

SOLUTION OF AN EXTRAORDINARY DIFFERENTIAL EQUATION BY ADOMIAN DECOMPOSITION METHOD

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The aim of the present analysis is to apply the Adomian decomposition method for the solution of a fractional differential equation as an alternative method of Laplace transform.

1. Introduction

Large classes of linear and nonlinear differential equations, both ordinary as well as partial, can be solved by the Adomian decomposition method [3, 4, 6]. This method is much more simpler in computation and quicker in convergence than any other method available in the open literature.

The application of the fractional differential equation in physical problems is available in the book of Bracewell [11]. Recently, the solution of the fractional differential equation has been obtained through the Adomian decomposition method by the researchers in [7, 14].

In this paper, we solve a differential equation containing a fractional derivative of order half along with an ordinary first-order derivative using the Adomian decomposition method. Then the solution obtained by this method is verified with that of the transformed ordinary differential equation derived from the original fractional differential equation.

For the sake of convenience, we first of all give definitions of fractional integral and fractional derivative introduced by Riemann-Liouville as discussed in [18, 20, 21].

Definition 1.1 (fractional integral). Let $q > 0$ denote a real number. Assuming $f(x)$ to be a function of class $C^{(n)}$ (the class of functions with continuous n th derivatives), the fractional integral of a function f of order $-q$ is given by

$$\frac{d^{-q} f(x)}{dx^{-q}} = \frac{1}{\Gamma(q)} \int_0^x \frac{f(t) dt}{(x-t)^{1-q}}, \quad (1.1)$$

$d^{-q} f(x)/dx^{-q}$ is also denoted by $I_0^q f$ [7].

Definition 1.2 (fractional derivative). Let $q > 0$ denote a real number and n the smallest integer exceeding q such that $n - q > 0$ ($n = 0$ if $q < 0$). Assuming $f(x)$ to be a function of class $C^{(n)}$ (the class of functions with continuous n th derivatives), the fractional derivative of a function f of order q is given by

$$\frac{d^q f(x)}{dx^q} = \frac{d^n}{dx^n} \left(\frac{d^{-(n-q)} f(x)}{dx^{-(n-q)}} \right) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dx^n} \int_0^x \frac{f(t)dt}{(x-t)^{1-n+q}}, \tag{1.2}$$

$d^q f(x)/dx^q$ is also denoted by $I_0^{\frac{x}{-q}} f$ [7].

Definition 1.3 (Mittag-Leffler function). A two-parameter function of the Mittag-Leffler type is defined by the series expansion [2, 21]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad (\alpha > 0, \beta > 0). \tag{1.3}$$

1.1. The decomposition method. We consider an equation in the form

$$Lu + Ru + Nu = g, \tag{1.4}$$

where L is an easily or trivially invertible linear operator, R is the remaining linear part, and N represents a nonlinear operator.

The general solution of the given equation is decomposed into the sum

$$u = \sum_{n=0}^{\infty} u_n, \tag{1.5}$$

where u_0 is the solution of the linear part.

Our approach will be to write any nonlinear term in terms of the Adomian A_n polynomials. It has been derived by Adomian that $Nu = \sum_{n=0}^{\infty} A_n$, where the A_n are special polynomials obtained for the particular nonlinearity $Nu = f(u)$ and generated by Adomian [3, 4, 5]. These A_n polynomials depend, of course, on the particular nonlinearity.

The A_n are given as

$$\begin{aligned} A_0 &= f(u_0), \\ A_1 &= u_1 \left(\frac{d}{du_0} \right) f(u_0), \\ A_2 &= u_2 \left(\frac{d}{du_0} \right) f(u_0) + \left(\frac{u_1^2}{2!} \right) \left(\frac{d^2}{du_0^2} \right) f(u_0), \\ A_3 &= u_3 \left(\frac{d}{du_0} \right) f(u_0) + u_1 u_2 \left(\frac{d^2}{du_0^2} \right) f(u_0) + \left(\frac{u_1^3}{3!} \right) \left(\frac{d^3}{du_0^3} \right) f(u_0), \\ &\vdots \end{aligned} \tag{1.6}$$

and can be found from the formula (for $n \geq 1$)

$$A_n = \sum_{\nu=1}^n c(\nu, n) f^{(\nu)}(u_0), \tag{1.7}$$

where the $c(\nu, n)$ are products (or sums of products) of ν components of u whose subscripts sum to n , divided by the factorial of the number of repeated subscripts [5].

Therefore, the general solution becomes

$$u = u_0 - L^{-1}R \sum_{n=0}^{\infty} u_n - L^{-1}Nu = u_0 - L^{-1}R \sum_{n=0}^{\infty} u_n - L^{-1} \sum_{n=0}^{\infty} A_n, \tag{1.8}$$

where $u_0 = \phi + L^{-1}g$ and $L\phi = 0$.

