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Research Article

Implicit One-Step Block Hybrid Third-Derivative Method for the Direct Solution of Initial Value Problems of Second-Order Ordinary Differential Equations

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A new one-step block method with generalized three hybrid points for solving initial value problems of second-order ordinary differential equations directly is proposed. In deriving this method, a power series approximate function is interpolated at $\{x_n, x_{n+r}\}$ while its second and third derivatives are collocated at all points $\{x_n, x_{n+r}, x_{n+s}, x_{n+t}, x_{n+1}\}$ in the given interval. The proposed method is then tested on initial value problems of second-order ordinary differential equations solved by other methods previously. The numerical results confirm the superiority of the new method to the existing methods in terms of accuracy.

1. Introduction

Numerous problems such as chemical kinetics, orbital dynamics, circuit and control theory, and Newton's second law applications involve second-order ordinary differential equations (ODEs). Normally, those equations have no analytical solutions. To approximate the solution of such problems several numerical methods were developed on the hands of many scholars such as [1–3].

Block methods for solving ODEs were first proposed by Milne ([4]). Later [5] adopted Milne's methods to provide starting values for predictor-corrector scheme. However, the block methods have some drawbacks and this led to the introduction of hybrid methods. According to [6], hybrid methods were initially introduced to overcome zero-stability barrier that occurred in block methods in Dahlquists ([7]). Besides the ability to change step size, the other benefit of these methods is utilizing data off-step points which contribute to the accuracy of the methods.

To increase the accuracy of the numerical methods further, researchers such as [8, 9] proposed high method derivative to overcome stiffness in ODEs. The former presented another type of hybrid methods called second-derivative methods, while the later proposed a Simpson's type second-derivative method for the solution of a stiff system of first-order IVPs. These scholars motivated us to develop a new generalized three-hybrid one-step third-derivative implicit method for solving second-order ODEs directly using the approach of interpolation and collocation for the general use to improve the efficiency of the approximate solution.

This article is organized as follows: in the coming section we demonstrate the derivation of the method, where we consider three off-step points through the approach of interpolation and collocation. The details of the analysis of the method are discussed in Section 3 which include zero stability, order, consistency, and convergence. In Section 4 some numerical problems are solved and the performance of the developed method is compared with other methods mentioned in literature. Finally, the conclusion is discussed in Section 5.

2. Development of the Method

An approximate power series basis function taking the form

$$p(x) = \sum_{j=0}^{2v+u-1} a_j \left(\frac{x - x_n}{h}\right)^j, \tag{1}$$

where $u = 2$ and $v = 5$ are the number of interpolation and collocation points, respectively, is considered to be a solution to the following ODE:

$$\begin{aligned} y'' &= f\left(x, y, y'\right), \\ y\left(a\right) &= p_0, \\ y_0'\left(a\right) &= p_0', \\ & \qquad x \in [a, b]. \end{aligned} \tag{2}$$

On derivation of (1) twice and thrice we obtain

$$\begin{aligned} p''\left(x\right) &= \sum_{j=2}^{2v+u-1} \frac{a_j j!}{h^2\left(j-2\right)!}\left(\frac{x-x_n}{h}\right)^{j-2} \\ &= f\left(x, y, y'\right), \\ p'''\left(x\right) &= \sum_{j=3}^{2v+u-1} \frac{a_j j!}{h^3\left(j-3\right)!}\left(\frac{x-x_n}{h}\right)^{j-3} \\ &= g\left(x, y, y'\right). \end{aligned} \tag{3}$$

Interpolating (1) at $x_{n+\widehat{u}} = x_n + \widehat{u}h$, $\widehat{u} = \{0, r\}$ and collocating (3) at all points $x_{n+\widehat{v}} = x_n + \widehat{v}h$, $\widehat{v} = \{0, r, s, t, 1\}$, where $\{r, s, t\} \in (0, 1)$, a system of equations in matrix form is produced as below:

$$AX = U, \tag{4}$$

where

$$A = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \\ a_{11} \end{bmatrix},$$

