DECAY RATES FOR SOLUTIONS OF A TIMOSHENKO SYSTEM WITH A MEMORY CONDITION AT THE BOUNDARY

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We consider a Timoshenko system with memory condition at the boundary and we study the asymptotic behavior of the corresponding solutions. We prove that the energy decay with the same rate of decay of the relaxation functions, that is, the energy decays exponentially when the relaxation functions decays exponentially and polynomially when the relaxation functions decays polynomially.

1. Introduction

The main purpose of this work is to study the asymptotic behavior of the solutions of a Timoshenko system with boundary conditions of memory type. To formalize this problem, take Ω an open bounded set of $\mathbb{R}^n$ with smooth boundary Γ and assume that Γ can be divided into two parts

$$\Gamma = \Gamma_0 \cup \Gamma_1 \quad \text{with } \bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset. \tag{1.1}$$

Denote by $\nu(x)$ the unit normal vector at $x \in \Gamma$ outside of Ω and consider the following initial boundary value problem:

$$u_{tt} - \Delta u - \alpha \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} + \beta u = 0 \quad \text{in } \Omega \times (0, \infty), \tag{1.2}$$

$$v_{tt} - \Delta v + \alpha \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} + f(v) = 0 \quad \text{in } \Omega \times (0, \infty), \tag{1.3}$$

$$u = v = 0 \quad \text{on } \Gamma_0 \times (0, \infty), \tag{1.4}$$

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$$u+\int_0^t g_1(t-s)\frac{\partial u}{\partial \nu}(s)\,ds=0 \quad \text{on } \Gamma_1\times(0,\infty), \tag{1.5}$$

$$v+\int_0^t g_2(t-s)\frac{\partial v}{\partial \nu}(s)\,ds=0 \quad \text{on } \Gamma_1\times(0,\infty), \tag{1.6}$$

$$(u(0,x),v(0,x))=(u_0(x),v_0(x)), \quad (u_t(0,x),v_t(0,x))=(u_1(x),v_1(x)) \quad \text{in } \Omega. \tag{1.7}$$

Here, u is the deflection of the beam from its equilibrium and v is the total rotatory angle of the beam at x, for those precise physical meaning, see Timoshenko [13]. We will assume in the sequel that α is a sufficiently small positive number, $\beta > n\alpha$, and the relaxation functions g_i are positive and nondecreasing and the function $f\in C^1(\mathbb{R})$ satisfies

$$f(s)s\geq 0, \quad \forall s\in\mathbb{R}. \tag{1.8}$$

Additionally, we suppose that f is superlinear, that is,

$$f(s)s\geq(2+\delta)F(s), \quad F(z)=\int_0^z f(s)\,ds, \quad \forall s\in\mathbb{R}, \tag{1.9}$$

for some $\delta>0$ with the following growth conditions:

$$|f(x)-f(y)|\leq c\left(1+|x|^{\rho-1}+|y|^{\rho-1}\right)|x-y|, \quad \forall x,y\in\mathbb{R}, \tag{1.10}$$

for some $c>0$ and $\rho\geq 1$ such that $(n-2)\rho\leq n$. The integral equations (1.5) and (1.6) describe the memory effects which can be caused, for example, by the interaction with another viscoelastic element. Also, we will assume that there exists $x_0\in\mathbb{R}^n$ such that

$$\begin{aligned}\Gamma_0&=\{x\in\Gamma:\nu(x)\cdot(x-x_0)\leq 0\},\\ \Gamma_1&=\{x\in\Gamma:\nu(x)\cdot(x-x_0)>0\}.\end{aligned} \tag{1.11}$$

As an example of a set Ω satisfying those properties, we can consider the domain shown in Figure 1.1.

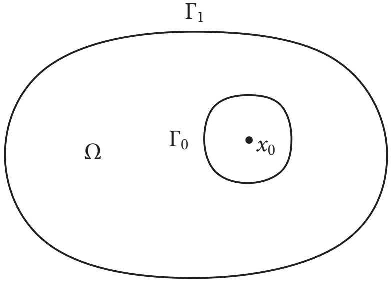

Figure 1.1

Let $m(x) = x - x_0$. Note that the compactness of Γ_1 implies that there exists a small positive constant δ_0 such that

$$0 < \delta_0 \leq m(x) \cdot \nu(x), \quad \forall x \in \Gamma_1. \tag{1.12}$$

Frictional dissipative boundary condition for the Timoshenko system was studied by several authors, see, for example, [4, 6, 11, 12] among others. Concerning the memory condition at the boundary we can cite the following works: in [1], Ciarletta established theorems of existence, uniqueness, and asymptotic stability for a linear model of heat conduction. In this case the memory condition describes a boundary that can absorb heat and due to the hereditary term, can retain part of it. In [3], Fabrizio and Morro considered a linear electromagnetic model and proved the existence, uniqueness, and asymptotic stability of the solutions. In [7], Muñoz Rivera and Andrade showed exponential stability for a nonhomogeneous anisotropic system when the resolvent kernel of the memory is of exponential type. They used multiplier technics and a compactness argument.

Nonlinear one-dimensional wave equation with memory condition on the boundary was studied by Qin [9]. He showed existence, uniqueness, and stability of global solutions provided the initial data is small in $H^3 \times H^2$. This result was improved by Muñoz Rivera and Andrade [8]. They only supposed small initial data in $H^2 \times H^1$. See also de Lima Santos [2].

