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Research Article

Generalized Hölder's and Minkowski's Inequalities for Jackson's q-Integral and Some Applications to the Incomplete q-Gamma Function

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We establish some generalized Hölder's and Minkowski's inequalities for Jackson's q-integral. As applications, we derive some inequalities involving the incomplete q-Gamma function.

1. Introduction

The classical Hölder's and Minkowski's inequalities are usually defined as follows.

Definition 1. Let $\alpha > 1$ and $1/\alpha + 1/\beta = 1$. Then the discrete and integral forms of Hölder's inequality are given as

$$\sum_{i=1}^{n} \left| a_i b_i \right| \leq \left(\sum_{i=1}^{n} \left| a_i \right|^{\alpha} \right)^{1/\alpha} \left(\sum_{i=1}^{n} \left| b_i \right|^{\beta} \right)^{1/\beta} \tag{1}$$

for sequences a_i, b_i and

$$\begin{aligned} &\int_a^b \left| f(t)\, g(t) \right| dt \\ &\quad \leq \left(\int_a^b \left| f(t) \right|^{\alpha} dt \right)^{1/\alpha} \left(\int_a^b \left| g(t) \right|^{\beta} dt \right)^{1/\beta} \end{aligned} \tag{2}$$

for continuous function f and g on $[a, b]$.

Definition 2. Let $u > 1$. Then the discrete and integral forms of Minkowski's inequality are given as

$$\left(\sum_{i=1}^{n} \left| a_i + b_i \right|^{u} \right)^{1/u} \leq \left(\sum_{i=1}^{n} \left| a_i \right|^{u} \right)^{1/u} + \left(\sum_{i=1}^{n} \left| b_i \right|^{u} \right)^{1/u} \tag{3}$$

for sequences a_i, b_i and

$$\begin{aligned} &\left(\int_a^b \left| f(t) + g(t) \right|^{u} dt \right)^{1/u} \\ &\quad \leq \left(\int_a^b \left| f(t) \right|^{u} dt \right)^{1/u} + \left(\int_a^b \left| g(t) \right|^{u} dt \right)^{1/u} \end{aligned} \tag{4}$$

for continuous function f and g on $[a, b]$.

These fundamental results are well known in the literature and have been studied intensively by several researchers. Their role in mathematics and related disciplines is invaluable. For instance, they play a pivotal role in classical real and complex analysis, probability theory, statistics, numerical analysis, and so on. Over the past years, various refinements, extensions, and applications have appeared in the literature. In the present work, our objective is to provide some generalized Hölder's and Minkowski's inequalities for Jackson's q-integral. As applications, we derive some inequalities involving the incomplete q-Gamma function. Let us begin with the following auxiliary results.

2. Auxiliary Results

We begin with the following inequality which is well known as Young's inequality.

Lemma 3. *For $x, y \geq 0$, $a > 1$, and $1/a + 1/b = 1$, the inequality*

$$xy \leq \frac{x^a}{a} + \frac{y^b}{b} \tag{5}$$

is satisfied.

Inequality (5) can be generalized as follows.

Lemma 4. *For $k = 1, 2, \ldots, n$, let $x_k \geq 0$ and $\alpha_k > 1$ such that $\sum_{k=1}^{n}(1/\alpha_k) = 1$. Then the inequality*

$$\prod_{k=1}^{n} x_k \leq \sum_{k=1}^{n} \frac{x_k^{\alpha_k}}{\alpha_k} \tag{6}$$

is valid.

Inequality (6) can be written in the following form which is known as the weighted AM-GM inequality.

Lemma 5. *For $k = 1, 2, \ldots, n$, let $Q_k \geq 0$ and $\alpha_k > 1$ such that $\sum_{k=1}^{n}(1/\alpha_k) = 1$. Then the inequality*

$$\prod_{k=1}^{n} Q_k^{1/\alpha_k} \leq \sum_{k=1}^{n} \frac{Q_k}{\alpha_k} \tag{7}$$

is valid.

Lemma 6 (generalized Hölder's inequality for sums). *Let $i = 1, 2, 3, \ldots, n$ and $k = 1, 2, 3, \ldots, m$ such that the sums exist. Then the inequality*

$$\sum_{i=1}^{n} \left| \prod_{k=1}^{m} Q_{i,k} \right| \leq \prod_{k=1}^{m} \left(\sum_{i=1}^{n} \left| Q_{i,k} \right|^{\alpha_k} \right)^{1/\alpha_k} \tag{8}$$

is valid for $\alpha_k > 1$ such that $\sum_{k=1}^{m}(1/\alpha_k) = 1$.

