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*Research Article*

# Nonnegative Infinite Matrices that Preserve $(p, q)$-Convexity of Sequences

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This paper deals with matrix transformations that preserve the $(p, q)$-convexity of sequences. The main result gives the necessary and sufficient conditions for a nonnegative infinite matrix $A$ to preserve the $(p, q)$-convexity of sequences. Further, we give examples of such matrices for different values of $p$ and $q$.

## 1. Introduction

If $p > 0, q > 0$, then the sequence $\{x_n\}$ of real numbers is said to be $(p, q)$-convex if

$$\Delta_{p,q}(x_n) = x_n - (p+q)x_{n-1} + pqx_{n-2} \geq 0 \tag{1}$$

for $n \geq 2$. The operator $\Delta_{p,q}$ generates the second-order difference $\Delta^2$ when $p = q = 1$. Several authors [1–3] have proved various results on the convex sequences defined by $\Delta^2 x_n \geq 0$. Other authors [4, 5] have studied the classes of sequences satisfying $\Delta_{1,q}(x_n) \geq 0$. Also, the necessary and sufficient conditions for a sequence to be a $(p, q)$-convex sequence can be found in [6]. Moreover, some inequalities on $(p, q)$-convex sequences are given in [7, 8].

In [9–11], the authors discuss the matrix transformations that preserve $(p, q)$-convexity of sequences in the case of a lower triangular matrix with a particular type of matrix transformation. But the question of a general infinite matrix preserving $(p, q)$-convexity has not been considered anywhere in the literature. This paper deals with the necessary and sufficient conditions for a nonnegative infinite matrix to preserve $(p, q)$-convexity in both settings when $p \neq q$ and $p = q$.

## 2. Preliminaries

For any given sequence $\{x_n\}$, we can find a corresponding sequence $\{c_k\}$ such that

$$\begin{aligned} c_0 &= x_0, \\ c_1 &= x_1 - (p+q)c_0 \end{aligned} \tag{2}$$

and, for $k \geq 2$,

$$c_k = x_k - \sum_{i=0}^{k-1} \left(p^{k-i} + p^{k-i-1}q + \cdots + pq^{k-i-1} + q^{k-i}\right) c_i, \tag{3}$$

which implies that $\{x_n\}$ can be represented by

$$\begin{aligned} x_0 &= c_0, \\ x_1 &= c_1 + (p+q)c_0, \end{aligned} \tag{4}$$

and, for $n \geq 2$,

$$\begin{aligned} x_n &= c_n + (p+q)c_{n-1} + \left(p^2 + pq + q^2\right)c_{n-2} + \cdots \\ &\quad + \left(p^n + p^{n-1}q + \cdots + pq^{n-1} + q^n\right)c_0 \\ &= c_n + \sum_{i=1}^{n} \left(p^i + p^{i-1}q + \cdots + pq^{n-i} + q^i\right) c_{n-i}. \end{aligned} \tag{5}$$

As a consequence, we get the following lemma. A variation of this lemma can be found in [6].

**Lemma 1.** *If the sequence $\{x_n\}$ is given by representation (5), then $\Delta_{p,q}(x_n) = c_n$. Thus, the sequence $\{x_n\}$ is $(p,q)$-convex if and only if $c_n \geq 0$ for $n \geq 2$.*

*Proof.* It suffices to show that $\Delta_{p,q}(x_n) = x_n - (p+q)x_{n-1} + pqx_{n-2} = c_n$ for $n \geq 2$. Using (5),

$$\begin{aligned}\Delta_{p,q}(x_n) = &\Bigg(c_n + (p+q)c_{n-1} + (p^2+pq+q^2)c_{n-2} \\ &+ \cdots + \left(\sum_0^n p^{n-k}q^k\right)c_0\Bigg) - (p+q)\Bigg(c_{n-1} \\ &+ (p+q)c_{n-2} + (p^2+pq+q^2)c_{n-3} + \cdots \\ &+ \left(\sum_0^{n-1} p^{n-k-1}q^k\right)c_0\Bigg) + pq\Bigg(c_{n-2} + (p+q)c_{n-3} \\ &+ (p^2+pq+q^2)c_{n-4} + \cdots + \left(\sum_0^{n-2} p^{n-k-2}q^k\right)c_0\Bigg).\end{aligned} \tag{6}$$

On the right side, we see that the coefficient of $c_n = 1$, and the coefficient of $c_{n-r} = 0$ for $r = 1, 2, \ldots, n$. Thus,

$$\Delta_{p,q}(x_n) = c_n \quad \text{for } n \geq 2. \tag{7}$$

Hence, we have the previous lemma.