In this paper, we show that the solutions of the coupled system (1.2)–(1.7) decays uniformly in time with the same rate of decay of the relaxation functions. More precisely, denoting by k_1 and k_2 the resolvent kernels of $-g_1'/g_1(0)$ and $-g_2'/g_2(0)$, respectively, we show that the solution decays exponentially to zero provided k_1 and k_2 decays exponentially to zero. When the resolvent kernels k_1 and k_2 decays polynomially, we show that the corresponding solution also decays polynomially to zero. The method used is based on the construction of a suitable Lyapunov functional $\mathcal{L}$ satisfying

$$\frac{d}{dt}\mathcal{L}(t) \leq -c_1\mathcal{L}(t) + c_2 e^{-\gamma t} \tag{1.13}$$

or

$$\frac{d}{dt}\mathcal{L}(t) \leq -c_1\mathcal{L}(t)^{1+1/\alpha} + \frac{c_2}{(1+t)^{\alpha+1}} \tag{1.14}$$

for some positive constants c_1, c_2, γ, and α. Note that, because of condition (1.4) the solution of system (1.2)–(1.7) must belong to the following space:

$$V := \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_0\}. \tag{1.15}$$

The notations we use in this paper are standard and can be found in Lions' book [5]. In the sequel, by c (sometimes $c_1, c_2, \ldots$) we denote various positive constants independent of t and on the initial data. The organization of this paper is as follows. In Section 2, we establish an existence and regularity result. In Section 3, we prove the uniform rate of exponential decay. Finally, in Section 4, we prove the uniform rate of polynomial decay.

2. Existence and regularity

In this section, we study the existence and regularity of solutions for the Timoshenko system (1.2)–(1.7). First, we use (1.5) and (1.6) to estimate the terms $\partial u/\partial \nu$ and $\partial v/\partial \nu$ on Γ_1. Denoting by

$$(g * \varphi)(t) = \int_0^t g(t-s)\varphi(s)\,ds, \tag{2.1}$$

the convolution product operator and differentiating (1.5) and (1.6), we arrive to the following Volterra equations:

$$\begin{aligned} \frac{\partial u}{\partial \nu} + \frac{1}{g_1(0)} g_1' * \frac{\partial u}{\partial \nu} &= -\frac{1}{g_1(0)} u_t, \\ \frac{\partial v}{\partial \nu} + \frac{1}{g_2(0)} g_2' * \frac{\partial v}{\partial \nu} &= -\frac{1}{g_2(0)} v_t. \end{aligned} \tag{2.2}$$

Applying the Volterra's inverse operator, we get

$$\begin{aligned} \frac{\partial u}{\partial \nu} &= -\frac{1}{g_1(0)} \{u_t + k_1 * u_t\}, \\ \frac{\partial v}{\partial \nu} &= -\frac{1}{g_2(0)} \{v_t + k_2 * v_t\}, \end{aligned} \tag{2.3}$$

where the resolvent kernels satisfy

$$k_i + \frac{1}{g_i(0)} g_i' * k_i = -\frac{1}{g_i(0)} g_i' \quad \text{for } i = 1, 2. \tag{2.4}$$

Denoting by $\tau_1 = 1/g_1(0)$ and $\tau_2 = 1/g_2(0)$ the normal derivatives of u and v can be written as

$$\begin{aligned} \frac{\partial u}{\partial \nu} &= -\tau_1 \{u_t + k_1(0)u - k_1(t)u_0 + k_1' * u\}, \\ \frac{\partial v}{\partial \nu} &= -\tau_2 \{v_t + k_2(0)v - k_2(t)v_0 + k_2' * v\}. \end{aligned} \tag{2.5}$$

Reciprocally, taking initial data such that $u_0 = v_0 = 0$ on Γ_1, identities (2.5) imply (1.5) and (1.6). Since we are interested in relaxation functions of exponential or polynomial type and identities (2.5) involve the resolvent kernels k_i, we want to know if k_i has the same properties. The following lemma answers this question. Let h be a relaxation function and k its resolvent kernel, that is,

$$k(t) - k * h(t) = h(t). \tag{2.6}$$

Lemma 2.1. *If h is a positive continuous function, then k also is a positive continuous function. Moreover,*

(1) *if there exist positive constants c_0 and γ with $c_0 < \gamma$ such that*

$$h(t) \le c_0 e^{-\gamma t}, \tag{2.7}$$

then the function k satisfies

$$k(t) \le \frac{c_0(\gamma - \epsilon)}{\gamma - \epsilon - c_0} e^{-\epsilon t}, \tag{2.8}$$

for all $0 < \epsilon < \gamma - c_0$.

(2) *Given $p > 1$, denote by $c_p := \sup_{t \in \mathbb{R}^+} \int_0^t (1+t)^p (1+t-s)^{-p} (1+s)^{-p}\, ds$. If there exists a positive constant c_0 with $c_0 c_p < 1$ such that*

$$h(t) \le c_0 (1+t)^{-p}, \tag{2.9}$$

then the function k satisfies

$$k(t) \le \frac{c_0}{1 - c_0 c_p} (1+t)^{-p}. \tag{2.10}$$

Proof. Note that $k(0) = h(0) > 0$. Now, we take $t_0 = \inf\{t \in \mathbb{R}^+ : k(t) = 0\}$, so $k(t) > 0$ for all $t \in [0, t_0[$. If $t_0 \in \mathbb{R}^+$, from (2.6) we get that $-k * h(t_0) = h(t_0)$ but this is contradictory. Therefore $k(t) > 0$ for all $t \in \mathbb{R}_0^+$. Now, fix ϵ, such that $0 < \epsilon < \gamma - c_0$ and denote by

$$k_\epsilon(t) := e^{\epsilon t} k(t), \qquad h_\epsilon(t) := e^{\epsilon t} h(t). \tag{2.11}$$