X

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & r & r^2 & r^3 & r^4 & \cdots & r^{(2v+u-1)} \\ 0 & 0 & \dfrac{2!}{0!h^2} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \dfrac{2!}{0!h^2} & \dfrac{3!r}{1!h^2} & \dfrac{4!r^2}{2!h^2} & \cdots & \dfrac{(2v+u-1)!r^{(2v+u-3)}}{(2v+u-3)!h^2} \\ 0 & 0 & \dfrac{2!}{0!h^2} & \dfrac{3!s}{1!h^2} & \dfrac{4!s^2}{2!h^2} & \cdots & \dfrac{(2v+u-1)!s^{(2v+u-3)}}{(2v+u-3)!h^2} \\ 0 & 0 & \dfrac{2!}{0!h^2} & \dfrac{3!t}{1!h^2} & \dfrac{4!t^2}{2!h^2} & \cdots & \dfrac{(2v+u-1)!t^{(2v+u-3)}}{(2v+u-3)!h^2} \\ 0 & 0 & \dfrac{2!}{0!h^2} & \dfrac{3!}{1!h^2} & \dfrac{4!}{2!h^2} & \cdots & \dfrac{(2v+u-1)!}{(2v+u-3)!h^2} \\ 0 & 0 & 0 & \dfrac{3!}{0!h^3} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \dfrac{3!r}{0!h^3} & \dfrac{4!r^2}{1!h^3} & \cdots & \dfrac{(2v+u-1)!r^{(2v+u-4)}}{(2v+u-4)!h^3} \\ 0 & 0 & 0 & \dfrac{3!s}{0!h^3} & \dfrac{4!s^2}{1!h^3} & \cdots & \dfrac{(2v+u-1)!s^{(2v+u-4)}}{(2v+u-4)!h^3} \\ 0 & 0 & 0 & \dfrac{3!t}{0!h^3} & \dfrac{4!t^2}{1!h^3} & \cdots & \dfrac{(2v+u-1)!t^{(2v+u-4)}}{(2v+u-4)!h^3} \\ 0 & 0 & 0 & \dfrac{3!}{0!h^3} & \dfrac{4!}{1!h^3} & \cdots & \dfrac{(2v+u-1)!}{(2v+u-4)!h^3} \end{bmatrix},$$

$$U = \begin{bmatrix} y_n \\ y_{n+r} \\ f_n \\ f_{n+r} \\ f_{n+s} \\ f_{n+t} \\ f_{n+1} \\ g_n \\ g_{n+r} \\ g_{n+s} \\ g_{n+t} \\ g_{n+1} \end{bmatrix}. \tag{5}$$

Using matrix manipulation to solve (4) for the unknown coefficients a_j's and then substituting them back into (1) yield

$$\begin{aligned} p\left(x\right) = \sum_{i=0,r} \alpha_i y_{n+i} + h^2 \left[\sum_{i=r,s,t} \beta_i f_{n+i} + \sum_{i=0}^{1} \beta_i f_{n+i}\right] \\ + h^3 \left[\sum_{i=r,s,t} \gamma_i g_{n+i} + \sum_{i=0}^{1} \gamma_i g_{n+i}\right], \end{aligned} \tag{6}$$

where $n = 0, 1, 2, \ldots, N-1$, $h = x_n - x_{n-1}$ is the constant step size for the partition π_N of the interval $[a, b]$ which

is given by $\pi_N = [a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b]$, α_i, β_i, and γ_i are undetermined constants listed in Appendix I in Supplementary Material available online at https://doi.org/10.1155/2017/8510948, $f_{n+i} = f(x + ih)$, and $g_{n+i} = df(x_{n+i}, y_{n+i}, y'_{n+i})/dx$ whose first partial derivative is

$$p'(x) = \frac{1}{h}\sum_{i=0,r}\alpha'_i y_{n+i} + h\left[\sum_{i=r,s,t}\beta'_i f_{n+i} + \sum_{i=0}^{1}\beta'_i f_{n+i}\right] + h^2\left[\sum_{i=r,s,t}\gamma'_i g_{n+i} + \sum_{i=0}^{1}\gamma'_i g_{n+i}\right]. \tag{7}$$