Proof. The inequality is obvious if $\sum_{i=1}^{n} |Q_{i,k}|^{\alpha_k} = 0$ for each k. So we assume $\sum_{i=1}^{n} |Q_{i,k}|^{\alpha_k} \neq 0$ and let

$$x_{i,k} = \frac{|Q_{i,k}|}{\left(\sum_{i=1}^{n} |Q_{i,k}|^{\alpha_k} \right)^{1/\alpha_k}}. \tag{9}$$

Then by the generalized Young's inequality (6), we obtain

$$\frac{\prod_{k=1}^{m} |Q_{i,k}|}{\prod_{k=1}^{m} \left(\sum_{i=1}^{n} |Q_{i,k}|^{\alpha_k} \right)^{1/\alpha_k}} \leq \sum_{k=1}^{m} \frac{1}{\alpha_k} \frac{|Q_{i,k}|^{\alpha_k}}{\sum_{i=1}^{n} |Q_{i,k}|^{\alpha_k}}. \tag{10}$$

By adding these n inequalities, we obtain

$$\begin{aligned} \frac{\sum_{i=1}^{n} \left| \prod_{k=1}^{m} Q_{i,k} \right|}{\prod_{k=1}^{m} \left(\sum_{i=1}^{n} |Q_{i,k}|^{\alpha_k} \right)^{1/\alpha_k}} &\leq \sum_{k=1}^{m} \frac{1}{\alpha_k} \frac{\sum_{i=1}^{n} |Q_{i,k}|^{\alpha_k}}{\sum_{i=1}^{n} |Q_{i,k}|^{\alpha_k}} \\ &= \sum_{k=1}^{m} \frac{1}{\alpha_k} = 1 \end{aligned} \tag{11}$$

which gives inequality (8). □

Lemma 7 (generalized Minkowski's inequality for sums). *Let $i = 1, 2, 3, \ldots, n$ and $k = 1, 2, 3, \ldots, m$ such that the sums exist. Then the inequality*

$$\left(\sum_{i=1}^{n} \left| \sum_{k=1}^{m} Q_{i,k} \right|^{u} \right)^{1/u} \leq \sum_{k=1}^{m} \left(\sum_{i=1}^{n} |Q_{i,k}|^{u} \right)^{1/u} \tag{12}$$

is valid for $u > 1$.

Proof. We prove this by mathematical induction on m. If $m = 2$, then (12) reduces to the classical Minkowski's inequality (3). Thus (12) is valid for $m = 2$. Next, assume that (12) holds for some $m \geq 2$. Based on this assumption, we want to show that (12) holds for $m + 1$. We proceed as follows:

$$\begin{aligned} \left(\sum_{i=1}^{n} \left| \sum_{k=1}^{m+1} Q_{i,k} \right|^{u} \right)^{1/u} &= \left(\sum_{i=1}^{n} \left| \sum_{k=1}^{m} Q_{i,k} + Q_{i,m+1} \right|^{u} \right)^{1/u} \\ &\leq \left(\sum_{i=1}^{n} \left| \sum_{k=1}^{m} Q_{i,k} \right|^{u} \right)^{1/u} \\ &\quad + \left(\sum_{i=1}^{n} |Q_{i,m+1}|^{u} \right)^{1/u} \\ &\leq \sum_{k=1}^{m} \left(\sum_{i=1}^{n} |Q_{i,k}|^{u} \right)^{1/u} \\ &\quad + \left(\sum_{i=1}^{n} |Q_{i,m+1}|^{u} \right)^{1/u} \\ &= \sum_{k=1}^{m+1} \left(\sum_{i=1}^{n} |Q_{i,k}|^{u} \right)^{1/u}. \end{aligned} \tag{13}$$

Thus (12) holds for $m + 1$ and this completes the proof. □

Remark 8. Inequalities (8) and (12) are already known in the literature. See, for instance, [1–3]. Here we offer simple proofs of the results.