Also, in (5), the representation of $x_n$ in terms of $c_n$ can be written as follows:

$$\begin{aligned}x_n &= c_n + \sum_{i=0}^{n-1}\left(p^{n-i} + p^{n-i-1}q + \cdots + q^{n-i}\right)c_i \\ &= \begin{cases} c_n + \displaystyle\sum_{i=0}^{n-1}\left(\frac{p^{n-i+1} - q^{n-i+1}}{p-q}\right)c_i, & \text{if } p \neq q \\ c_n + \displaystyle\sum_{i=0}^{n-1}(n-i+1)p^{n-i}c_i, & \text{if } p = q \end{cases} \\ &= \begin{cases} \displaystyle\sum_{i=0}^{n}\left(\frac{p^{n-i+1} - q^{n-i+1}}{p-q}\right)c_i, & \text{if } p \neq q \\ \displaystyle\sum_{i=0}^{n}(n-i+1)p^{n-i}c_i & \text{if } p = q. \end{cases}\end{aligned} \tag{8}$$

Now, we give below some definitions. Let $A = [a_{n,k}]$ be a nonnegative infinite matrix defining a sequence to sequence transformation by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{n,k}x_k. \tag{9}$$

Then, we define the matrices $[\alpha_{n,k}]$ and $[\beta_{n,k}]$ as

$$\begin{aligned}\alpha_{n,k} &= \sum_{j=k}^{\infty} p^{j-k}a_{n,j} = a_{n,k} + pa_{n,k+1} + p^2a_{n,k+2} + \cdots, \\ \beta_{n,i} &= \sum_{k=i}^{\infty} q^{k-i}\alpha_{n,k} = \alpha_{n,i} + q\alpha_{n,i+1} + q^2\alpha_{n,i+2} + \cdots \\ &= \sum_{k=i}^{\infty} q^{k-i}\left(\sum_{j=k}^{\infty} p^{j-k}a_{n,j}\right).\end{aligned} \tag{10}$$

Interchanging the order of summation, we get, for each $n = 0, 1, 2, \ldots$, and $i = 0, 1, 2, \ldots$,

$$\begin{aligned}\beta_{n,i} &= \sum_{j=i}^{\infty}\left(\sum_{k=i}^{j} q^{k-i}p^{j-k}\right)a_{n,j} \\ &= \sum_{j=i}^{\infty}\left(p^{j-i} + qp^{j-i-1} + q^2p^{j-i-2} + \cdots + q^{j-i}\right)a_{n,j} \\ &= \begin{cases} \displaystyle\sum_{j=i}^{\infty}\left(\frac{p^{j-i+1} - q^{j-i+1}}{p-q}\right)a_{n,j}, & \text{if } p \neq q \\ \displaystyle\sum_{j=i}^{\infty}(j-i+1)p^{j-i}a_{n,j}, & \text{if } p = q. \end{cases}\end{aligned} \tag{11}$$

Furthermore, for $n \geq 2$,

$$\begin{aligned}\Delta_{p,q}(\beta_{n,i}) &= \beta_{n,i} - (p+q)\beta_{n-1,i} + pq\beta_{n-2,i} \\ &= \begin{cases} \displaystyle\sum_{j=i}^{\infty}\left(\frac{p^{j-i+1} - q^{j-i+1}}{p-q}\right)\Delta_{p,q}(a_{n,j}), & \text{if } p \neq q \\ \displaystyle\sum_{j=i}^{\infty}(j-i+1)p^{j-i}\Delta_{p,q}(a_{n,j}), & \text{if } p = q. \end{cases}\end{aligned} \tag{12}$$

In order for the matrix $[\beta_{n,i}]$ to be well-defined, we need the matrix $[a_{n,k}]$ to satisfy certain conditions which will depend on the values of $p$ and $q$.

*(I) When $p \neq q$*, due to symmetry of $p$ and $q$ in the definition of $\beta_{n,i}$, it is sufficient to consider the following cases:

$$\begin{aligned}&\text{(a)} \quad 0 < p, q < 1 \\ &\text{(b)} \quad 0 < p < 1,\ q = 1 \\ &\text{(c)} \quad p > 1,\ q = 1 \\ &\text{(d)} \quad p > 1,\ 0 < q < 1 \\ &\text{(e)} \quad p, q > 1\end{aligned} \tag{13}$$

*Case (a).* For $0 < p, q < 1$, we require the matrix $A$ to satisfy that, for each $n$,

$$\sum_{k=1}^{\infty} ka_{n,k} < \infty. \tag{14}$$

Thus, using (11) and $p, q < 1$, we have

$$\begin{aligned}\beta_{n,i} &= \sum_{j=i}^{\infty}\left(p^{j-i} + qp^{j-i-1} + \cdots + q^{j-i}\right)a_{n,j} \\ &< \sum_{j=i}^{\infty}(j-i+1)\,a_{n,j} = \sum_{j=i}^{\infty}(j-i)\,a_{n,j} + \sum_{j=i}^{\infty}a_{n,j} < \infty \\ &\qquad \text{by (14)}.\end{aligned} \tag{15}$$

Thus, $\beta_{n,i}$ is well-defined.

*Case (b).* For $0 < p < 1,\ q = 1$, we require the matrix $A$ to satisfy that, for each $n$,

$$\sum_{k=0}^{\infty} a_{n,k} < \infty. \tag{16}$$

Then using (11), we have

$$\begin{aligned}\beta_{n,i} &= \sum_{j=i}^{\infty}\left(\frac{1-p^{j-i+1}}{1-p}\right)a_{n,j} \\ &= \frac{1}{1-p}\left((1-p)\,a_{n,i} + \left(1-p^2\right)a_{n,i+1} + \cdots\right) \\ &< \frac{1}{1-p}\left(a_{n,i} + a_{n,i+1} + \cdots\right), \quad \text{since } 0 < p < 1 \\ &< \infty \quad \text{by (16)}.\end{aligned} \tag{17}$$

Thus, $\beta_{n,i}$ is well-defined.