To identify the terms in $\sum_{n=1}^{\infty} u_n$, it has been derived by Adomian that

$$u_{n+1} = -L^{-1}Ru_n - L^{-1}A_n, \quad n \geq 0. \tag{1.9}$$

From (1.9), we can write $u_1 = -L^{-1}Ru_0 - L^{-1}A_0$. Thus u_1 can be calculated in terms of the known u_0 .

Now,

$$\begin{aligned} u_2 &= -L^{-1}Ru_1 - L^{-1}A_1, \\ u_3 &= -L^{-1}Ru_2 - L^{-1}A_2, \end{aligned} \tag{1.10}$$

and so on.

Hence all the terms of u are now calculated and the general solution is obtained as

$$u = \sum_{n=0}^{\infty} u_n. \tag{1.11}$$

Recently, the Adomian decomposition method was reviewed and a mathematical model of Adomian polynomials was introduced in [1].

2. Solution of an extraordinary differential equation

A relationship involving one or more derivatives of an unknown function f with respect to its independent variable x is known as an ordinary differential equation. Similar relationship involving at least one differintegral of noninteger order may be termed as an extraordinary differential equation. Such an equation is solved when an explicit expression for f is exhibited. As with ordinary differential equations, the solutions of extraordinary differential equations often involve integrals and contain arbitrary constants as discussed in [20]. These types of equations are also known as fractional differential equations. The application of extraordinary differential equation is now available in many physical and

technical areas [21]. It can be mentioned here that the simplified fractional-order differential equation appearing in applied problems is of the form

$$\begin{aligned} \frac{d^\alpha y(t)}{dt^\alpha} + Ay(t) &= f(t) \quad (t > 0), \\ y^{(k)}(0) &= 0 \quad (k = 0, 1, \dots, n-1), \end{aligned} \quad (2.1)$$

where $n-1 < \alpha \leq n$. For $0 < \alpha \leq 2$, this equation is called the *relaxation-oscillation* equation [21]. Moreover, the Bagley-Torvik equation [10]

$$\begin{aligned} A \frac{d^2 y(t)}{dt^2} + B \frac{d^{3/2} y(t)}{dt^{3/2}} + Cy(t) &= f(t) \quad (t > 0), \\ y(0) = 0, \quad y'(0) &= 0 \end{aligned} \quad (2.2)$$

(where $A \neq 0$ and $B, C \in \mathbb{R}$) arises in the modelling of the motion of a rigid plate immersed in a Newtonian fluid [21]. Another equation in the form of

$$\begin{aligned} A \frac{d^2 y(t)}{dt^2} + B \frac{d^{1/2} y(t)}{dt^{1/2}} + Cy(t) &= f(t), \quad (t > 0) \\ y(0) = 0, \quad y'(0) &= 0 \end{aligned} \quad (2.3)$$

(where $A \neq 0$ and $B, C \in \mathbb{R}$) introduced by Caputo [12] arises in the modelling of the motion of a single degree-of-freedom oscillator with damping system [22].

Phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry, and materials science are described by differential equations of fractional order (see [8, 9, 13, 15, 16, 17, 19]).

Here we apply the Adomian decomposition method for solving an extraordinary differential equation and then compare this solution with that obtained by an alternative method.

In the present paper, we consider the Adomian decomposition method for solving the following extraordinary differential equation:

$$\frac{dy}{dx} + \frac{d^{1/2}y}{dx^{1/2}} - 2y = 0. \quad (2.4)$$

We suppose that $L \equiv d/dx$.

Therefore, by the Adomian decomposition method, we can write

$$y = c - L^{-1} \left(\frac{d^{1/2}y}{dx^{1/2}} \right) + 2L^{-1}y, \quad (2.5)$$

where c is an arbitrary constant. This implies that

$$y = c - \frac{d^{-1/2}y}{dx^{-1/2}} + 2 \frac{d^{-1}y}{dx^{-1}}. \quad (2.6)$$

In the light of the Adomian decomposition method, we assume $y(x) = y_0(x) + y_1(x) + y_2(x) + \dots$ to be the solution of (2.4), where

$$\begin{aligned}
 y_0(x) &= c, \\
 y_1(x) &= 2 \frac{d^{-1}y_0(x)}{dx^{-1}} - \frac{d^{-1/2}y_0(x)}{dx^{-1/2}} = \left(2x - 2 \frac{\sqrt{x}}{\sqrt{\pi}}\right)c, \\
 y_2(x) &= 2 \frac{d^{-1}y_1(x)}{dx^{-1}} - \frac{d^{-1/2}y_1(x)}{dx^{-1/2}} = \left(2x^2 - \frac{16}{3} \frac{x^{3/2}}{\sqrt{\pi}} + x\right)c.
 \end{aligned}
 \tag{2.7}$$