Evaluating (6) at the noninterpolating points $\{x_{n+s}, x_{n+t}, x_{n+1}\}$ and (7) at all points x_{n+i}, $i = \{0, r, s, t, 1\}$, produces the following general equations in block form:

$$A^{[0]}Y_m^{[1]} = A^{[1]}Y_m^{[0]} + \sum_{i=0}^{1}B^{[i]}F_m^{[i]} + \sum_{i=0}^{1}D^{[i]}G_m^{[i]}, \tag{8}$$

where $A^{[0]}$ is an 8×8 identity matrix and

$$A^{[1]} = \begin{bmatrix} 0&0&0&1&0&0&0&rh \\ 0&0&0&1&0&0&0&sh \\ 0&0&0&1&0&0&0&th \\ 0&0&0&1&0&0&0&h \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \end{bmatrix},$$

$$B^{[0]} = \begin{bmatrix} 0&0&0&0&0&0&0&B_{18}^{[0]} \\ 0&0&0&0&0&0&0&B_{28}^{[0]} \\ 0&0&0&0&0&0&0&B_{38}^{[0]} \\ 0&0&0&0&0&0&0&B_{48}^{[0]} \\ 0&0&0&0&0&0&0&B_{58}^{[0]} \\ 0&0&0&0&0&0&0&B_{68}^{[0]} \\ 0&0&0&0&0&0&0&B_{78}^{[0]} \\ 0&0&0&0&0&0&0&B_{88}^{[0]} \end{bmatrix},$$

$$B^{[1]} = \begin{bmatrix} B_{11}^{[1]}&B_{12}^{[1]}&B_{13}^{[1]}&B_{14}^{[1]} \\ B_{21}^{[1]}&B_{22}^{[1]}&B_{23}^{[1]}&B_{24}^{[1]} \\ B_{31}^{[1]}&B_{32}^{[1]}&B_{33}^{[1]}&B_{34}^{[1]} \\ B_{41}^{[1]}&B_{42}^{[1]}&B_{43}^{[1]}&B_{44}^{[1]} \\ B_{51}^{[1]}&B_{52}^{[1]}&B_{53}^{[1]}&B_{54}^{[1]} \\ B_{61}^{[1]}&B_{62}^{[1]}&B_{63}^{[1]}&B_{64}^{[1]} \\ B_{71}^{[1]}&B_{72}^{[1]}&B_{73}^{[1]}&B_{74}^{[1]} \\ B_{81}^{[1]}&B_{82}^{[1]}&B_{83}^{[1]}&B_{84}^{[1]} \end{bmatrix},$$

$$D^{[0]} = \begin{bmatrix} 0&0&0&0&0&0&0&D_{18}^{[0]} \\ 0&0&0&0&0&0&0&D_{28}^{[0]} \\ 0&0&0&0&0&0&0&D_{38}^{[0]} \\ 0&0&0&0&0&0&0&D_{48}^{[0]} \\ 0&0&0&0&0&0&0&D_{58}^{[0]} \\ 0&0&0&0&0&0&0&D_{68}^{[0]} \\ 0&0&0&0&0&0&0&D_{78}^{[0]} \\ 0&0&0&0&0&0&0&D_{88}^{[0]} \end{bmatrix},$$

