1}^{\infty} k|f_k(1)|^2 < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} k|f_{-k}(1)|^2 < \infty.$$

Thus

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} f_k(1)r^{|k|}e^{ik\theta} \in \mathcal{D} \oplus \overline{\mathcal{D}}.$$

So, each $f \in W^{1,2}(\mathbb{D})$ can be represented as

$$f = f - \sum_{k\in\mathbb{Z}\setminus\{0\}} f_k(1) r^{|k|} e^{ik\theta} + \sum_{k\in\mathbb{Z}\setminus\{0\}} f_k(1) r^{|k|} e^{ik\theta}.$$

The previous argument actually shows

$$P\left(f - \sum_{k\in\mathbb{Z}\setminus\{0\}} f_k(1) r^{|k|} e^{ik\theta}\right) = 0,$$

and thus

$$f - \sum_{k\in\mathbb{Z}\setminus\{0\}} f_k(1) r^{|k|} e^{ik\theta} \perp \mathcal{D}, \quad f - \sum_{k\in\mathbb{Z}\setminus\{0\}} f_k(1) r^{|k|} e^{ik\theta} \perp \overline{\mathcal{D}}.$$

This in turn reveals the desired decomposition:

$$W^{1,2}(\mathbb{D}) = \Lambda \oplus \mathcal{D} \oplus \overline{\mathcal{D}};$$

see also [7, Theorem 1] for an analogous argument. □

2.2. T_ϕ on $\mathcal{D}$ with $\phi \in W^{1,2}(\mathbb{D})$

The Toeplitz operator can be defined for a Sobolev symbol. To be more precise, for $\phi \in W^{1,2}(\mathbb{D})$, define the Toeplitz operator T_ϕ on $\mathcal{D}$ as

$$T_\phi f(w) = \int_{\mathbb{D}} \left(\frac{\partial(\phi f)}{\partial z}\right) \overline{\left(\frac{\partial R_w(z)}{\partial z}\right)} \, dA(z), \quad (f, w) \in \mathcal{D} \times \mathbb{D}. \tag{2.1}$$

Theorem 2.2. *Let*

$$\phi \in W^{1,2}(\mathbb{D}), \quad \Phi(re^{i\theta}) = P[\phi|_{\mathbb{T}}](re^{i\theta}) = \sum_{k\in\mathbb{Z}} \phi_k(1) r^{|k|} e^{ik\theta}, \quad \psi = \phi - P[\phi|_{\mathbb{T}}].$$

Then

(i)
$$T_\phi(z^n)(w) = \int_{\mathbb{T}} we^{-i\theta} \frac{(z^n\phi)(e^{i\theta})}{1 - we^{-i\theta}} \frac{d\theta}{2\pi} = \sum_{k=1}^{\infty} \phi_{k-n}(1) w^k, \quad n \in \mathbb{Z}_+.$$

(ii) $T_\psi(z^n) = 0$, $n \in \mathbb{Z}_+$.

(iii) *If T_ϕ is a bounded operator on $\mathcal{D}$, then it is determined by the boundary function $\phi|_{\mathbb{T}}$.*

(iv) *T_ϕ is bounded on $\mathcal{D}$ if and only if $\Phi \in L^\infty(\mathbb{D})$ and $\left|\frac{\partial \Phi}{\partial z}\right|^2 dA$ is a Carleson measure for $\mathcal{D}$.*

(v) *The following are equivalent:*

(a) $T_\phi = 0$ *on* $\mathcal{D}$,

(b) $\phi|_{\mathbb{T}} = 0$,

(c) T_ϕ *is compact on* $\mathcal{D}$.

Proof. (i) This (i.e., [7, Proposition 2]) follows from Lemma 2.1.

(ii) This follows from $\psi = \phi - P[\phi|_{\mathbb{T}}]$ and (i) above.

(iii) This follows from either (i) or the Green's first identity.