Multiplying (2.6) by $e^{\epsilon t}$ we get $k_\epsilon(t) = h_\epsilon(t) + k_\epsilon * h_\epsilon(t)$, hence

$$\begin{aligned} \sup_{s \in [0,t]} k_\epsilon(s) &\le \sup_{s \in [0,t]} h_\epsilon(s) + \left(\int_0^\infty c_0 e^{(\epsilon - \gamma)s}\, ds \right) \sup_{s \in [0,t]} k_\epsilon(s) \\ &\le c_0 + \frac{c_0}{(\gamma - \epsilon)} \sup_{s \in [0,t]} k_\epsilon(s). \end{aligned} \tag{2.12}$$

Therefore,

$$k_\epsilon(t) \le \frac{c_0(\gamma - \epsilon)}{\gamma - \epsilon - c_0}, \tag{2.13}$$

which implies our first assertion. To show the second part consider the following notations:

$$k_p(t) := (1+t)^p k(t), \qquad h_p(t) := (1+t)^p h(t). \tag{2.14}$$

Multiplying (2.6) by $(1+t)^p$, we get

$$k_p(t) = h_p(t) + \int_0^t k_p(t-s)(1+t-s)^{-p}(1+t)^p h(s)\,ds, \tag{2.15}$$

hence

$$\sup_{s\in[0,t]} k_p(s) \le \sup_{s\in[0,t]} h_p(s) + c_0 c_p \sup_{s\in[0,t]} k_p(s) \le c_0 + c_0 c_p \sup_{s\in[0,t]} k_p(s). \tag{2.16}$$

Therefore,

$$k_p(t) \le \frac{c_0}{1-c_0c_p}, \tag{2.17}$$

which proves our second assertion. □

Remark 2.2. The finiteness of the constant c_p can be found in [10, Lemma 7.4].

Due to Lemma 2.1, in the remainder of this paper, we will use (2.5) instead of (1.5) and (1.6). Denote by

$$(g\Box\varphi)(t) := \int_0^t g(t-s)\left|\varphi(t)-\varphi(s)\right|^2 ds. \tag{2.18}$$

The next lemma gives an identity for the convolution product.

Lemma 2.3. *For $g,\varphi \in C^1([0,\infty[:\mathbb{R})$,*

$$(g*\varphi)\varphi_t = -\frac{1}{2}g(t)\left|\varphi(t)\right|^2 + \frac{1}{2}g'\Box\varphi - \frac{1}{2}\frac{d}{dt}\left[g\Box\varphi - \left(\int_0^t g(s)\,ds\right)|\varphi|^2\right]. \tag{2.19}$$

The proof of this lemma follows by differentiating the term $g\Box\varphi$.

The well-posedness of system (1.2)–(1.7) is given by the following theorem.

Theorem 2.4. *Let $k_i \in C^2(\mathbb{R}^+)$ be such that*

$$k_i, -k_i', k_i'' \ge 0 \quad \textit{for } i = 1,2. \tag{2.20}$$

If $(u_0,v_0) \in (H^2(\Omega)\cap V)^2$ and $(u_1,v_1) \in V\times V$ satisfy the compatibility conditions

$$\begin{aligned} \frac{\partial u_0}{\partial \nu} + \tau_1 u_1 = 0 \quad \text{on } \Gamma_1, \\ \frac{\partial v_0}{\partial \nu} + \tau_2 v_1 = 0 \quad \text{on } \Gamma_1, \end{aligned} \tag{2.21}$$

then there exists only one strong solution (u,v) of the Timoshenko system (1.2)–(1.7) satisfying

$$u,v \in L^\infty(0,T;H^2(\Omega)\cap V) \cap W^{1,\infty}(0,T;V) \cap W^{2,\infty}(0,T;L^2(\Omega)). \tag{2.22}$$

This theorem can be proved using the standard Galerkin method, for this reason we omit it here.

3. Exponential decay

In this section, we study the asymptotic behavior of the solutions of system (1.2)–(1.7) when the resolvent kernels k_1 and k_2 are exponentially decreasing, that is, there exist positive constants b_1 and b_2 such that

$$k_i(0) > 0, \quad k_i'(t) \le -b_1 k_i(t), \quad k_i''(t) \ge -b_2 k_i'(t), \quad \text{for } i = 1, 2. \tag{3.1}$$

Note that these conditions imply that

$$k_i(t) \le k_i(0) e^{-b_1 t} \quad \text{for } i = 1, 2. \tag{3.2}$$

Our point of departure will be to establish some inequalities for the strong solution of Timoshenko system (1.2)–(1.7). For this end, we introduce the functional

$$\begin{aligned} E(t) := E(t,u,v) = {} & \frac{1}{2}\int_\Omega |u_t|^2 + (\beta - \alpha n)|u|^2 + |\nabla u|^2\,dx \\ & + \frac{\alpha}{2}\sum_{i=1}^n \int_\Omega \left|\frac{\partial v}{\partial x_i} - u\right|^2 dx + \frac{1}{2}\int_\Omega |v_t|^2 + (1-\alpha)|\nabla v|^2 + 2F(v)\,dx \\ & + \frac{\tau_1}{2}\int_{\Gamma_1} (k_1(t)|u|^2 - k_1' \Box u)\,d\Gamma_1 + \frac{\tau_2}{2}\int_{\Gamma_1} (k_2(t)|v|^2 - k_2' \Box v)\,d\Gamma_1. \end{aligned} \tag{3.3}$$