3. Generalized q-Hölder's and q-Minkowski's Inequalities

Jackson's q-integral from 0 to a and that from 0 to ∞ are, respectively, defined as

$$\int_0^a f(t)\, d_q t = (1 - q)\, a \sum_{k=0}^{\infty} f\left(aq^k\right) q^k, \tag{14}$$

$$\int_0^{\infty} f(t)\, d_q t = (1 - q) \sum_{-\infty}^{\infty} f\left(q^k\right) q^k \tag{15}$$

provided that the sums in (14) and (15) converge absolutely [4]. In a generic interval $[a,b]$, the q-integral takes the following form:

$$\int_a^b f(t)\, d_q t = \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t. \tag{16}$$

Theorem 9 (generalized q-Hölder's inequality). *Let $f_1, f_2, \ldots, f_n$ be functions such that the integrals exist. Then the inequality*

$$\int_0^a \left|\prod_{i=1}^n f_i(t)\right| d_q t \le \prod_{i=1}^n \left(\int_0^a |f_i(t)|^{\alpha_i}\, d_q t\right)^{1/\alpha_i} \tag{17}$$

holds for $\alpha_i > 1$ such that $\sum_{i=1}^n (1/\alpha_i) = 1$.

Proof. Let $\alpha_i > 1$ such that $\sum_{i=1}^n (1/\alpha_i) = 1$. Then by relation (14) and inequality (8), we obtain

$$\begin{aligned}\int_0^a \left|\prod_{i=1}^n f_i(t)\right| d_q t &= (1-q)\, a \sum_{k=0}^{\infty}\prod_{i=1}^n \left|f_i\left(aq^k\right)\right| q^k \\ &\le (1-q)\, a \prod_{i=1}^n \left(\sum_{k=0}^{\infty}\left|f_i\left(aq^k\right)\right|^{\alpha_i} q^k\right)^{1/\alpha_i} \\ &= \left[(1-q)\, a\right]^{\sum_{i=1}^n (1/\alpha_i)} \prod_{i=1}^n \left(\sum_{k=0}^{\infty}\left|f_i\left(aq^k\right)\right|^{\alpha_i} q^k\right)^{1/\alpha_i} \\ &= \prod_{i=1}^n \left[(1-q)\, a\right]^{1/\alpha_i} \prod_{i=1}^n \left(\sum_{k=0}^{\infty}\left|f_i\left(aq^k\right)\right|^{\alpha_i} q^k\right)^{1/\alpha_i} \\ &= \prod_{i=1}^n \left((1-q)\, a \sum_{k=0}^{\infty}\left|f_i\left(aq^k\right)\right|^{\alpha_i} q^k\right)^{1/\alpha_i} \\ &= \prod_{i=1}^n \left(\int_0^a |f_i(t)|^{\alpha_i}\, d_q t\right)^{1/\alpha_i}\end{aligned} \tag{18}$$

which concludes the proof. □

Remark 10. Let $n = 2$, $\alpha_1 = p$, $\alpha_2 = t$, $f_1 = f$, and $f_2 = g$ in Theorem 9. Then we obtain the result of Lemma 2.1 of [5].

Theorem 11 (generalized q-Minkowski's inequality). *Let $f_1, f_2, \ldots, f_n$ be functions such that the integrals exist. Then the inequality*

$$\left(\int_0^a \left|\sum_{i=1}^n f_i(t)\right|^u d_q t\right)^{1/u} \le \sum_{i=1}^n \left(\int_0^a |f_i(t)|^u\, d_q t\right)^{1/u} \tag{19}$$

holds for $u > 1$.

Proof. Similarly we apply the principle of mathematical induction. For $n = 2$, inequality (19) reduces to q-Minkowski's inequality obtained in Remark 4 of [6]. Assume that (19) holds for $n \ge 2$. Based on this assumption, we show that (19) holds for $n+1$. That is,

$$\begin{aligned}&\left(\int_0^a \left|\sum_{i=1}^{n+1} f_i(t)\right|^u d_q t\right)^{1/u} \\ &\quad= \left(\int_0^a \left|\sum_{i=1}^n f_i(t) + f_{n+1}(t)\right|^u d_q t\right)^{1/u} \\ &\quad\le \left(\int_0^a \left|\sum_{i=1}^n f_i(t)\right|^u d_q t\right)^{1/u} \\ &\qquad+ \left(\int_0^a |f_{n+1}(t)|^u\, d_q t\right)^{1/u} \\ &\quad\le \sum_{i=1}^n \left(\int_0^a |f_i(t)|^u\, d_q t\right)^{1/u} \\ &\qquad+ \left(\int_0^a |f_{n+1}(t)|^u\, d_q t\right)^{1/u} \\ &\quad= \sum_{i=1}^{n+1} \left(\int_0^a |f_i(t)|^u\, d_q t\right)^{1/u}.\end{aligned} \tag{20}$$

Thus (19) holds for $n+1$. This completes the proof. □

4. Some Applications to the Incomplete q-Gamma Function

In this section, we derive some inequalities involving the incomplete q-Gamma function. We shall use the notations $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ and $\aleph = \{0, 2, 4, 6, \ldots\}$ subsequently.