(iv) Suppose that T_ϕ is bounded on $\mathcal{D}$. An application of (2.1) gives that $T_\Phi = T_\phi$ is bounded on $\mathcal{D}$. Motivated somewhat by [6, Theorems 2.2–2.3], we consider

$$\Phi_1(z) = \sum_{k=1}^{\infty} \phi_k(1) z^k, \quad \overline{\Phi_2(z)} = \sum_{k=1}^{\infty} \phi_{-k}(1) \overline{z}^k, \quad c = \phi_0(1).$$

Then

$$\Phi_1, \Phi_2 \in \mathcal{D} \quad \text{and} \quad \Phi = \Phi_1 + \overline{\Phi_2} + c.$$

For $a \in \mathbb{D}$, let

$$h_a(z) = \frac{1-|a|^2}{1-\overline{a}z} z.$$

Then $\|h_a\|_{\mathcal{D}} = 1$, and $h_a'(z) = k_a(z)$ is the normalized reproducing kernel for $L_a^2(\mathbb{D})$. Moreover, we have

$$\tag{2.2} \begin{aligned} \langle T_\Phi h_a, h_a \rangle_{\mathcal{D}} &= \left\langle \frac{\partial \Phi}{\partial z} h_a, k_a \right\rangle_{L^2(\mathbb{D})} + \langle \Phi k_a, k_a \rangle_{L^2(\mathbb{D})} \\ &= \langle \Phi_1' h_a, k_a \rangle_{L_a^2} + \Phi(a) \\ &= a(1-|a|^2)\Phi_1'(a) + \Phi(a). \end{aligned}$$

If

$$\begin{aligned} \mathcal{B} &= \left\{ f \in H(\mathbb{D}) : \sup_{z \in \mathbb{D}} (1-|z|^2)|f'(z)| < \infty \right\}, \\ \mathcal{B}_0 &= \left\{ f \in \mathcal{B} : \lim_{|z| \to 1} (1-|z|^2)|f'(z)| = 0 \right\} \end{aligned}$$

are the Bloch space and its little one respectively, then $\mathcal{D}$ is contained in $\mathcal{B}_0$ (cf. [21]). So, it follows that

$$(1-|a|^2)|\Phi_1'(a)| \to 0 \quad \text{as } |a| \to 1,$$

and Φ is bounded thanks to (2.2). Notice that for each $g, h \in \mathcal{D}$,

$$\langle T_\Phi g, h \rangle_{\mathcal{D}} = \langle \Phi_1' g + \Phi g', h \rangle_{L^2(\mathbb{D})}.$$

Thus

$$\|\Phi_1' g\|_{L^2(\mathbb{D})} \le \|T_\Phi g\|_{\mathcal{D}} + \|\Phi g'\|_{L^2(\mathbb{D})} \le C\|g\|_{\mathcal{D}}.$$

As a result, we find that

$$|\Phi_1'|^2\, dA = \left|\frac{\partial \Phi}{\partial z}\right|^2 dA$$

is a Carleson measure for $\mathcal{D}$. The converse part of (iv) is evident.

(v) The equivalence (a) $\Leftrightarrow$ (b) can be found in [20, Theorem 3.1]. Trivially, (a) implies (c), so, it is enough to show that (c) implies (b). Using

$$\Phi = P[\phi|_{\mathbb{T}}]$$

and noticing that h_a converges to 0 weakly in $\mathcal{D}$ as $|a| \to 1$, we get that if (c) holds then an application of (2.2) yields

$$|\Phi(a)| \to 0 \quad \text{as } |a| \to 1,$$

and thus (b) follows from $\phi|_{\mathbb{T}} = \Phi|_{\mathbb{T}} = 0$. □

3. T_ϕ on $\mathcal{D}$ with $\phi \in M(W^{1,2}(\mathbb{D}))$

3.1. Structure of $M(W^{1,2}(\mathbb{D}))$

This part can be seen from the following implications (cf. [19]):

Lemma 3.1. *Let $H(\mathbb{D})$ be the class of all holomorphic functions on $\mathbb{D}$.*