Lemma 3.1. *Any strong solution (u, v) of system (1.2)–(1.7) satisfies*

$$\begin{aligned} \frac{d}{dt}E(t) \le {} & -\frac{\tau_1}{2}\int_{\Gamma_1} |u_t|^2\,d\Gamma_1 + \frac{\tau_1}{2}k_1^2(t)\int_{\Gamma_1} |u_0|^2\,d\Gamma_1 \\ & + \frac{\tau_1}{2}k_1'(t)\int_{\Gamma_1} |u|^2\,d\Gamma_1 - \frac{\tau_1}{2}\int_{\Gamma_1} k_1'' \Box u\,d\Gamma_1 \\ & - \frac{\tau_2}{2}\int_{\Gamma_1} |v_t|^2\,d\Gamma_1 + \frac{\tau_2}{2}k_2^2(t)\int_{\Gamma_1} |v_0|^2\,d\Gamma_1 \\ & + \frac{\tau_2}{2}k_2'(t)\int_{\Gamma_1} |v|^2\,d\Gamma_1 - \frac{\tau_2}{2}\int_{\Gamma_1} k_2'' \Box v\,d\Gamma_1. \end{aligned} \tag{3.4}$$

Proof. Multiplying (1.2) by u_t and integrating by parts over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left\{|u_t|^2 + |\nabla u|^2 + \beta|u|^2\right\}dx - \alpha\sum_{i=1}^n \int_\Omega \frac{\partial v}{\partial x_i} u_t\,dx = \int_{\Gamma_1} \frac{\partial u}{\partial \nu} u_t\,d\Gamma_1. \tag{3.5}$$

Similarly, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}\left\{|v_t|^2+|\nabla v|^2+2F(v)\right\}dx+\alpha\sum_{i=1}^{n}\int_{\Omega}\frac{\partial u}{\partial x_i}v_t\,dx=\int_{\Gamma_1}\frac{\partial v}{\partial \nu}v_t\,d\Gamma_1. \tag{3.6}$$

Summing the above identities, substituting the boundary terms by (2.5), and using Lemma 2.3 our conclusion follows. □

Let $\theta > 0$ be a small constant and define the following functional:

$$\psi(t)=\int_{\Omega}\left\{m\cdot\nabla u+\left(\frac{n}{2}-\theta\right)u\right\}u_t\,dx+\int_{\Omega}\left\{m\cdot\nabla v+\left(\frac{n}{2}-\theta\right)v\right\}v_t\,dx. \tag{3.7}$$

The following lemma plays an important role for the construction of the Lyapunov functional.

Lemma 3.2. *For any strong solution of system (1.2)–(1.7),*

$$\begin{aligned}
\frac{d}{dt}\psi(t)\le{}&\frac{1}{2}\int_{\Gamma_1}m\cdot\nu\left(|u_t|^2+|v_t|^2\right)d\Gamma_1-\theta\int_{\Omega}|u_t|^2+|v_t|^2\,dx\\
&-\frac{(1-\theta)}{2}\int_{\Omega}|\nabla u|^2\,dx-\frac{(1-\theta)}{2}\int_{\Omega}|\nabla v|^2\,dx\\
&-\left(\frac{n\delta}{2}-\theta(2+\delta)\right)\int_{\Omega}F(v)\,dx\\
&-c\sum_{i=1}^{n}\int_{\Omega}\left|\frac{\partial v}{\partial x_i}-u\right|^2dx+\int_{\Gamma_1}\frac{\partial u}{\partial\nu}m\cdot\nabla u\,d\Gamma_1\\
&+\int_{\Gamma_1}\frac{\partial v}{\partial\nu}m\cdot\nabla v\,d\Gamma_1-\frac{1}{2}\int_{\Gamma_1}m\cdot\nu|\nabla u|^2\,d\Gamma_1\\
&-\frac{1}{2}\int_{\Gamma_1}m\cdot\nu|\nabla v|^2\,d\Gamma_1-\frac{\beta}{2}\int_{\Gamma_1}m\cdot\nu|u|^2\,d\Gamma_1.
\end{aligned} \tag{3.8}$$

Proof. From (1.2) we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\Omega}u_t\left\{m\cdot\nabla u+\left(\frac{n}{2}-\theta\right)u\right\}dx\\
&\quad=\int_{\Omega}u_t m\cdot\nabla u_t\,dx+\left(\frac{n}{2}-\theta\right)\int_{\Omega}|u_t|^2\,dx+\int_{\Omega}\Delta u m\cdot\nabla u\,dx\\
&\qquad+\left(\frac{n}{2}-\theta\right)\int_{\Omega}\Delta u u\,dx+\alpha\sum_{i=1}^{n}\int_{\Omega}\frac{\partial v}{\partial x_i}\left\{m\cdot\nabla u+\left(\frac{n}{2}-\theta\right)u\right\}dx\\
&\qquad-\beta\int_{\Omega}u\left\{m\cdot\nabla u+\left(\frac{n}{2}-\theta\right)u\right\}dx.
\end{aligned} \tag{3.9}$$