$$D^{[1]} = \begin{bmatrix} D_{11}^{[1]}&D_{12}^{[1]}&D_{13}^{[1]}&D_{14}^{[1]} \\ D_{21}^{[1]}&D_{22}^{[1]}&D_{23}^{[1]}&D_{24}^{[1]} \\ D_{31}^{[1]}&D_{32}^{[1]}&D_{33}^{[1]}&D_{34}^{[1]} \\ D_{41}^{[1]}&D_{42}^{[1]}&D_{43}^{[1]}&D_{44}^{[1]} \\ D_{51}^{[1]}&D_{52}^{[1]}&D_{53}^{[1]}&D_{54}^{[1]} \\ D_{61}^{[1]}&D_{62}^{[1]}&D_{63}^{[1]}&D_{64}^{[1]} \\ D_{71}^{[1]}&D_{72}^{[1]}&D_{73}^{[1]}&D_{74}^{[1]} \\ D_{81}^{[1]}&D_{82}^{[1]}&D_{83}^{[1]}&D_{84}^{[1]} \end{bmatrix}. \tag{9}$$

Their entries are listed in Appendix II in Supplementary Material, while the vectors $Y_m^{[0]}, Y_m^{[1]}, F_m^{[0]}, F_m^{[1]}, G_m^{[0]}, G_m^{[1]}$ are defined as follows:

$$Y_m^{[0]} = \begin{bmatrix} y_{n-t} \\ y_{n-s} \\ y_{n-r} \\ y_n \\ y'_{n-t} \\ y'_{n-s} \\ y'_{n-r} \\ y'_n \end{bmatrix},$$

$$Y_m^{[1]} = \begin{bmatrix} y_{n+r} \\ y_{n+s} \\ y_{n+t} \\ y_{n+1} \\ y'_{n+r} \\ y'_{n+s} \\ y'_{n+t} \\ y'_{n+1} \end{bmatrix}, \quad F_m^{[0]} = \begin{bmatrix} f_{n-7} \\ f_{n-6} \\ f_{n-5} \\ f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix}, \quad F_m^{[1]} = \begin{bmatrix} f_{n+r} \\ f_{n+s} \\ f_{n+t} \\ f_{n+1} \end{bmatrix}, \quad G_m^{[0]} = \begin{bmatrix} g_{n-7} \\ g_{n-6} \\ g_{n-5} \\ g_{n-4} \\ g_{n-3} \\ g_{n-2} \\ g_{n-1} \\ g_n \end{bmatrix}, \quad G_m^{[1]} = \begin{bmatrix} g_{n+r} \\ g_{n+s} \\ g_{n+t} \\ g_{n+1} \end{bmatrix}. \tag{10}$$

3. Analysis of the Method

3.1. Zero Stability

Definition 1. The hybrid block method formula (8) is said to be zero stable if no root of the first characteristic equation $\rho(R)$ has modulus greater than one; that is, $|R_m| \leqslant 1$ and if $R_m = 1$ then the multiplicity of R_m must not exceed two.

To illustrate that the root of the first characteristic equation satisfies the prior definition we assume that $\{r, s, t\} \in (0, 1)$ and hence

$$\rho(R) = \det\left[RA^{[0]} - A^{[1]}\right] = \begin{vmatrix} R & 0 & 0 & -1 & 0 & 0 & 0 & -rh \\ 0 & R & 0 & -1 & 0 & 0 & 0 & -sh \\ 0 & 0 & R & -1 & 0 & 0 & 0 & -th \\ 0 & 0 & 0 & R-1 & 0 & 0 & 0 & -h \\ 0 & 0 & 0 & 0 & R & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & R & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & R & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & R-1 \end{vmatrix} = 0 \tag{11}$$

which imply that

$$R^6(R-1)^2 = 0 \Longrightarrow R_i = \begin{cases} 0, & \text{if } i = 1\,(1)\,6 \\ 1, & \text{if } i = 7, 8. \end{cases} \tag{12}$$

As a result, the developed method is zero stable.