(i) *If $\phi \in M(W^{1,2}(\mathbb{D}))$, then $\phi \in L^\infty(\mathbb{D})$.*

(ii) *If $\phi \in M(W^{1,2}(\mathbb{D})) \cap H(\mathbb{D})$, then*

$$\phi \in M(\mathcal{D}) = \{\phi \in H(\mathbb{D}) : M_\phi f = \phi f \in \mathcal{D}, f \in \mathcal{D}\},$$

equivalently, $\phi \in H^\infty(\mathbb{D}) \cap \mathcal{X}(D)$, where

$$H^\infty(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \sup_{z\in\mathbb{D}} |f(z)| < \infty \right\},$$
$$\mathcal{X}(D) = \left\{ f \in \mathcal{D} : |f'|^2\, dA \text{ is a Carleson measure for } \mathcal{D} \right\}.$$

(iii) *If $\phi \in M(W^{1,2}(\mathbb{D})) \cap H(\mathbb{D})$ and $u \in W^{1,2}(\mathbb{D})$, then $P(\overline{\phi} P(u)) = P(\overline{\phi} u)$.*

Proof. (i) Note that $M_\phi \colon f \to \phi f$ is the pointwise multiplication associated with ϕ. So it obeys

$$\|\phi^2\|_{1,2} \le \|M_\phi\| \|\phi\|_{1,2} \le \|M_\phi\|^2.$$

This gives by induction

$$\|\phi^n\|_{1,2} \le \|M_\phi\|^n,$$

and then

$$\|\phi^n\|_{L^2(\mathbb{D})}^{1/n} \le [C\|\phi^n\|_{1,2}]^{1/n} \le C^{1/n}\|M_\phi\|.$$

Consequently, letting $n \to \infty$ yields

$$\|\phi\|_{L^\infty(\mathbb{D})} \le \|M_\phi\|.$$

(ii) If $f \in \mathcal{D}$, then $\phi f \in W^{1,2}(\mathbb{D})$. Since ϕf is holomorphic with $(\phi f)(0) = 0$, it follows that $\phi \in M(\mathcal{D})$.

(iii) This result was proved in [11, Lemma 3] for $\phi \in W^{1,\infty}(\mathbb{D}) \cap H(\mathbb{D})$. The same proof works for $\phi \in M(W^{1,2}(\mathbb{D})) \cap H(\mathbb{D})$. □

3.2. Finite rank $\prod_{j=1}^n T_{\phi_j}$ on $\mathcal{D}$ with $\phi_j \in M(W^{1,2}(\mathbb{D}))$

First of all, it should be pointed out that if

$$\phi \in M(W^{1,2}(\mathbb{D})), \quad \Phi = P[\phi|_{\mathbb{T}}],$$

then $T_\phi = T_\Phi$ is bounded on $\mathcal{D}$.

Next, we take a look at T_ϕ on $\mathcal{D}$ via the action of T_ϕ on $H^2(\mathbb{D})$ with $\phi \in M(W^{1,2}(\mathbb{D}))$. Given $\phi \in L^\infty(\mathbb{T})$, let $t_\phi\colon H^2(\mathbb{D}) \to H^2(\mathbb{D})$ be the Toeplitz operator defined by

$$t_\phi f = P_{H^2}(\phi f),$$

where P_{H^2} is the orthogonal projection from $L^2(\mathbb{T})$ to $H^2(\mathbb{D})$. Then

$$t_\phi f(z) = \langle \phi f, Q_z\rangle_{L^2(\mathbb{T})} = \int_{\mathbb{T}} \frac{(\phi f)(e^{i\theta})}{1 - ze^{-i\theta}}\frac{d\theta}{2\pi} = \sum_{k=0}^\infty (\phi f)_k(1) z^k. \tag{3.1}$$

By Lemma 2.2, $\Phi \in L^\infty(\mathbb{D})$, thus $\phi|_{\mathbb{T}} \in L^\infty(\mathbb{T})$ and so we have the corresponding Toeplitz operator $t_{\phi|_{\mathbb{T}}}$ on $H^2(\mathbb{D})$. For convenience, we will write t_ϕ for $t_{\phi|_{\mathbb{T}}}$.