For the cases (c), (d), and (e), we require the matrix $A$ to satisfy that, for each $n$,

$$\sum_{k=0}^{\infty} p^k a_{n,k} < \infty. \tag{18}$$

*Case (c).* When $p > 1,\ q = 1$, we have, as in the case (b),

$$\begin{aligned}\beta_{n,i} &= \sum_{j=i}^{\infty}\left(\frac{p^{j-i+1}-1}{p-1}\right)a_{n,j} \\ &= \frac{p}{p-1}\sum_{j=i}^{\infty}\left(p^{j-i} - \frac{1}{p}\right)a_{n,j} < \frac{p}{p-1}\sum_{j=i}^{\infty}p^{j-i}a_{n,j} \\ &\le \frac{p}{p-1}\sum_{j=i}^{\infty}p^{j}a_{n,j} < \infty \quad \text{by (18)}.\end{aligned} \tag{19}$$

Thus, $\beta_{n,i}$ is well-defined.

*Case (d).* When $p > 1,\ 0 < q < 1$, from (11),

$$\beta_{n,i} = \frac{1}{p-q}\sum_{j=i}^{\infty}\left(p^{j-i+1} - q^{j-i+1}\right)a_{n,j}. \tag{20}$$

Since $q < p$, using (18), we have $\sum_{j=i}^{\infty} q^{j-i}a_{n,j} < \sum_{j=i}^{\infty} p^{j-i}a_{n,j} < \infty$. Therefore,

$$\beta_{n,i} = \frac{p}{p-q}\sum_{j=i}^{\infty}p^{j-i}a_{n,j} - \frac{q}{p-q}\sum_{j=i}^{\infty}q^{j-i}a_{n,j} < \infty. \tag{21}$$

Thus $\beta_{n,i}$ is well-defined.

*Case (e).* When $p, q > 1$, we can assume without loss of generality that $p > q$.

Proceeding as in case (d), we see that $\beta_{n,i}$ is well-defined in this case also.

*(II) When $p = q$*, we consider the following cases:

$$\begin{aligned}&\text{(f)} \quad 0 < p < 1 \\ &\text{(g)} \quad p = 1 \\ &\text{(h)} \quad p > 1.\end{aligned} \tag{22}$$

*Case (f).* For $0 < p < 1$, we require the matrix $A$ to satisfy that, for each $n$,

$$\sum_{k=1}^{\infty} k a_{n,k} < \infty. \tag{23}$$

Then, using (11), we have

$$\begin{aligned}\beta_{n,i} &= \sum_{j=i}^{\infty}(j-i+1)\,p^{j-i}a_{n,j} < \sum_{j=i}^{\infty}(j-i+1)\,a_{n,j} \\ &= \sum_{j=i}^{\infty}(j-i)\,a_{n,j} + \sum_{j=i}^{\infty}a_{n,j} < \infty \quad \text{by (23)}.\end{aligned} \tag{24}$$

Thus, $\beta_{n,i}$ is well-defined.

*Case (g).* When $p = 1$, $\Delta_{p,q}$-convexity reduces to the well-known second-order convexity $\Delta^2$, which has been investigated in detail in [3].

*Case (h).* For $p > 1$, we require the matrix $A$ to satisfy that, for each $n$,

$$\sum_{k=1}^{\infty} k p^k a_{n,k} < \infty. \tag{25}$$

Then, using (11), we have

$$\begin{aligned}\beta_{n,i} &= \sum_{j=i}^{\infty}(j-i+1)\,p^{j-i}a_{n,j} \\ &\le \sum_{j=i}^{\infty}(j-i)\,p^{j-i}a_{n,j} + \sum_{j=i}^{\infty}p^{j}a_{n,j} < \infty \quad \text{by (25)}.\end{aligned} \tag{26}$$

Thus, $\beta_{n,i}$ is well-defined. □

## 3. Main Results

In this section, we prove the necessary and sufficient conditions for a nonnegative infinite matrix $A$ to transform a $(p,q)$-convex sequence into a $(p,q)$-convex sequence showing that each column of the corresponding matrix $[\beta_{n,k}]$ is a $(p,q)$-convex sequence.

First, we consider the values of $p$ and $q$, where $p \neq q$ results in the cases listed in (13).

**Theorem 2.** *For $p \neq q$, a nonnegative infinite matrix $A$ satisfying (14), (16), or (18), corresponding to the cases listed in (13), preserves $(p,q)$-convexity of sequences if and only if, for $n = 2, 3, 4, \ldots$,*

(i) $\Delta_{p,q}(\beta_{n,0}) = 0$

(ii) $\Delta_{p,q}(\beta_{n,1}) = 0$

(iii) $\Delta_{p,q}(\beta_{n,i}) \geq 0$ *for* $i \geq 2$

*where the matrix $[\beta_{n,i}]$ is defined by*

$$\beta_{n,i} = \sum_{j=i}^{\infty} \left( \frac{p^{j-i+1} - q^{j-i+1}}{p-q} \right) a_{n,j}. \tag{27}$$

*Proof.* First, we prove a result on the transformed sequence of any $(p,q)$-convex sequence $\{x_n\}$. Now, we have, from (8),

$$x_n = \sum_{i=0}^{n} \left( \frac{p^{n-i+1} - q^{n-i+1}}{p-q} \right) c_i, \tag{28}$$

where $c_i \geq 0$ for $i \geq 2$ by Lemma 1. Then, the $n$th term of the transformed sequence is

$$(Ax)_n = \sum_{k=0}^{\infty} a_{n,k} x_k = \sum_{k=0}^{\infty} a_{n,k} \sum_{i=0}^{k} \left( \frac{p^{k-i+1} - q^{k-i+1}}{p-q} \right) c_i. \tag{29}$$