3.2. Order of the Method. The linear operator Δ associated with the hybrid block methods formula (8) is defined as

$$\Delta\{y(x); h\} = A^{[0]}Y_m^{[1]} - A^{[1]}Y_m^{[0]} - \sum_{i=0}^{1} B^{[i]}F_m^{[i]} - \sum_{i=0}^{1} D^{[i]}G_m^{[i]}. \tag{13}$$

Expanding the above equation in Taylor series and combining like terms we wind up with

$$\Delta\{y(x); h\} = \widehat{C}_0 h^0 y(x) + \widehat{C}_1 h^1 y'(x) + \widehat{C}_2 h^2 y''(x) + \cdots + \widehat{C}_{p+2} h^{p+2} y^{(p+2)}(x) + \cdots \tag{14}$$

According to [6, 10] method (8) is said to be of order p if

$$\widehat{C}_0 = \widehat{C}_1 = \cdots = \widehat{C}_{p+1} = 0, \quad \widehat{C}_{p+2} \neq 0. \tag{15}$$

The term $\widehat{C}_{p+2}$ is called the error constant and the local truncation error is given by

$$t_{n+k} = \widehat{C}_{p+2} y^{(p+2)} h^{p+2} + O\left(h^{p+3}\right). \tag{16}$$

Comparing like terms of $y^{(i)}$ and h^i in (14) produces the coefficients $\widehat{C}_0 = \widehat{C}_1 = \cdots = \widehat{C}_{11} = 0$ with vector of error constants

$$\widehat{C}_{12} = \left[\widehat{C}_{12}^{[1]} \ \widehat{C}_{12}^{[2]} \ \widehat{C}_{12}^{[3]} \ \widehat{C}_{12}^{[4]} \ \widehat{C}_{12}^{[5]} \ \widehat{C}_{12}^{[6]} \ \widehat{C}_{12}^{[7]} \ \widehat{C}_{12}^{[8]}\right]^T, \tag{17}$$

where

$$\widehat{C}_{12}^{[1]} = \frac{h^{12}r^6y^{12}}{100590336000}\Big(\big(14r^6 - 42r^5s - 42r^5t - 42r^5 + 33r^4s^2 + 132r^4st + 132r^4s + 33r^4t^2 + 132r^4t + 33r^4 - 110r^3s^2t - 110r^3s^2 - 110r^3st^2 - 440r^3st - 110r^3s - 110r^3t^2 - 110r^3t + 99r^2s^2t^2 + 396r^2s^2t + 99r^2s^2 + 396r^2st^2 + 396r^2st + 99r^2t^2 - 396rs^2t^2 - 396rs^2t - 396rst^2 + 462s^2t^2\big)\Big),$$

$$\widehat{C}_{12}^{[2]} = \frac{h^{12}s^6y^{12}}{100590336000}\Big(\big(33r^2s^4 - 110r^2s^3t - 110r^2s^3 + 99r^2s^2t^2 + 396r^2s^2t + 99r^2s^2 - 396r^2st^2 - 396r^2st + 462r^2t^2 - 42rs^5 + 132rs^4t + 132rs^4 - 110rs^3t^2 - 440rs^3t - 110rs^3 + 396rs^2t^2 + 396rs^2t - 396rst^2 + 14s^6 - 42s^5t - 42s^5 + 33s^4t^2 + 132s^4t + 33s^4 - 110s^3t^2 - 110s^3t + 99s^2t^2\big)\Big),$$

$$\widehat{C}_{12}^{[3]} = \frac{h^{12}t^6y^{12}}{100590336000}\Big(\big(99r^2s^2t^2 - 396r^2s^2t + 462r^2s^2 - 110r^2st^3 + 396r^2st^2 - 396r^2st + 33r^2t^4 - 110r^2t^3 + 99r^2t^2 - 110rs^2t^3 + 396rs^2t^2 - 396rs^2t + 132rst^4 - 440rst^3 + 396rst^2 - 42rt^5 + 132rt^4 - 110rt^3 + 33s^2t^4 - 110s^2t^3 + 99s^2t^2 - 42st^5 + 132st^4 - 110st^3 + 14t^6 - 42t^5 + 33t^4\big)\Big),$$