El-Shahed and Salem [7] defined the incomplete q-Gamma function for $q \in (0,1)$ as

$$\gamma_q(x,z) = \int_0^z t^{x-1} E_q^{-qt}\, d_q t, \quad x > 0,\ z > 0 \tag{21}$$

and the complementary incomplete q-Gamma function as

$$\Gamma_q(x,z) = \int_z^{1/(1-q)} t^{x-1} E_q^{-qt}\, d_q t, \quad x > 0,\ z \ge 0, \tag{22}$$

where $E_q^t = \sum_{n=0}^{\infty} q^{n(n-1)/2}(t^n/[n]_q)$ is a q-analogue of the classical exponential function, $[u]_q = (1-q^u)/(1-q)$, and $[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$. One can easily see that

$$\begin{aligned}\Gamma_q(x,0) &= \Gamma_q(x), \\ \gamma_q(x,z) + \Gamma_q(x,z) &= \Gamma_q(x),\end{aligned} \tag{23}$$

where $\Gamma_q(x)$ is the q-Gamma function. Also, the following identities are satisfied:

$$\begin{aligned}
\Gamma_q(1,z) &= E_q^{-z}, \\
\Gamma_q(x+1,z) &= [x]_q \Gamma_q(x,z) + z^x E_q^{-z}, \\
\gamma_q(1,z) &= 1 - E_q^{-z}, \\
\gamma_q(x+1,z) &= [x]_q \gamma_q(x,z) - z^x E_q^{-z}.
\end{aligned} \tag{24}$$

Moreover,

$$\Gamma_q(0,z) = \int_z^{1/(1-q)} t^{-1} E_q^{-qt} d_q t = E_1(z,q), \tag{25}$$

where $E_1(z,q)$ is the q-exponential integral [8].

Remark 12. The functions $\gamma_q(x,z)$ and $\Gamma_q(x,z)$ can be viewed as both functions of x (for fixed z) and functions of z (for fixed x). For the purpose of this paper, we shall concentrate on $\gamma_q(x,z)$ as functions of x.

By differentiating (21) m times, we obtain

$$\gamma_q^{(m)}(x,z) = \int_0^z t^{x-1} (\ln t)^m E_q^{-qt} d_q t, \quad m \in \mathbb{N}_0, \tag{26}$$

where $\gamma_q^{(0)}(x,z) = \gamma_q(x,z)$.

Theorem 13. *For $i = 1,2,\ldots,n$, let $\alpha_i > 1$, $\sum_{i=1}^n (1/\alpha_i) = 1$, $m_i \in \aleph$, and $\sum_{i=1}^n (m_i/\alpha_i) \in \aleph$. Then the inequality*

$$\gamma_q^{(\sum_{i=1}^n (m_i/\alpha_i))}\left(\sum_{i=1}^n \frac{x_i}{\alpha_i}, z\right) \le \prod_{i=1}^n \left(\gamma_q^{(m_i)}(x_i,z)\right)^{1/\alpha_i} \tag{27}$$

holds for $x_i > 0$.

Proof. By (26) and the generalized q-Hölder's inequality (17), we obtain

$$\begin{aligned}
&\gamma_q^{(\sum_{i=1}^n (m_i/\alpha_i))}\left(\sum_{i=1}^n \frac{x_i}{\alpha_i}, z\right) \\
&= \int_0^z t^{\sum_{i=1}^n (x_i/\alpha_i)-1} (\ln t)^{\sum_{i=1}^n (m_i/\alpha_i)} E_q^{-qt} d_q t \\
&= \int_0^z t^{\sum_{i=1}^n ((x_i-1)/\alpha_i)} (\ln t)^{\sum_{i=1}^n (m_i/\alpha_i)} E_q^{-qt \sum_{i=1}^n (1/\alpha_i)} d_q t \\
&= \int_0^z \prod_{i=1}^n \left(t^{(x_i-1)/\alpha_i} (\ln t)^{m_i/\alpha_i} E_q^{-qt(1/\alpha_i)}\right) d_q t \\
&\le \prod_{i=1}^n \left[\int_0^z t^{x_i-1} (\ln t)^{m_i} E_q^{-qt}\right]^{1/\alpha_i} d_q t \\
&= \prod_{i=1}^n \left(\gamma_q^{(m_i)}(x_i,z)\right)^{1/\alpha_i}
\end{aligned} \tag{28}$$