Interchanging the order of summation,

$$\begin{aligned}(Ax)_n &= \sum_{i=0}^{\infty} c_i \sum_{k=i}^{\infty} \frac{p^{k-i+1} - q^{k-i+1}}{p-q} a_{n,k} \\ &= c_0 \sum_{k=0}^{\infty} \frac{p^{k+1} - q^{k+1}}{p-q} a_{n,k} + c_1 \sum_{k=1}^{\infty} \frac{p^{k} - q^{k}}{p-q} a_{n,k} \\ &\quad + \sum_{i=2}^{\infty} c_i \sum_{k=i}^{\infty} \frac{p^{k-i+1} - q^{k-i+1}}{p-q} a_{n,k}.\end{aligned} \tag{30}$$

From (11), we have

$$(Ax)_n = c_0 \beta_{n,0} + c_1 \beta_{n,1} + \sum_{i=2}^{\infty} c_i \beta_{n,i}. \tag{31}$$

Then, for $n \geq 2$,

$$\begin{aligned}\Delta_{p,q}(Ax)_n &= (Ax)_n - (p+q)(Ax)_{n-1} + pq(Ax)_{n-2} \\ &= \left( c_0 \beta_{n,0} + c_1 \beta_{n,1} + \sum_{i=2}^{\infty} c_i \beta_{n,i} \right) \\ &\quad - (p+q) \left( c_0 \beta_{n-1,0} + c_1 \beta_{n-1,1} + \sum_{i=2}^{\infty} c_i \beta_{n-1,i} \right) \\ &\quad + pq \left( c_0 \beta_{n-2,0} + c_1 \beta_{n-2,1} + \sum_{i=2}^{\infty} c_i \beta_{n-2,i} \right) \\ &= c_0 \left[ \beta_{n,0} - (p+q) \beta_{n-1,0} + pq \beta_{n-2,0} \right] \\ &\quad + c_1 \left[ \beta_{n,1} - (p+q) \beta_{n-1,1} + pq \beta_{n-2,1} \right] \\ &\quad + \sum_{i=2}^{\infty} c_i \left[ \beta_{n,i} - (p+q) \beta_{n-1,i} + pq \beta_{n-2,i} \right].\end{aligned} \tag{32}$$

Thus, for any $(p,q)$-convex sequence $\{x_n\}$,

$$\begin{aligned}\Delta_{p,q}(Ax)_n &= c_0 \Delta_{p,q}(\beta_{n,0}) + c_1 \Delta_{p,q}(\beta_{n,1}) \\ &\quad + \sum_{i=2}^{\infty} c_i \Delta_{p,q}(\beta_{n,i}).\end{aligned} \tag{33}$$

Now, to prove the sufficiency of the conditions given in the theorem, assume that (i), (ii), and (iii) are true. Then, by (33),

$$\Delta_{p,q}(Ax)_n \geq 0. \tag{34}$$

Thus, the sequence $(Ax)_n$ is also $(p,q)$-convex.

Conversely, assume that the matrix $A$ preserves $(p,q)$-convexity of the sequences. Suppose that the condition (i) fails to hold. Then there exists an integer $N \geq 2$ such that

$$\Delta_{p,q}(\beta_{N,0}) = L \neq 0. \tag{35}$$

Consider the following sequence:

$$u = \left\{ -L, \frac{-(p^2 - q^2)}{p-q} L, \frac{-(p^3 - q^3)}{p-q} L, \ldots \right\}. \tag{36}$$

Then $\{u_n\}$ is a $(p,q)$-convex sequence because, using (2) and Lemma 1,

$$\begin{aligned}c_0 &= u_0 = -L, \\ c_1 &= u_1 - (p+q) c_0 = 0\end{aligned} \tag{37}$$

and, for $i \geq 2$,

$$\begin{aligned}c_i &= \Delta_{p,q}(u_i) = u_i - (p+q) u_{i-1} + pq u_{i-2} \\ &= \frac{-(p^{i+1} - q^{i+1})}{p-q} L + (p+q) \frac{(p^i - q^i)}{p-q} L \\ &\quad - pq \frac{(p^{i-1} - q^{i-1})}{p-q} L = 0.\end{aligned} \tag{38}$$

Thus, from (33), for the transformed sequence $\{(Au)_n\}$,

$$\Delta_{p,q}(Au)_N = c_0\Delta_{p,q}(\beta_{N,0}) + c_1\Delta_{p,q}(\beta_{N,1}) + \sum_{i=2}^{\infty} c_i\Delta_{p,q}(\beta_{N,i}) = -L^2 < 0, \tag{39}$$

which contradicts that the transformed sequence $\{(Au)_n\}$ must be $(p,q)$-convex.