$$\widehat{C}_{12}^{[4]} = \frac{h^{12}y^{12}}{100590336000}\Big(\big(462r^2s^2t^2 - 396r^2s^2t + 99r^2s^2 - 396r^2st^2 + 396r^2st - 110r^2s + 99r^2t^2 - 110r^2t + 33r^2 - 396rs^2t^2 + 396rs^2t - 110rs^2 + 396rst^2 - 440rst + 132rs - 110rt^2 + 132rt - 42r + 99s^2t^2 - 110s^2t + 33s^2 - 110st^2 + 132st - 42s + 33t^2 - 42t + 14\big)\Big),$$

$$\widehat{C}_{12}^{[5]} = \frac{h^{11}r^5y^{12}}{50295168000}\Big(\big(28r^6 - 77r^5s - 77r^5t - 77r^5 + 55r^4s^2 + 220r^4st + 220r^4s + 55r^4t^2 + 220r^4t + 55r^4 - 165r^3s^2t - 165r^3s^2 - 165r^3st^2 - 660r^3st - 165r^3s - 165r^3t^2 - 165r^3t + 132r^2s^2t^2 + 528r^2s^2t + 132r^2s^2 + 528r^2st^2 + 528r^2st + 132r^2t^2 - 462rs^2t^2 - 462rs^2t - 462rst^2 + 462s^2t^2\big)\Big),$$

$$\widehat{C}_{12}^{[6]} = \frac{h^{11}s^5y^{12}}{50295168000}\Big(\big(55r^2s^4 - 165r^2s^3t - 165r^2s^3 + 132r^2s^2t^2 + 528r^2s^2t + 132r^2s^2 - 462r^2st^2 - 462r^2st + 462r^2t^2 - 77rs^5 + 220rs^4t + 220rs^4 - 165rs^3t^2 - 660rs^3t - 165rs^3 + 528rs^2t^2 + 528rs^2t - 462rst^2 + 28s^6 - 77s^5t - 77s^5 + 55s^4t^2 + 220s^4t + 55s^4 - 165s^3t^2 - 165s^3t + 132s^2t^2\big)\Big),$$

$$\widehat{C}_{12}^{[7]} = \frac{h^{11}t^5y^{12}}{50295168000}\Big(\big(132r^2s^2t^2 - 462r^2s^2t + 462r^2s^2 - 165r^2st^3 + 528r^2st^2 - 462r^2st + 55r^2t^4 - 165r^2t^3 + 132r^2t^2 - 165rs^2t^3 + 528rs^2t^2 - 462rs^2t + 220rst^4 - 660rst^3 + 528rst^2 - 77rt^5 + 220rt^4 - 165rt^3 + 55s^2t^4 - 165s^2t^3 + 132s^2t^2 - 77st^5 + 220st^4 - 165st^3 + 28t^6 - 77t^5 + 55t^4\big)\Big),$$

$$\widehat{C}_{12}^{[8]} = \frac{h^{11}y^{12}}{50295168000}\Big(\big(462r^2s^2t^2 - 462r^2s^2t + 132r^2s^2 - 462r^2st^2 + 528r^2st - 165r^2s + 132r^2t^2 - 165r^2t + 55r^2 - 462rs^2t^2 + 528rs^2t - 165rs^2 + 528rst^2 - 660rst + 220rs - 165rt^2 + 220rt - 77r + 132s^2t^2 - 165s^2t + 55s^2 - 165st^2 + 220st - 77s + 55t^2 - 77t + 28\big)\Big), \tag{18}$$

which conclude that the order of the developed method is $p = 10$.

3.3. Consistency

Definition 2. A block method is said to be consistent if its order p is greater than one.

Table 1: Comparison of the proposed method with [11].