which completes the proof. □

Remark 14. Let $n = 2$, $\alpha_1 = a$, $\alpha_2 = b$, $x_1 = x$, and $x_2 = y$ in Theorem 13. Then, we obtain

$$\begin{aligned}
&\gamma_q^{(m_1/a+m_2/b)}\left(\frac{x}{a} + \frac{y}{b}, z\right) \\
&\quad \le \left(\gamma_q^{(m_1)}(x,z)\right)^{1/a} \left(\gamma_q^{(m_2)}(y,z)\right)^{1/b}.
\end{aligned} \tag{29}$$

Remark 15. Let $m_1 = m_2 = m$ in (29). Then we obtain

$$\gamma_q^{(m)}\left(\frac{x}{a} + \frac{y}{b}, z\right) \le \left(\gamma_q^{(m)}(x,z)\right)^{1/a} \left(\gamma_q^{(m)}(y,z)\right)^{1/b} \tag{30}$$

which implies that the function $\gamma_q^{(m)}(x,z)$ is logarithmically convex. Also, since $\gamma_q^{(0)}(x,z) = \gamma_q(x,z)$, then it follows that $\gamma_q(x,z)$ is also logarithmically convex.

Remark 16. Let $a = b = 2$ and $x = y$ in (29). Then we obtain the Turan-type inequality

$$\gamma_q^{(m_1)}(x,z)\,\gamma_q^{(m_2)}(x,z) \ge \left(\gamma_q^{((m_1+m_2)/2)}(x,z)\right)^2. \tag{31}$$

Corollary 17. *Let $m \in \aleph$. Then the inequality*

$$\gamma_q^{(m)}\left(\frac{x+y}{2}, z\right) \le \frac{\gamma_q^{(m)}(x,z) + \gamma_q^{(m)}(y,z)}{2} \tag{32}$$

holds for $x, y > 0$.

Proof. Let $a = b = 2$ in (30). Then, by the AM-GM inequality, we have

$$\begin{aligned}
\gamma_q^{(m)}\left(\frac{x+y}{2}, z\right) &\le \sqrt{\gamma_q^{(m)}(x,z)\,\gamma_q^{(m)}(y,z)} \\
&\le \frac{\gamma_q^{(m)}(x,z) + \gamma_q^{(m)}(y,z)}{2}.
\end{aligned} \tag{33}$$

□

Theorem 18. *For $i = 1,2,\ldots,n$, let $m_i \in \aleph$. Then the inequality*

$$\left(\sum_{i=1}^n \gamma_q^{(m_i)}(x_i,z)\right)^{1/u} \le \sum_{i=1}^n \left(\gamma_q^{(m_i)}(x_i,z)\right)^{1/u} \tag{34}$$

holds for $x_i > 0$ and $u \ge 1$.

Proof. Note that $\sum_{i=1}^{n} a_i^u \leq (\sum_{i=1}^{n} a_i)^u$ for $a_i \geq 0$ and $u \geq 1$. Then by the generalized q-Minkowski's inequality (19), we obtain

$$\begin{aligned}
&\left(\sum_{i=1}^{n}\gamma_q^{(m_i)}(x_i,z)\right)^{1/u} \\
&= \left(\sum_{i=1}^{n}\int_0^z t^{x_i-1}(\ln t)^{m_i}E_q^{-qt}d_qt\right)^{1/u} \\
&= \left(\int_0^z\left[\sum_{i=1}^{n}\left(t^{(x_i-1)/u}(\ln t)^{m_i/u}E_q^{-qt/u}\right)^u\right]d_qt\right)^{1/u} \\
&\leq \left(\int_0^z\left[\sum_{i=1}^{n}\left(t^{(x_i-1)/u}(\ln t)^{m_i/u}E_q^{-qt/u}\right)\right]^u d_qt\right)^{1/u} \\
&\leq \sum_{i=1}^{n}\left(\int_0^z\left(t^{(x_i-1)/u}(\ln t)^{m_i/u}E_q^{-qt/u}\right)^u d_qt\right)^{1/u} \\
&= \sum_{i=1}^{n}\left(\int_0^z t^{x_i-1}(\ln t)^{m_i}E_q^{-qt}d_qt\right)^{1/u} \\
&= \sum_{i=1}^{n}\left(\gamma_q^{(m_i)}(x_i,z)\right)^{1/u}
\end{aligned} \tag{35}$$