Next, suppose that the condition (ii) is not true. This case can be settled by a similar argument by considering the following sequence:

$$v = \left\{0, -L, \frac{-(p^2-q^2)}{p-q}L, \frac{-(p^3-q^3)}{p-q}L, \dots\right\}, \tag{40}$$

which implies that

$$\begin{aligned} c_0 &= 0, \\ c_1 &= -L, \\ c_i &= 0 \quad \text{for } i \geq 2. \end{aligned} \tag{41}$$

Now, suppose that the condition (iii) is not true. Then there exists an integer $j \geq 2$ such that the $j$th column-sequence of the matrix $[\beta_{n,k}]$ is not $(p,q)$-convex. That is, for some $N \geq 2$,

$$\Delta_{p,q}(\beta_{N,j}) = L < 0. \tag{42}$$

Now, consider the following sequence:

$$x = \Big\{\underset{\substack{\downarrow\\ x_0}}{0}, \dots, \underset{\substack{\downarrow\\ x_{j-1}}}{0}, \underset{\substack{\downarrow\\ x_j}}{1}, \underset{\substack{\downarrow\\ x_{j+1}}}{\frac{p^2-q^2}{p-q}}, \frac{p^3-q^3}{p-q}, \dots\Big\}. \tag{43}$$

Then, $\{x_n\}$ is a $(p,q)$-convex sequence, because, using (2) and Lemma 1, we get

$$\begin{aligned} c_i &= 0 \quad \text{for } 0 \leq i \leq j-1; \\ c_j &= 1; \\ c_{j+1} &= x_{j+1} - (p+q)x_j + pqx_{j-1} = 0; \end{aligned} \tag{44}$$

and, for $i \geq j+2$,

$$c_i = \Delta_{p,q}(x_i) = 0 \quad \text{as in (38)}. \tag{45}$$

But, from (33),

$$\begin{aligned} \Delta_{p,q}(Ax)_N &= c_0\Delta_{p,q}(\beta_{N,0}) + c_1\Delta_{p,q}(\beta_{N,1}) \\ &\quad + \sum_{i=2}^{\infty} c_i\Delta_{p,q}(\beta_{N,i}) = c_j\Delta_{p,q}(\beta_{N,j}) = L \\ &< 0, \end{aligned} \tag{46}$$

which again contradicts that $\{Ax\}$ is a $(p,q)$-convex sequence. This completes the proof. □

Theorem 2 generalizes the necessary and sufficient conditions given in [9, Theorem 2, p. 8] in the case of $p = 1$ and $q > 0$ with $q \neq 1$.

Next, we consider the values of $p$ and $q$ where $p = q$ results in the cases listed in (22).

**Theorem 3.** *For $p = q$, a nonnegative infinite matrix $A$ satisfying (23) or (25), corresponding to the cases listed in (22), preserves $(p,p)$-convexity of sequences if and only if, for $n = 2, 3, 4, \dots$,*

(i) $\Delta_{p,p}(\beta_{n,0}) = 0$

(ii) $\Delta_{p,p}(\beta_{n,1}) = 0$

(iii) $\Delta_{p,p}(\beta_{n,i}) \geq 0$ *for* $i = 2, 3, \dots$,

*where the matrix $[\beta_{n,i}]$ is defined by*

$$\beta_{n,i} = \sum_{j=i}^{\infty} (j-i+1)\, p^{j-i} a_{n,j}. \tag{47}$$

*Proof.* First we prove a result on the transformed sequence of any $(p,p)$-convex sequence $\{x_n\}$. Now, we have, from (8),

$$x_n = \sum_{i=0}^{n} (n-i+1)\, p^{n-i} c_i, \tag{48}$$

where $c_i \geq 0$ for $i \geq 2$ by Lemma 1. Then, the $n$th term of the transformed sequence is

$$(Ax)_n = \sum_{k=0}^{\infty} a_{n,k} x_k = \sum_{k=0}^{\infty} a_{n,k}\left(\sum_{i=0}^{k}(k-i+1)\, p^{k-i} c_i\right). \tag{49}$$

Interchanging the order of summation,

$$\begin{aligned} (Ax)_n &= \sum_{i=0}^{\infty} c_i \sum_{k=i}^{\infty} (k-i+1)\, p^{k-i} a_{n,k} \\ &= c_0\sum_{k=0}^{\infty}(k+1)\, p^k a_{n,k} + c_1\sum_{k=1}^{\infty} k p^{k-1} a_{n,k} \\ &\quad + \sum_{i=2}^{\infty} c_i \sum_{k=i}^{\infty}(k-i+1)\, p^{k-i} a_{n,k}. \end{aligned} \tag{50}$$

From (11), we have

$$(Ax)_n = c_0\beta_{n,0} + c_1\beta_{n,1} + \sum_{i=2}^{\infty} c_i\beta_{n,i}. \tag{51}$$

Then, for $n \geq 2$,

$$\begin{aligned} \Delta_{p,p}(Ax)_n &= (Ax)_n - 2p(Ax)_{n-1} + p^2(Ax)_{n-2} \\ &= \left(c_0\beta_{n,0} + c_1\beta_{n,1} + \sum_{i=2}^{\infty} c_i\beta_{n,i}\right) \\ &\quad - 2p\left(c_0\beta_{n-1,0} + c_1\beta_{n-1,1} + \sum_{i=2}^{\infty} c_i\beta_{n-1,i}\right) \\ &\quad + p^2\left(c_0\beta_{n-2,0} + c_1\beta_{n-2,1} + \sum_{i=2}^{\infty} c_i\beta_{n-2,i}\right). \end{aligned} \tag{52}$$

Thus, for any $(p, p)$-convex sequence $\{x_n\}$,

$$\Delta_{p,p}(Ax)_n = c_0\Delta_{p,p}(\beta_{n,0}) + c_1\Delta_{p,p}(\beta_{n,1}) + \sum_{i=2}^{\infty} c_i\Delta_{p,p}(\beta_{n,i}). \tag{53}$$

Now, to prove the sufficiency of the conditions given in the theorem, assume that (i), (ii), and (iii) are true. Then by (53),

$$\Delta_{p,p}(Ax)_n \geq 0. \tag{54}$$

Thus, the sequence $(Ax)_n$ is also $(p, p)$-convex.