Similarly,

$$\begin{aligned}
 y_3(x) &= \left(\frac{4}{3}x^3 - \frac{32}{5} \frac{x^{5/2}}{\sqrt{\pi}} + 3x^2 - \frac{4}{3} \frac{x^{3/2}}{\sqrt{\pi}}\right)c, \\
 y_4(x) &= \left(\frac{2}{3}x^4 - \frac{512}{105} \frac{x^{7/2}}{\sqrt{\pi}} + 4x^3 - \frac{64}{15} \frac{x^{5/2}}{\sqrt{\pi}} + \frac{1}{2}x^2\right)c, \\
 y_5(x) &= \left(\frac{4}{15}x^5 - \frac{512}{189} \frac{x^{9/2}}{\sqrt{\pi}} + \frac{5}{3}x^3 - \frac{128}{21} \frac{x^{7/2}}{\sqrt{\pi}} - \frac{8}{15} \frac{x^{5/2}}{\sqrt{\pi}} + \frac{10}{3}x^4\right)c,
 \end{aligned}
 \tag{2.8}$$

and so on.

Therefore, the solution of (2.4) is

$$y(x) = c \left(1 + 3x + \frac{11}{2}x^2 - 2 \frac{\sqrt{x}}{\sqrt{\pi}} - \frac{20}{3} \frac{x^{3/2}}{\sqrt{\pi}} - \frac{56}{5} \frac{x^{5/2}}{\sqrt{\pi}} + \dots\right).
 \tag{2.9}$$

It can be written as

$$y(x) = \frac{c}{3} \left(2 \sum_{k=0}^{\infty} \frac{(-1)^k 2^k x^{k/2}}{\Gamma(k/2 + 1)} + \sum_{k=0}^{\infty} \frac{x^{k/2}}{\Gamma(k/2 + 1)}\right).
 \tag{2.10}$$

As in [2, 21], the explicit formula for $E_{1/2,1}(z)$ is

$$E_{1/2,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k/2 + 1)} = e^{z^2} (1 + \operatorname{erf}(z)).
 \tag{2.11}$$

That is,

$$E_{1/2,1}(z) = e^{z^2} \operatorname{erfc}(-z).
 \tag{2.12}$$

Therefore, the solution of (2.4) becomes

$$y(x) = \frac{c}{3} (2e^{4x} \operatorname{erfc}(2\sqrt{x}) + e^x \operatorname{erfc}(-\sqrt{x})).
 \tag{2.13}$$

3. Verification of the solution

We can convert (2.4) into an ordinary differential equation.

Applying $d^{-1/2}/dx^{-1/2}$ to both sides of (2.4), we get

$$\frac{d^{1/2}y}{dx^{1/2}} + y - cx^{-1/2} - 2\frac{d^{-1/2}y}{dx^{-1/2}} = 0. \quad (3.1)$$

This implies that

$$2y - \frac{dy}{dx} + y - cx^{-1/2} - 2\frac{d^{-1/2}y}{dx^{-1/2}} = 0. \quad (3.2)$$

After differentiating (3.2), we get

$$2\frac{dy}{dx} - \frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{c}{2}x^{-3/2} - 2\frac{d^{1/2}y}{dx^{1/2}} = 0. \quad (3.3)$$

This implies that

$$2\frac{dy}{dx} - \frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{c}{2}x^{-3/2} - 2\left(2y - \frac{dy}{dx}\right) = 0 \quad (3.4)$$

or

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = \frac{c}{2}x^{-3/2}. \quad (3.5)$$

The solution of (3.5) is

$$y(x) = c_1e^{4x} + c_2e^x + \frac{c}{3}[-2e^{4x} - e^x + 2e^{4x}\operatorname{erfc}(2\sqrt{x}) + e^x\operatorname{erfc}(-\sqrt{x})], \quad (3.6)$$

where c_1 , c_2 , and c are arbitrary constants.

The function $y(x)$ given in (3.6) will be a solution of (2.4) if $c_1 = 2c/3$ and $c_2 = c/3$.

Therefore, the solution becomes

$$\begin{aligned} y(x) &= \frac{2c}{3}e^{4x} + \frac{c}{3}e^x + \frac{c}{3}[-2e^{4x} - e^x + 2e^{4x}\operatorname{erfc}(2\sqrt{x}) + e^x\operatorname{erfc}(-\sqrt{x})] \\ &= \frac{c}{3}[2e^{4x}\operatorname{erfc}(2\sqrt{x}) + e^x\operatorname{erfc}(-\sqrt{x})]. \end{aligned} \quad (3.7)$$

Solution (3.7) completely matches solution (2.13) obtained by the Adomian method.

4. Conclusion

It is observed that although the extraordinary differential equation can be converted into an ordinary differential equation, the way of solving that ordinary differential equation is quite laborious and requires enough skill. Moreover, an extraordinary differential equation can also be solved by Laplace transform method. But sometimes it becomes

cumbersome to have the solution because of finding out its inverse. In order to avoid this, we can use the Adomian decomposition method, which gives a quite satisfactory result.

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