Performing an integration by parts, we get

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega u_t\Big\{m\cdot\nabla u+\Big(\frac{n}{2}-\theta\Big)u\Big\}\,dx
&\le \frac{1}{2}\int_{\Gamma_1} m\cdot\nu\,|u_t|^2\,d\Gamma_1-\theta\int_\Omega |u_t|^2\,dx+\int_{\Gamma_1}\frac{\partial u}{\partial\nu}m\cdot\nabla u\,d\Gamma_1\\
&\quad-\frac{1}{2}\int_{\Gamma_1} m\cdot\nu|\nabla u|^2\,d\Gamma_1-(1-\theta)\int_\Omega|\nabla u|^2\,dx+\frac{\alpha c}{2}\int_\Omega\{|\nabla u|^2+|\nabla v|^2\}\,dx\\
&\quad+\alpha\Big(\frac{n}{2}-\theta\Big)\sum_{i=1}^{n}\int_\Omega\frac{\partial v}{\partial x_i}u\,dx-\frac{\beta}{2}\int_{\Gamma_1} m\cdot|u|^2\,d\Gamma_1+\beta\theta\int_\Omega|u|^2\,dx.
\end{aligned}
\tag{3.10}
$$

Similarly, using (1.3) instead of (1.2) we get

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega v_t\Big(m\cdot\nabla v+\Big(\frac{n}{2}-\theta\Big)v\Big)\,dx
&\le \frac{1}{2}\int_{\Gamma_1} m\cdot\nu\,|v_t|^2\,d\Gamma_1-\theta\int_\Omega |v_t|^2\,dx+\int_{\Gamma_1}\frac{\partial v}{\partial\nu}m\cdot\nabla v\,d\Gamma_1\\
&\quad-\frac{1}{2}\int_{\Gamma_1} m\cdot\nu|\nabla v|^2\,d\Gamma_1-(1-\theta)\int_\Omega|\nabla v|^2\,dx-\Big(\frac{n}{2}-\theta\Big)(2+\delta)\int_\Omega F(v)\,dx\\
&\quad+n\int_\Omega F(v)\,dx+\frac{\alpha c}{2}\int_\Omega\{|\nabla u|^2+|\nabla v|^2\}\,dx+\alpha\Big(\frac{n}{2}-\theta\Big)\sum_{i=1}^{n}\int_\Omega\frac{\partial v}{\partial x_i}u\,dx.
\end{aligned}
\tag{3.11}
$$

Summing these two last inequalities, using Poincaré's inequality and taking θ small enough our conclusion follows. □

We introduce the Lyapunov functional

$$
\mathscr{L}(t)=NE(t)+\psi(t),
\tag{3.12}
$$

with $N>0$. Using Young's inequality and taking N large enough we find that

$$
q_0E(t)\le\mathscr{L}(t)\le q_1E(t),
\tag{3.13}
$$

for some positive constants q_0 and q_1. We will show later that the functional $\mathscr{L}$ satisfies the inequality of the following lemma.

Lemma 3.3. *Let f be a real positive function of class C^1. If there exist positive constants γ_0,γ_1, and c_0 such that*

$$
f'(t)\le-\gamma_0f(t)+c_0e^{-\gamma_1t},
\tag{3.14}
$$

then there exist positive constants γ and c such that

$$f(t) \le (f(0)+c)e^{-\gamma t}. \tag{3.15}$$

Proof. First, suppose that $\gamma_0 < \gamma_1$. Define $F(t)$ by

$$F(t) := f(t) + \frac{c_0}{\gamma_1 - \gamma_0} e^{-\gamma_1 t}. \tag{3.16}$$

Then

$$F'(t) = f'(t) - \frac{\gamma_1 c_0}{\gamma_1 - \gamma_0} e^{-\gamma_1 t} \le -\gamma_0 F(t). \tag{3.17}$$

Integrating from 0 to t we arrive to

$$F(t) \le F(0)e^{-\gamma_0 t} \Longrightarrow f(t) \le \left(f(0) + \frac{c_0}{\gamma_1 - \gamma_0} \right) e^{-\gamma_0 t}. \tag{3.18}$$

Now, we will assume that $\gamma_0 \ge \gamma_1$, and we get

$$f'(t) \le -\gamma_1 f(t) + c_0 e^{-\gamma_1 t} \Longrightarrow \left[e^{\gamma_1 t} f(t) \right]' \le c_0. \tag{3.19}$$

Integrating from 0 to t, we obtain

$$f(t) \le (f(0) + c_0 t)e^{-\gamma_1 t}. \tag{3.20}$$

Since $t \le (\gamma_1 - \epsilon)e^{(\gamma_1 - \epsilon)t}$ for any $0 < \epsilon < \gamma_1$ we conclude that

$$f(t) \le \left[f(0) + c_0(\gamma_1 - \epsilon) \right] e^{-\epsilon t}. \tag{3.21}$$

This completes the proof. □

Finally, we will show the main result of this section.

Theorem 3.4. *Take $(u_0, v_0) \in V^2$ and $(u_1, v_1) \in [L^2(\Omega)]^2$. If the resolvent kernels k_1 and k_2 satisfy (3.1), then there exist positive constants α_1 and γ_1 such that*

$$E(t) \le \alpha_1 e^{-\gamma_1 t} E(0), \quad \forall t \ge 0. \tag{3.22}$$

Proof. We will prove this result for strong solutions, that is, for solutions with initial data $(u_0, v_0) \in (H^2(\Omega) \cap V)^2$ and $(u_1, v_1) \in V^2$ satisfying the compatibility conditions (2.21). Our conclusion follows by standard density arguments.