In [2], Aleman and Vukotic proved that if the product of n Toeplitz operators on $H^2(\mathbb{D})$ has finite rank then at least one of symbols of the operator is zero almost everywhere. Such a finite rank property can be carried over to $\mathcal{D}$ through the following result whose (i) and (ii) extend [12, Theorem 1.2] and whose (iii) extends [20, Lemma 3.3].

Theorem 3.2. *For $n \in \mathbb{Z}_+$ let $\phi, \phi_1, \ldots, \phi_n \in M(W^{1,2}(\mathbb{D}))$.*

(i) *If $f \in \mathcal{D}$, then $T_\phi f = zt_\phi g$, where $g = f/z$.*

(ii) *If $g \in H^2(\mathbb{D})$, then $(T_\phi^* h)' = t_\phi^* g$, where $h(z) = \int_0^z g(\zeta)\, d\zeta$.*

(iii) *If $T_{\phi_1}T_{\phi_2}\cdots T_{\phi_n}$ has finite rank on $\mathcal{D}$, then there exists some $i\in\{1,\dots,n\}$ such that $\phi_i=0$ almost everywhere on $\mathbb{T}$.*

Proof. (i) By (3.1), we have

$$t_\phi\frac{f}{z}=\int_{\mathbb{T}}\frac{\big(\phi\frac{f}{z}\big)(e^{i\theta})}{1-ze^{-i\theta}}\frac{d\theta}{2\pi}=\int_{\mathbb{T}}e^{-i\theta}\frac{(\phi f)(e^{i\theta})}{1-ze^{-i\theta}}\frac{d\theta}{2\pi}.$$

It follows from Lemma 2.1 that

$$T_\phi f=\int_{\mathbb{T}}e^{-i\theta}z\frac{(\phi f)(e^{i\theta})}{1-ze^{-i\theta}}\frac{d\theta}{2\pi}=zt_\phi\frac{f}{z}.$$

(ii) By

$$\langle t_\phi^*g,z^n\rangle_{H^2}=\langle g,t_\phi z^n\rangle_{H^2}=\sum_{k=0}^{\infty}g_k(1)\overline{(\phi z^n)_k(1)},$$

we have

$$t_\phi^*g=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}g_k(1)\overline{(\phi z^n)_k(1)}z^n.$$

Similarly, we get

$$\langle T_\phi^*h,z^n\rangle_{\mathcal{D}}=\langle h,T_\phi z^n\rangle_{\mathcal{D}}=\sum_{k=1}^{\infty}kh_k(1)\overline{(\phi z^n)_k(1)},\quad h\in\mathcal{D},$$

whence reaching

$$T_\phi^*h=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}kh_k(1)\overline{(\phi z^n)_k(1)}\frac{z^n}{n}=\sum_{l=0}^{\infty}\sum_{i=0}^{\infty}(i+1)h_{i+1}(1)\overline{(\phi z^l)_i(1)}\frac{z^{l+1}}{l+1}.$$

If

$$g\in H^2(\mathbb{D}),\quad h(z)=\int_0^z g(\zeta)\,d\zeta=\sum_{k=0}^{\infty}g_k(1)\frac{z^{k+1}}{k+1},$$

then

$$(i+1)h_{i+1}(1)=g_i(1),$$

hence

$$T_\phi^*h=\sum_{l=0}^{\infty}\sum_{i=0}^{\infty}g_i(1)\overline{(\phi z^l)_i(1)}\frac{z^{l+1}}{l+1},$$

and so $(T_\phi^*h)'=t_\phi^*g$.