Conversely, assume that the matrix $A$ preserves $(p, p)$-convexity of sequences.

Suppose that the condition (i) fails to hold. Then there exists an integer $N \geq 2$ such that

$$\Delta_{p,p}(\beta_{N,0}) = L \neq 0. \tag{55}$$

Consider the following sequence:

$$u = \{-L, -2pL, -3p^2L, \ldots\}. \tag{56}$$

It is easy to see, using (2) and Lemma 1, that $u$ is a $(p, p)$-convex sequence with

$$c_0 = u_o = -L, \quad c_i = 0 \quad \text{for } i \geq 1. \tag{57}$$

Thus, from (53), for the transformed sequence $\{(Au)_n\}$,

$$\Delta_{p,p}(Au)_N = c_0\Delta_{p,p}(\beta_{N,0}) + c_1\Delta_{p,p}(\beta_{N,1}) + \sum_{i=2}^{\infty} c_i\Delta_{p,p}(\beta_{N,i}) = -L^2 < 0, \tag{58}$$

which contradicts that $\{(Au)_n\}$ must be $(p, p)$-convex.

Next, suppose that the condition (ii) is not true. This case can be settled by a similar argument by considering the following sequence:

$$v = \{0, -L, -2pL, -3p^2L, \ldots\}, \tag{59}$$

which implies that

$$c_0 = 0, \quad c_1 = -L, \quad c_i = 0 \quad \text{for } i \geq 2. \tag{60}$$

Now, suppose that the condition (iii) is not true. Then there exists an integer $j \geq 2$ such that the $j$th column-sequence of the matrix $[\beta_{n,k}]$ is not $(p, p)$-convex. That is, for some $N \geq 2$,

$$\Delta_{p,p}(\beta_{N,j}) = L < 0. \tag{61}$$

Consider the $(p, p)$-convex sequence:

$$x = \{\underset{x_0}{\underset{\downarrow}{0}}, \ldots, \underset{x_{j-1}}{\underset{\downarrow}{0}}, \underset{x_j}{\underset{\downarrow}{1}}, \underset{x_{j+1}}{\underset{\downarrow}{2p}}, 3p^2, \ldots\}. \tag{62}$$

We see that, as in the proof of Theorem 2,

$$\Delta_{p,p}(Ax)_N = L < 0, \tag{63}$$

which contradicts that $\{Ax\}$ is a $(p, p)$-convex sequence.

We see that the result on the convexity of sequences given in [3, p. 331] is a particular case of Theorem 3 when $p = q = 1$. Also, this theorem generalizes the necessary and sufficient conditions for a triangular matrix given in [9, p. 4]. □

## 4. Examples

We give below examples of $(p, q)$-convexity preserving matrices for each of the cases (a) through (h) given in (13) and (22).

*Example for Case (a).* Considering $0 < p, q < 1$, and $p \neq q$, we can assume, without loss of generality, that $p < q$. Let the matrix $A = [a_{n,k}]$ be defined by

$$a_{n,k} = \begin{cases} p^n, & \text{if } k = 0, \\ \dfrac{p^n q^k}{k}, & \text{if } k \geq 1. \end{cases} \tag{64}$$

Then, for each $n$,

$$\sum_{k=1}^{\infty} k a_{n,k} = \sum_{k=1}^{\infty} p^n q^k = p^n\left(\frac{q}{1-q}\right) < \infty. \tag{65}$$

Thus, by (14), $\beta_{n,i}$ is well-defined for $n = 0, 1, 2, \ldots$ and $i = 0, 1, 2, \ldots$. The matrix $A$ satisfies the three conditions of Theorem 2 because, for $n \geq 2$, using (12),

$$\Delta_{p,q}(\beta_{n,i}) = \sum_{j=i}^{\infty}\left(\frac{p^{j-i+1} - q^{j-i+1}}{p-q}\right)\Delta_{p,q}(a_{n,j}), \tag{66}$$

in which

$$\begin{aligned} \Delta_{p,q}(a_{n,j}) &= a_{n,j} - (p+q)a_{n-1,j} + pqa_{n-2,j} \\ &= \begin{cases} p^n - (p+q)p^{n-1} + pqp^{n-2}, & \text{if } j = 0, \\ \dfrac{q^j}{j}\left(p^n - (p+q)p^{n-1} + pqp^{n-2}\right), & \text{if } j \geq 1 \end{cases} \\ &= 0. \end{aligned} \tag{67}$$

Therefore, the matrix $A$ preserves $(p, q)$-convexity of sequences.