Using Lemmas 3.1 and 3.2, condition (1.12), and Young's inequality we get

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) \le N\Big\{ &-\frac{\tau_1\beta_1}{2}\int_{\Gamma_1}|u_t|^2\,d\Gamma_1 + \frac{\tau_1\beta_1}{2}k_1^2(t)\int_{\Gamma_1}|u_0|^2\,d\Gamma_1 \\
&+\frac{\tau_1\beta_1}{2}k_1'(t)\int_{\Gamma_1}|u|^2\,d\Gamma_1 - \frac{\tau_1\beta_1}{2}\int_{\Gamma_1}k_1''\Box u\,d\Gamma_1 \\
&-\frac{\tau_2\beta_2}{2}\int_{\Gamma_1}|v_t|^2\,d\Gamma_1 + \frac{\tau_2\beta_2}{2}k_2^2(t)\int_{\Gamma_1}|v_0|^2\,d\Gamma_1 \\
&+\frac{\tau_2\beta_2}{2}k_2'(t)\int_{\Gamma_1}|v|^2\,d\Gamma_1 - \frac{\tau_2\beta_2}{2}\int_{\Gamma_1}k_2''\Box v\,d\Gamma_1\Big\} \\
&+\frac{1}{2}\int_{\Gamma_1}m\cdot\nu\left(|u_t|^2+|v_t|^2\right)d\Gamma_1 - \theta\int_\Omega |u_t|^2+|v_t|^2\,dx \\
&-(1-\theta)\frac{\beta_1}{2}\int_\Omega|\nabla u|^2\,dx-(1-\theta)\frac{\beta_2}{2}\int_\Omega|\nabla v|^2\,dx \\
&-c\sum_{i=1}^n\int_\Omega\left|\frac{\partial v}{\partial x_i}-u\right|^2dx-\left(\frac{n\delta}{2}-\theta(2+\delta)\right)\int_\Omega F(v)\,dx \\
&+\frac{\epsilon c}{2}\int_{\Gamma_1}\left|\frac{\partial u}{\partial \nu}\right|^2d\Gamma_1+\frac{\epsilon}{2\delta_0}\int_{\Gamma_1}m\cdot\nu|\nabla u|^2\,d\Gamma_1 \\
&+\frac{c}{2\epsilon}\int_{\Gamma_1}\left|\frac{\partial v}{\partial \nu}\right|^2d\Gamma_1+\frac{\epsilon}{2\delta_0}\int_{\Gamma_1}m\cdot\nu|\nabla v|^2\,d\Gamma_1 \\
&-\frac{1}{2}\int_{\Gamma_1}m\cdot\nu|\nabla u|^2\,d\Gamma_1-\frac{1}{2}\int_{\Gamma_1}m\cdot\nu|\nabla v|^2\,d\Gamma_1,
\end{aligned}\tag{3.23}$$

for any $\epsilon > 0$. Choosing N large enough, fixing $\epsilon = \delta_0$, and using the inequalities

$$\begin{aligned}
\int_{\Gamma_1}\left|\frac{\partial u}{\partial \nu}\right|^2d\Gamma_1 &\le c\int_{\Gamma_1}|u_t|^2+k_1^2|u|^2+k_1(0)|k_1'|\Box u+k_1^2|u|^2\,d\Gamma_1, \\
\int_{\Gamma_1}\left|\frac{\partial v}{\partial \nu}\right|^2d\Gamma_1 &\le c\int_{\Gamma_1}|v_t|^2+k_2^2|v|^2+k_2(0)|k_2'|\Box v+k_2^2|v|^2\,d\Gamma_1,
\end{aligned}\tag{3.24}$$

we arrive to

$$\frac{d}{dt}\mathcal{L}(t) \le -q_2E(t)+cR^2(t)E(0),\tag{3.25}$$

where $R(t) = k_1(t)+k_2(t)$ and $q_2 > 0$ is a small constant. Here we have used assumptions (3.1) in order to obtain the following estimates:

$$\begin{aligned}
-\frac{\tau_1}{2}\int_{\Gamma_1} k_1''\Box u\,d\Gamma_1 &\le c_1\int_{\Gamma_1} k_1'\Box u\,d\Gamma_1,\\
-\frac{\tau_2}{2}\int_{\Gamma_1} k_2''\Box v\,d\Gamma_1 &\le c_2\int_{\Gamma_1} k_2'\Box v\,d\Gamma_1,\\
\frac{\tau_1}{2}\int_{\Gamma_1} k_1'|u|^2\,d\Gamma_1 &\le -c_3\int_{\Gamma_1} k_1|u|^2\,d\Gamma_1,\\
\frac{\tau_2}{2}\int_{\Gamma_1} k_2'|v|^2\,d\Gamma_1 &\le -c_4\int_{\Gamma_1} k_2|v|^2\,d\Gamma_1,
\end{aligned} \tag{3.26}$$

for some boundary terms in (3.23). Finally, in view of (3.13) we conclude that

$$\frac{d}{dt}\mathscr{L}(t) \le -\frac{q_2}{q_1}\mathscr{L}(t) + cR^2(t)E(0). \tag{3.27}$$

From the exponential decay of k_1, k_2, and Lemma 3.3 there exist positive constants c and γ_1 such that

$$\mathscr{L}(t) \le \{\mathscr{L}(0)+c\}e^{-\gamma_1 t}, \quad \forall t \ge 0. \tag{3.28}$$

From inequality (3.13) our conclusion follows. □

4. Polynomial rate of decay

Here our attention will be focused on the uniform rate of decay when the resolvent kernels k_1 and k_2 decay polynomially like $(1+t)^{-p}$. In this case we will show that the solution also decays polynomially with the same rate. Therefore, we will assume that the resolvent kernels k_1 and k_2 satisfy

$$k_i(0) > 0, \quad k_i'(t) \le -b_1\big[k_i(t)\big]^{1+1/p}, \quad k_i''(t) \ge b_2\big[-k_i'(t)\big]^{1+1/(p+1)}, \quad \text{for } i = 1,2, \tag{4.1}$$

for some $p > 1$ and some positive constants b_1 and b_2. The following lemmas will play an important role in the sequel.