(iii) If $f\in\mathcal{D}$, then an application of (i) gives

$$T_{\phi_1}T_{\phi_2}\cdots T_{\phi_n}f=zt_{\phi_1}t_{\phi_2}\cdots t_{\phi_n}\frac{f}{z}.$$

Since the space

$$\frac{\mathcal{D}}{z} := \left\{\frac{f}{z} : f \in \mathcal{D}\right\}$$

is dense in $H^2(\mathbb{D})$, it follows from the hypothesis that the operator $t_{\phi_1} t_{\phi_2} \cdots t_{\phi_n}$ has finite rank on $H^2(\mathbb{D})$. Thus by the remark right after [2, Theorem A], we obtain that ϕ_i vanishes almost everywhere on $\mathbb{T}$ for some $i \in \{1, \ldots, n\}$. □

Corollary 3.3. *Let $\phi \in M(W^{1,2}(\mathbb{D}))$. Then:*

(i) *both* $\dim\ker T_\phi \le \dim\ker t_\phi$ *and* $\dim\ker t_\phi^* \le \dim\ker T_\phi^*$ *hold;*

(ii) $\operatorname{ran} t_\phi$ *is dense in* $H^2(\mathbb{D})$ *whenever* $\operatorname{ran} T_\phi$ *is dense in* $\mathcal{D}$*;*

(iii) $\operatorname{ran} T_\phi^*$ *is dense in* $\mathcal{D}$ *whenever* $\operatorname{ran} t_\phi^*$ *is dense in* $H^2(\mathbb{D})$.

Proof. (i) Let $\{f_1, \ldots, f_n\}$ be any linearly independent vectors in $\ker T_\phi$. If

$$g_i = \frac{f_i}{z}, \quad i \in \{1, \ldots, n\},\ n \in \mathbb{Z}_+,$$

then Theorem 3.2(i) is used to imply

$$T_\phi f_i = z t_\phi g_i = 0,$$

hence

$$\{g_i\}_{i=1}^n \subseteq \ker t_\phi.$$

Since $f_1, \ldots, f_n$ are linearly independent, $g_1, \ldots, g_n$ are linearly independent and

$$\dim\ker T_\phi \le \dim\ker t_\phi.$$

Similarly, by Theorem 3.2(ii), we have

$$\dim\ker t_\phi^* \le \dim\ker T_\phi^*.$$

Obviously, (ii) and (iii) follow from (i). □

Remark 3.4. (b) and (c) also follow easily from the following two inequalities for integer $k \ge 0$:

$$\left\| t_\phi \frac{f}{z} - z^k \right\|_{H^2} = \|T_\phi f - z^{k+1}\|_{H^2} \le \|T_\phi f - z^{k+1}\|_{\mathcal{D}}, \quad f \in \mathcal{D}$$

and

$$\left\| T_\phi^* \left(\int_0^z g(\zeta)\, d\zeta \right) - \frac{z^{k+1}}{k+1} \right\|_{\mathcal{D}} = \|t_\phi^* g - z^k\|_{L_a^2} \le \|t_\phi^* g - z^k\|_{H^2}, \quad g \in H^2(\mathbb{D}).$$

3.3. Compactness of $\sum_{i=1}^{n}\prod_{j=1}^{m} T_{\phi_{ij}}$ on $\mathcal{D}$ with $\phi_{ij} \in M(W^{1,2}(\mathbb{D}))$

For a pair $(e^{i\theta}, \alpha) \in \mathbb{T} \times (1, \infty)$, let

$$\Gamma_\alpha(e^{i\theta}) = \{z \in \mathbb{D} : |z - e^{i\theta}| < \alpha(1 - |z|)\}$$

be the nontangential region with vertex $e^{i\theta}$. A function f, defined on $\mathbb{D}$, is said to have nontangential limit L at $e^{i\theta}$, if $f(z) \to L$ as $z \to e^{i\theta}$ within any nontangential region $\Gamma_\alpha(e^{i\theta})$ (see [17]). It is known that if $f \in L^1(\mathbb{T})$, then $P[f]$ has nontangential limit $f(e^{i\theta})$ at almost all points of $\mathbb{T}$. Recall that

$$\phi_a(z) = \frac{a - z}{1 - \bar{a}z}, \quad a \in \mathbb{D}$$

and

$$P_\zeta(z) = \frac{1 - |z|^2}{|\zeta - z|^2}, \quad \zeta \in \mathbb{T}$$

are the Möbius transform and the Possion kernel, respectively.