*Example for Case (b).* Considering $0 < p < 1$, $q = 1$, let the matrix $A = [a_{n,k}]$ be defined by

$$a_{n,k} = p^k. \tag{68}$$

Then, for each $n$,

$$\sum_{k=0}^{\infty} a_{n,k} = \sum_{k=0}^{\infty} p^k = \frac{1}{1-p} < \infty. \tag{69}$$

Thus, by (16), $\beta_{n,i}$ is well-defined for $n = 0, 1, 2, \ldots$ and $i = 0, 1, 2, \ldots$. The matrix $A$ satisfies the three conditions of Theorem 2 because, for $n \geq 2$, using (12),

$$\Delta_{p,1}(\beta_{n,i}) = \sum_{j=i}^{\infty} \left( \frac{p^{j-i+1} - 1}{p - 1} \right) \Delta_{p,1}(a_{n,j}), \quad (70)$$

in which

$$\begin{aligned} \Delta_{p,1}(a_{n,j}) &= a_{n,j} - (p+1) a_{n-1,j} + p a_{n-2,j} \\ &= p^j - (p+1) p^j + p^{j+1} = 0. \end{aligned} \quad (71)$$

Therefore, the matrix $A$ preserves $(p, 1)$-convexity of sequences.

*Example for Case (c).* Considering $p > 1$, $q = 1$, let matrix $A = [a_{n,k}]$ be defined by

$$a_{n,k} = \frac{1}{p^{2k}}. \quad (72)$$

Then, for each $n$,

$$\sum_{k=0}^{\infty} p^k a_{n,k} = \sum_{k=0}^{\infty} \frac{1}{p^k} = \frac{1}{1 - 1/p} < \infty. \quad (73)$$

Thus, by (18), $\beta_{n,i}$ is well-defined for $n = 0, 1, 2, \ldots$ and $i = 0, 1, 2, \ldots$. The matrix $A$ satisfies the three conditions of Theorem 2 because, for $n \geq 2$, as in the previous example (b),

$$\begin{aligned} \Delta_{p,1}(a_{n,j}) &= a_{n,j} - (p+1) a_{n-1,j} + p a_{n-2,j} \\ &= \frac{1}{p^{2j}} - (p+1) \frac{1}{p^{2j}} + p \frac{1}{p^{2j}} = 0. \end{aligned} \quad (74)$$

Therefore, the matrix $A$ preserves $(p, 1)$-convexity of sequences.

*Example for Case (d).* Considering $p > 1$, $0 < q < 1$, let matrix $A = [a_{n,k}]$ be defined by

$$a_{n,k} = p^{n-2k}. \quad (75)$$

Then, for each $n$,

$$\sum_{k=0}^{\infty} p^k a_{n,k} = \sum_{k=0}^{\infty} p^{n-k} = \frac{p^{n+1}}{p - 1} < \infty. \quad (76)$$

Thus, by (18), $\beta_{n,i}$ is well-defined for $n = 0, 1, 2, \ldots$ and $i = 0, 1, 2, \ldots$. The matrix $A$ satisfies the three conditions of Theorem 2 because, for $n \geq 2$, using (12),

$$\Delta_{p,q}(\beta_{n,i}) = \sum_{j=i}^{\infty} \left( \frac{p^{j-i+1} - q^{j-i+1}}{p - q} \right) \Delta_{p,q}(a_{n,j}), \quad (77)$$

in which

$$\begin{aligned} \Delta_{p,q}(a_{n,j}) &= a_{n,j} - (p+q) a_{n-1,j} + pq a_{n-2,j} \\ &= \frac{1}{p^{2j}} \left( p^n - (p+q) p^{n-1} + pq p^{n-2} \right) = 0. \end{aligned} \quad (78)$$

Therefore, the matrix $A$ preserves $(p, q)$-convexity of sequences.

*Example for Case (e).* Considering $p, q > 1$ and $p \neq q$, we can assume, without loss of generality, that $p > q$. Let the matrix $A = [a_{n,k}]$ be defined by

$$a_{n,k} = p^{n-2k} q^k. \quad (79)$$

Then, for each $n$,

$$\sum_{k=0}^{\infty} p^k a_{n,k} = p^n \sum_{k=0}^{\infty} \left( \frac{q}{p} \right)^k = \frac{p^{n+1}}{p - q} < \infty. \quad (80)$$

Thus, by (18), $\beta_{n,i}$ is well-defined for $n = 0, 1, 2, \ldots$ and $i = 0, 1, 2, \ldots$. The matrix $A$ satisfies the three conditions of Theorem 2 because, for $n \geq 2$, as in the previous example (d),

$$\begin{aligned} \Delta_{p,q}(a_{n,j}) &= a_{n,j} - (p+q) a_{n-1,j} + pq a_{n-2,j} \\ &= \frac{q^j}{p^{2j}} \left( p^n - (p+q) p^{n-1} + pq p^{n-2} \right) = 0. \end{aligned} \quad (81)$$

Therefore, the matrix $A$ preserves $(p, q)$-convexity of sequences.

*Example for Case (f).* Considering $0 < p = q < 1$, let the matrix $A = [a_{n,k}]$ be defined by

$$a_{n,k} = \begin{cases} p^n, & \text{if } k = 0, \\ \dfrac{p^{n+k}}{k}, & \text{if } k \geq 1. \end{cases} \quad (82)$$

Then, for each $n$,

$$\sum_{k=1}^{\infty} k a_{n,k} = \sum_{k=1}^{\infty} p^{n+k} = p^n \sum_{k=1}^{\infty} p^k = p^{n+1} \left( \frac{1}{1 - p} \right) < \infty. \quad (83)$$

Thus, by (23), $\beta_{n,i}$ is well-defined for $n = 0, 1, 2, \ldots$ and $i = 0, 1, 2, \ldots$. The matrix $A$ satisfies the three conditions of Theorem 3 because, for $n \geq 2$, using (12),

$$\Delta_{p,p}(\beta_{n,i}) = \sum_{j=i}^{\infty} (j - i + 1) p^{j-i} \Delta_{p,p}(a_{n,j}), \quad (84)$$

in which

$$\begin{aligned} \Delta_{p,p}(a_{n,j}) &= a_{n,j} - 2p a_{n-1,j} + p^2 a_{n-2,j} \\ &= \begin{cases} p^n - 2p p^{n-1} + p^2 p^{n-2}, & \text{if } j = 0, \\ \dfrac{p^j}{j} \left( p^n - 2p p^{n-1} + p^2 p^{n-2} \right), & \text{if } j \geq 1 \end{cases} = 0. \end{aligned} \quad (85)$$

Therefore, the matrix $A$ preserves $(p, p)$-convexity of sequences.