Lemma 4.1. *Let (u,v) be a solution of system (1.2)–(1.7) and denote by $(\phi_1,\phi_2) = (u,v)$. Then, for $p > 1$, $0 < r < 1$, and $t \ge 0$,*

$$\begin{aligned}
&\left(\int_{\Gamma_1} |k_i'|\Box\phi_i\,d\Gamma_1\right)^{(1+(1-r)(p+1))/(1-r)(p+1)}\\
&\quad\le 2^{1/(1-r)(p+1)}\left(\int_0^t |k_i'(s)|^r\,ds\|\phi_i\|^2_{L^\infty(0,t;L^2(\Gamma_1))}\right)^{1/(1-r)(p+1)}\\
&\qquad\times\int_{\Gamma_1} |k_i'|^{1+1/(p+1)}\Box\phi_i\,d\Gamma_1,
\end{aligned} \tag{4.2}$$

while for $r=0$

$$\left(\int_{\Gamma_1}|k_i'|\,\Box\phi_i\,d\Gamma_1\right)^{(p+2)/(p+1)}$$
$$\leq 2\left(\int_0^t \|\phi_i(s,\cdot)\|^2_{L^2(\Gamma_1)}\,ds+t\|\phi_i(s,\cdot)\|^2_{L^2(\Gamma_1)}\right)^{p+1}$$
$$\times\int_{\Gamma_1}|k_i'|^{1+1/(p+1)}\Box\phi_i\,d\Gamma_1,\quad \text{for } i=1,2. \tag{4.3}$$

Proof. See [2]. □

Lemma 4.2. *Let* $f\geq 0$ *be a differentiable function satisfying*

$$f'(t)\leq -\frac{c_1}{f(0)^{1/\alpha}}f(t)^{1+1/\alpha}+\frac{c_2}{(1+t)^{\beta}}f(0)\quad \text{for } t\geq 0, \tag{4.4}$$

for some positive constants c_1, c_2, α, *and* β *such that*

$$\beta\geq\alpha+1. \tag{4.5}$$

Then there exists a constant $c>0$ *such that*

$$f(t)\leq\frac{c}{(1+t)^{\alpha}}f(0)\quad \text{for } t\geq 0. \tag{4.6}$$

Proof. See [2]. □

Theorem 4.3. *Take* $(u_0,v_0)\in V^2$ *and* $(u_1,v_1)\in[L^2(\Omega)]^2$. *If the resolvent kernels* k_1 *and* k_2 *satisfy conditions (4.1), then there exists a positive constant c such that*

$$E(t)\leq\frac{c}{(1+t)^{p+1}}E(0). \tag{4.7}$$

Proof. We will prove this result for strong solutions, that is, for solutions with initial data $(u_0,v_0)\in(H^2(\Omega)\cap V)^2$ and $(u_1,v_1)\in V^2$ satisfying the compatibility conditions (2.21). Our conclusion will follow by standard density arguments. We define the functional $\mathcal{L}$ as in (3.12) therefore we have the equivalence relation given in (3.13) again. Combining Lemmas 3.1 and 3.2 we get

$$\frac{d}{dt}\mathcal{L}(t)\leq -c_1\left\{\int_\Omega|u_t|^2+|u|^2+|\nabla u|^2+|v_t|^2+|\nabla v|^2+F(v)\,dx\right.$$
$$\left.+\sum_{i=1}^n\int_\Omega\left|\frac{\partial v}{\partial x_i}-u\right|^2dx\right\}-N\left\{\int_{\Gamma_1}k_1''\Box u+k_2''\Box v\,d\Gamma_1\right\}+c_2R^2(t)E(0), \tag{4.8}$$

for some positive constants c_1 and c_2. Using hypothesis (4.1) we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) \le -c_1\bigg\{\int_\Omega \left|u_t\right|^2+|u|^2+|\nabla u|^2+\left|v_t\right|^2+|\nabla v|^2+F(v)\,dx\\ +\sum_{i=1}^n\int_\Omega\left|\frac{\partial v}{\partial x_i}-u\right|^2dx\bigg\}\\ -N\bigg\{\int_{\Gamma_1}\left[-k_1'\right]^{1+1/(p+1)}\square u\,d\Gamma_1+\int_{\Gamma_1}\left[-k_2'\right]^{1+1/(p+1)}\square v\,d\Gamma_1\bigg\}\\ +c_2R^2(t)E(0).\end{aligned} \tag{4.9}$$

Denote by

$$\begin{aligned}\mathcal{N}(t):=\int_\Omega \left|u_t\right|^2+|u|^2+|\nabla u|^2+\left|v_t\right|^2+|\nabla v|^2+F(v)\,dx\\ +\sum_{i=1}^n\int_\Omega\left|\frac{\partial v}{\partial x_i}-u\right|^2dx+k_1(t)\int_{\Gamma_1}|u|^2\,d\Gamma_1+k_2(t)\int_{\Gamma_1}|v|^2\,d\Gamma_1.\end{aligned} \tag{4.10}$$