Lemma 3.5. *Let $n \in \mathbb{Z}_+$ and $i \in \{1, \ldots, n\}$.*

(i) *If $u_i \in L^\infty(\mathbb{T})$, then for almost every $\zeta \in \mathbb{T}$,*

$$t_{u_n \circ \phi_a} t_{u_{n-1} \circ \phi_a} \cdots t_{u_1 \circ \phi_a} 1 \to \prod_{i=1}^{n} u_i(\zeta) \quad \textit{in } H^2(\mathbb{D}) \textit{ as } a \to \zeta \textit{ nontangentially.}$$

(ii) *If p, q are polynomials, then*

$$\langle (zp)', q\rangle_{L_a^2} = \langle p, q\rangle_{H^2}.$$

Proof. (i) We will use induction argument to prove the lemma. Fix $\zeta \in \mathbb{T}$ such that $P[u_1], \ldots, P[u_n]$ have finite nontangential limit at ζ. Note that

$$\begin{aligned}\|u_1 \circ \phi_a - u_1(\zeta)\|_{L^2(\mathbb{T})}^2 &= \int_{\mathbb{T}} |u_1(\phi_a(t)) - u_1(\zeta)|^2 \frac{|dt|}{2\pi} \\ &= \int_{\mathbb{T}} |u_1(s) - u_1(\zeta)|^2 P_a(s) \frac{|ds|}{2\pi} \\ &= P[|u_1 - u_1(\zeta)|^2](a) \to 0 \quad (\text{as } a \to \zeta \text{ nontangentially}).\end{aligned}$$

Thus

$$t_{u_1 \circ \phi_a} 1 \to u_1(\zeta) \quad \text{in } H^2(\mathbb{D}) \text{ as } a \to \zeta \text{ nontangentially.}$$

Suppose

$$t_{u_{n-1} \circ \phi_a} \cdots t_{u_1 \circ \phi_a} 1 \to \prod_{i=1}^{n-1} u_i(\zeta) \quad \text{in } H^2(\mathbb{D}) \text{ as } a \to \zeta \text{ nontangentially.}$$

Note that

$$\begin{aligned} & t_{u_n\circ\phi_a}t_{u_{n-1}\circ\phi_a}\cdots t_{u_1\circ\phi_a}1-\prod_{i=1}^{n}u_i(\zeta) \\ &= t_{u_n\circ\phi_a}\left[t_{u_{n-1}\circ\phi_a}\cdots t_{u_1\circ\phi_a}1-\prod_{i=1}^{n-1}u_i(\zeta)\right]+\prod_{i=1}^{n-1}u_i(\zeta)\,[t_{u_n\circ\phi_a}1-u_n(\zeta)] \end{aligned}$$

and that

$$\|t_{u_n\circ\phi_a}\|_{B(H^2)}\le\|u_n\|_{L^\infty}.$$

So, it follows that

$$t_{u_n\circ\phi_a}t_{u_{n-1}\circ\phi_a}\cdots t_{u_1\circ\phi_a}1\to\prod_{i=1}^{n}u_i(\zeta)\quad\text{in } H^2(\mathbb{D}) \text{ as } a\to\zeta \text{ nontangentially.}$$

(ii) This follows from a straightforward computation. □

Given $a\in\mathbb{D}$, let $U_a\colon H^2(\mathbb{D})\to H^2(\mathbb{D})$ be defined by

$$U_af=(f\circ\phi_a)q_a,$$

where

$$q_a(z)=\frac{(1-|a|^2)^{1/2}}{1-\overline{a}z}$$

is the normalized reproducing kernel for $H^2(\mathbb{D})$. Since $U_aq_a=1$, it follows that U_a is a unitary operator with

$$U_a^*=U_a^{-1}=U_a\quad\text{and}\quad U_at_\phi U_a=t_{\phi\circ\phi_a},\quad \phi\in L^\infty(\mathbb{T}).$$

Moreover, given $\phi_1,\dots,\phi_n\in L^\infty(\mathbb{T})$, it is known that if $\prod_{i=1}^n t_{\phi_i}$ is compact on $H^2(\mathbb{D})$ then $\prod_{i=1}^n\phi_i|_{\mathbb{T}}=0$ (cf. [8]). Below is an analogue of this implication for $\mathcal{D}$.