*Examples for Case (g).* They can be found in [3], since $\Delta_{1,1}$ is the same as the second-order convexity $\Delta^2$.

*Example for Case (h).* Considering $p = q > 1$, let the matrix $A = [a_{n,k}]$ be defined by

$$a_{n,k} = \begin{cases} p^n(n+2), & \text{if } k = 0, \\ \dfrac{p^{n-2k}(n+2)}{k}, & \text{if } k \geq 1. \end{cases} \quad (86)$$

Therefore, for each $n$,

$$\sum_{k=1}^{\infty} kp^k a_{n,k} = \sum_{k=1}^{\infty} p^{n-k}(n+2) = p^n(n+2)\sum_{k=1}^{\infty}\left(\frac{1}{p}\right)^k = (n+2)\frac{p^n}{p-1} < \infty. \quad (87)$$

Thus, by (23), $\beta_{n,i}$ is well-defined for $n = 0, 1, 2, \ldots$ and $i = 0, 1, 2, \ldots$. The matrix $A$ satisfies the three conditions of Theorem 3 because, for $n \geq 2$, using (12),

$$\Delta_{p,p}(\beta_{n,i}) = \sum_{j=i}^{\infty}(j-i+1)p^{j-i}\Delta_{p,p}(a_{n,j}), \quad (88)$$

in which

$$\begin{aligned} \Delta_{p,p}(a_{n,j}) &= a_{n,j} - 2pa_{n-1,j} + p^2 a_{n-2,j} \\ &= \begin{cases} p^n(n+2) - 2p^n(n+1) + p^n n & \text{if } j = 0, \\ \dfrac{(n+2)p^{n-2j}}{j} - 2\dfrac{(n+1)p^{n-2j}}{j} + \dfrac{np^{n-2j}}{j}, & \text{if } j \geq 1. \end{cases} \\ &= 0. \end{aligned} \quad (89)$$

Therefore, the matrix $A$ preserves the convexity of sequences.

We conclude this paper by giving an example of an infinite matrix which does not preserve $(p, q)$-convexity of sequences.

It is interesting to notice that the Borel matrix preserves the $(1, 1)$-convexity of sequences [3, p. 336], but it does not preserve $(p, p)$-convexity when $p \neq 1$.

The Borel matrix $B = [b_{n,k}]$ is defined by

$$b_{n,k} = \frac{n^k}{e^n k!}. \quad (90)$$

Then, for each $n$,

$$\sum_{k=1}^{\infty} kb_{n,k} = \frac{n}{e^n}\sum_{k=1}^{\infty}\frac{n^{k-1}}{(k-1)!} = n < \infty, \quad (91)$$

$$\sum_{k=1}^{\infty} kp^k b_{n,k} = \frac{(np)}{e^n}\sum_{k=1}^{\infty}\frac{(np)^{k-1}}{(k-1)!} = \frac{(np)}{e^n}e^{np} < \infty. \quad (92)$$

Thus, for each of the cases, $0 < p < 1$ and $p > 1$, we see that (23) and (25) are satisfied and hence $\beta_{n,i}$ is well-defined for $n = 0, 1, 2, \ldots$ and $i = 0, 1, 2\ldots$.

From (11),

$$\beta_{n,i} = \sum_{j=i}^{\infty}(j-i+1)p^{j-i}b_{n,j}. \quad (93)$$

Therefore,

$$\begin{aligned} \beta_{n,0} &= \sum_{j=0}^{\infty}(j+1)p^j\frac{n^j}{e^n j!} \\ &= \frac{1}{e^n}\left(\sum_{j=0}^{\infty}\frac{j(pn)^j}{j!} + \sum_{j=0}^{\infty}\frac{(pn)^j}{j!}\right) \\ &= \frac{1}{e^n}\left(pne^{pn} + e^{pn}\right) = e^{n(p-1)}(pn+1), \end{aligned} \quad (94)$$

which implies that

$$\begin{aligned} \Delta_{p,p}(\beta_{n,0}) &= \beta_{n,0} - 2p\beta_{n-1,0} + p^2\beta_{n-2,0} \\ &= e^{n(p-1)}(pn+1) - 2pe^{(n-1)(p-1)}(p(n-1)+1) \\ &\quad + p^2 e^{(n-2)(p-1)}(p(n-2)+1) \\ &= \frac{e^{n(p-1)}}{e^{2p}}\left((pn+1)(e^p - pe)^2 + 2p^2 e(e^p - pe)\right) \\ &> 0, \end{aligned} \quad (95)$$

since $e^p - pe > 0$ when $p \neq 1$. Thus, the condition (i) of Theorem 3 fails in the case of Borel matrix.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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