Using the following estimates:

$$\begin{aligned}k_1(t)\int_{\Gamma_1}|u|^2\,d\Gamma_1 \le c\int_\Omega|\nabla u|^2\,dx,\\ k_2(t)\int_{\Gamma_1}|v|^2\,d\Gamma_1 \le c\int_\Omega|\nabla v|^2\,dx,\end{aligned} \tag{4.11}$$

inequality (4.9) can be written as

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) \le -c_1\mathcal{N}(t)+c_2R^2(t)E(0)\\ -N\bigg\{\int_{\Gamma_1}\left[-k_1'\right]^{1+1/(p+1)}\square u\,d\Gamma_1+\int_{\Gamma_1}\left[-k_2'\right]^{1+1/(p+1)}\square v\,d\Gamma_1\bigg\}.\end{aligned} \tag{4.12}$$

Fix $0<r<1$ such that $1/(p+1)<r<p/(p+1)$. Under this condition we have

$$\int_0^\infty \left|k_i'\right|^r \le c\int_0^\infty \frac{1}{(1+t)^{r(p+1)}}<\infty \quad \text{for } i=1,2. \tag{4.13}$$

Using this estimate and Lemma 4.1 we get

$$\begin{aligned}\int_{\Gamma_1}\left[-k_1'\right]^{1+1/(p+1)}\square u\,d\Gamma_1 \ge \frac{c}{E(0)^{1/(1-r)(p+1)}}\left(\int_{\Gamma_1}\left[-k_1'\right]\square u\,d\Gamma_1\right)^{1+1/(1-r)(p+1)},\\ \int_{\Gamma_1}\left[-k_2'\right]^{1+1/(p+1)}\square v\,d\Gamma_1 \ge \frac{c}{E(0)^{1/(1-r)(p+1)}}\left(\int_{\Gamma_1}\left[-k_2'\right]\square v\,d\Gamma_1\right)^{1+1/(1-r)(p+1)}.\end{aligned} \tag{4.14}$$

On the other hand, since the energy is bounded we have

$$\mathcal{N}(t)^{1+1/(1-r)(p+1)} \leq cE(0)^{1/(1-r)(p+1)}\mathcal{N}(t). \tag{4.15}$$

Substitution of (4.14) and (4.15) into (4.12) we arrive to

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) \leq & -\frac{c}{E(0)^{1/(1-r)(p+1)}}\mathcal{N}(t)^{1+1/(1-r)(p+1)} + cR^2(t)E(0)\\ & - \frac{c}{E(0)^{1/(1-r)(p+1)}}\left\{\left(\int_{\Gamma_1}[-k_1']\Box u\,d\Gamma_1\right)^{1+1/(1-r)(p+1)}\right.\\ & \left. + \left(\int_{\Gamma_1}[-k_2']\Box v\,d\Gamma_1\right)^{1+1/(1-r)(p+1)}\right\}.\end{aligned} \tag{4.16}$$

Taking into account inequality (3.13) we conclude that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{c}{\mathcal{L}(0)^{1/(1-r)(p+1)}}\mathcal{L}(t)^{1+1/(1-r)(p+1)} + cR^2(t)E(0). \tag{4.17}$$

Therefore, from Lemma 4.2 we conclude that

$$\mathcal{L}(t) \leq \frac{c}{(1+t)^{(1-r)(p+1)}}\mathcal{L}(0). \tag{4.18}$$

Since $(1-r)(p+1) > 1$ we get, for $t \geq 0$, the following estimates:

$$\begin{aligned}& t\|u\|_{L^2(\Gamma_1)} + t\|v\|_{L^2(\Gamma_1)} \leq t\mathcal{L}(t) < \infty,\\ & \int_0^t \|u\|_{L^2(\Gamma_1)} + \|v\|_{L^2(\Gamma_1)} \leq c\int_0^t \mathcal{L}(t) < \infty.\end{aligned} \tag{4.19}$$

Under this condition applying Lemma 4.1 for $r = 0$ we get

$$\begin{aligned}& \int_{\Gamma_1}[-k_1']^{1+1/(p+1)}\Box u\,d\Gamma_1 \geq \frac{c}{E(0)^{1/(p+1)}}\left(\int_{\Gamma_1}[-k_1']u\,d\Gamma_1\right)^{1+1/(p+1)},\\ & \int_{\Gamma_1}[-k_2']^{1+1/(p+1)}\Box v\,d\Gamma_1 \geq \frac{c}{E(0)^{1/(p+1)}}\left(\int_{\Gamma_1}[-k_2']v\,d\Gamma_1\right)^{1+1/(p+1)}.\end{aligned} \tag{4.20}$$

Using these inequalities instead of (4.14) and reasoning in the same way as above, we conclude that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{c}{\mathcal{L}(0)^{1/(p+1)}}\mathcal{L}(t)^{1+1/(p+1)} + cR^2(t)E(0). \tag{4.21}$$

Applying Lemma 4.2 again, we obtain

$$\mathscr{L}(t) \leq \frac{c}{(1+t)^{p+1}} \mathscr{L}(0). \tag{4.22}$$

Finally, from (3.13) we conclude

$$E(t) \leq \frac{c}{(1+t)^{p+1}} E(0), \tag{4.23}$$

which completes the present proof. □

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