Theorem 3.6. *For $(i,j)\in\{1,\dots,n\}\times\{1,\dots,m\}$ and $(n,m)\in\mathbb{Z}_+\times\mathbb{Z}_+$ let $\phi_{ij}\in M(W^{1,2}(\mathbb{D}))$. If $\sum_{i=1}^n\prod_{j=1}^m T_{\phi_{ij}}$ is compact on $\mathcal{D}$, then*

$$\sum_{i=1}^{n}\prod_{j=1}^{m}\phi_{ij}|_{\mathbb{T}}=0.$$

Proof. Let

$$T=\sum_{i=1}^{n}\prod_{j=1}^{m}T_{\phi_{ij}}\quad\text{and}\quad t=\sum_{i=1}^{n}\prod_{j=1}^{m}t_{\phi_{ij}}.$$

By Theorem 3.2, we have

$$T_{\phi_{ij}}f=zt_{\phi_{ij}}\frac{f}{z},\quad f\in\mathcal{D}.$$

This yields $Tf = zt\frac{f}{z}$. Thus, an application of Lemma 3.5(ii) yields

$$\begin{aligned}\langle Th_a, h_a\rangle_{\mathcal{D}} &= \left\langle zt\frac{h_a}{z}, h_a\right\rangle_{\mathcal{D}} = \left\langle \left(zt\frac{h_a}{z}\right)', (h_a)'\right\rangle_{L^2_a} = \left\langle t\frac{h_a}{z}, k_a\right\rangle_{H^2} \\ &= \left\langle t\left(\frac{1-|a|^2}{1-\overline{a}z}\right), \frac{1-|a|^2}{(1-\overline{a}z)^2}\right\rangle_{H^2} = \left\langle tq_a, \frac{(1-|a|^2)^{3/2}}{(1-\overline{a}z)^2}\right\rangle_{H^2} \\ &= \left\langle U_atU_aU_aq_a, U_a\frac{(1-|a|^2)^{3/2}}{(1-\overline{a}z)^2}\right\rangle_{H^2} = \langle U_atU_a1, 1-\overline{a}z\rangle_{H^2}.\end{aligned}$$

Since T is compact on $\mathcal{D}$ and h_a converges weakly to 0 in $\mathcal{D}$ as $|a| \to 1$, it follows that

$$\langle Th_a, h_a\rangle_{\mathcal{D}} = \langle U_atU_a1, 1-\overline{a}z\rangle_{H^2} \to 0 \quad \text{as } |a| \to 1. \tag{3.2}$$

Note that

$$U_atU_a1 = \sum_{i=1}^{n}\prod_{j=1}^{m} t_{\phi_{ij}\circ\phi_a}.$$

By Lemma 3.5(i) we have for almost every $\zeta \in \mathbb{T}$,

$$U_atU_a1 \to \sum_{i=1}^{n}\prod_{j=1}^{m}\phi_{ij}(\zeta) \quad \text{in } H^2(\mathbb{D}) \text{ as } a \to \zeta \text{ nontangentially},$$

whence

$$\langle U_atU_a1, 1-\overline{a}z\rangle_{H^2} \to \sum_{i=1}^{n}\prod_{j=1}^{m}\phi_{ij}(\zeta) \quad \text{as } a \to \zeta \text{ nontangentially}.$$

This, together with (3.2), derives

$$\sum_{i=1}^{n}\prod_{j=1}^{m}\phi_{ij}(\zeta) = 0 \quad \text{for almost every } \zeta \in \mathbb{T},$$

whence completing the argument. □

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