which completes the proof. □

Remark 19. In particular, let $n = 2$, $m_1 = m$, $m_2 = n$, $x_1 = x$, and $x_2 = y$ in Theorem 18. Then we obtain

$$\begin{aligned}
&\left(\gamma_q^{(m)}(x,z)+\gamma_q^{(n)}(y,z)\right)^{1/u} \\
&\quad\leq \left(\gamma_q^{(m)}(x,z)\right)^{1/u}+\left(\gamma_q^{(n)}(y,z)\right)^{1/u}.
\end{aligned} \tag{36}$$

Theorem 20. *For $i = 1, 2, \ldots, n$, let $\alpha_i > 1$, $\sum_{i=1}^{n}(1/\alpha_i) = 1$, $m_i \in \aleph$, and $\sum_{i=1}^{n}(m_i/\alpha_i) \in \aleph$. Then the inequality*

$$\begin{aligned}
&\exp\left\{\gamma_q^{(\sum_{i=1}^{n}(m_i/\alpha_i))}\left(\sum_{i=1}^{n}\frac{x_i}{\alpha_i},z\right)\right\} \\
&\quad\leq \prod_{i=1}^{n}\left(\exp\left\{\gamma_q^{(m_i)}(x_i,z)\right\}\right)^{1/\alpha_i}
\end{aligned} \tag{37}$$

is satisfied for $x_i > 0$.

Proof. By (26) we obtain

$$\begin{aligned}
&\gamma_q^{(\sum_{i=1}^{n}(m_i/\alpha_i))}\left(\sum_{i=1}^{n}\frac{x_i}{\alpha_i},z\right)-\sum_{i=1}^{n}\frac{\gamma_q^{(m_i)}(x_i,z)}{\alpha_i} \\
&= \int_0^z t^{\sum_{i=1}^{n}(x_i/\alpha_i)-1}(\ln t)^{\sum_{i=1}^{n}(m_i/\alpha_i)}E_q^{-qt}d_qt-\sum_{i=1}^{n}\frac{1}{\alpha_i} \\
&\cdot\int_0^z t^{x_i-1}(\ln t)^{m_i}E_q^{-qt}d_qt \\
&= \int_0^z\prod_{i=1}^{n}t^{(x_i-1)/\alpha_i}(\ln t)^{m_i/\alpha_i}E_q^{-qt}d_qt-\sum_{i=1}^{n}\frac{1}{\alpha_i} \\
&\cdot\int_0^z t^{x_i-1}(\ln t)^{m_i}E_q^{-qt}d_qt \\
&= \int_0^z\left[\prod_{i=1}^{n}\left(t^{x_i-1}(\ln t)^{m_i}\right)^{1/\alpha_i}-\sum_{i=1}^{n}\frac{1}{\alpha_i}t^{x_i-1}(\ln t)^{m_i}\right] \\
&\cdot E_q^{-qt}d_qt \leq 0
\end{aligned} \tag{38}$$

which results from the weighted AM-GM inequality (7). Hence

$$\gamma_q^{(\sum_{i=1}^{n}(m_i/\alpha_i))}\left(\sum_{i=1}^{n}\frac{x_i}{\alpha_i},z\right)\leq\sum_{i=1}^{n}\frac{\gamma_q^{(m_i)}(x_i,z)}{\alpha_i}. \tag{39}$$

Then, by exponentiating (39), we obtain the required result (37). □

Remark 21. Results of types (27), (34), and (37) which deal with the $(q \cdot k)$-Gamma function can also be found in [9].

5. Conclusion

In this study, we provided simple proofs of the discrete forms of some generalized Hölder's and Minkowski's inequalities. Based on these results, we established some generalized Hölder's and Minkowski's inequalities for Jackson's q-integral. Furthermore, by using the established results, we derived some new inequalities involving the incomplete q-Gamma function. We anticipate that the present results will find some applications in q-Calculus as well as other related disciplines.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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