X value	Exact solution	Computed solution for $r = 1/2$; $s = 2/3$; $t = 3/4$	Error in new method	Error for [11]
0.100	0.904837418035959520	0.904837418035948970	$1.054712E(-14)$	$2.05E(-11)$
0.200	0.818730753077981820	0.818730753077964060	$1.776357E(-14)$	$4.39E(-11)$
0.300	0.740818220681717770	0.740818220681694340	$2.342571E(-14)$	$6.55E(-11)$
0.400	0.670320046035639330	0.670320046035611350	$2.797762E(-14)$	$8.38E(-11)$
0.500	0.606530659712633420	0.606530659712602120	$3.130829E(-14)$	$9.86E(-11)$
0.600	0.548811636094026500	0.548811636093992530	$3.397282E(-14)$	$1.10E(-10)$
0.700	0.496585303791409530	0.496585303791373890	$3.563816E(-14)$	$1.19E(-10)$
0.800	0.449328964117221620	0.449328964117184870	$3.674838E(-14)$	$1.24E(-10)$
0.900	0.406569659740599170	0.406569659740561860	$3.730349E(-14)$	$1.28E(-10)$
1.000	0.367879441171442330	0.367879441171404920	$3.741452E(-14)$	$1.30E(-10)$

Table 2: Comparison of the proposed method with [12].

X value	Exact solution	Computed solution for $r = 1/4$; $s = 1/2$; $t = 3/4$	Error in new method	Error for [12]
0.100	1.050041729278491400	1.050041729278491200	$2.220446E(-16)$	$1.1102E(-15)$
0.200	1.100335347731075600	1.100335347731075300	$2.220446E(-16)$	$5.9952E(-15)$
0.300	1.151140435936466800	1.151140435936466100	$6.661338E(-16)$	$2.5535E(-14)$
0.400	1.202732554054082100	1.202732554054081000	$1.110223E(-15)$	$7.1054E(-14)$
0.500	1.255412811882995200	1.255412811882994800	$4.440892E(-16)$	$1.1568E(-13)$
0.600	1.309519604203111900	1.309519604203112800	$8.881784E(-16)$	$1.1990E(-13)$
0.700	1.365443754271396400	1.365443754271398000	$1.554312E(-15)$	$6.8567E(-13)$
0.800	1.423648930193601900	1.423648930193606400	$4.440892E(-15)$	$3.4754E(-12)$
0.900	1.484700278594052000	1.484700278594060600	$8.659740E(-16)$	$1.2219E(-11)$
1.000	1.549306144334055000	1.549306144334067700	$1.265654E(-14)$	$3.7282E(-11)$

Consistency property is achieved for the hybrid block method from the above analysis since the order $p = 10 \geq 1$.

3.4. Convergence

Theorem 3 (see [16]). *Consistency and zero stability are sufficient conditions for a linear multistep method to be convergent*

The hybrid block method equation (8) is convergent since it fulfills both the consistency and zero-stability conditions.

4. Numerical Examples

In this section, the efficiency and the performance of the general three-hybrid one-step implicit hybrid block method (8) with order $p = 10$ is investigated on five test problems. The first example is highly stiff linear IVP problem with step size $h = 1/10$, the second is nonlinear IVP with $h = 5/100$, the third is linear with $h = 1/100$, the fourth is a nonlinear system with $h = 1/10$, and finally the fifth is a nonlinear undamped Duffing equation with $h = \pi/5$. It is worth mentioning that this method works even for large interval and different values of step size. The values mentioned in this article are chosen just for the sake of comparison with the existing methods only.

Problem 4. $f(x, y, y') = -1001y' - 1000y$, $y(0) = 1$, $y'(0) = -1$, $0 \leq x \leq 1$.

Exact Solution. $y = e^{-x}$ with $h = 1/10$.

Source (see [11]). See Table 1.

Problem 5. $f(x, y, y') = x(y')^2$, $y(0) = 1$, $y'(0) = 1/2$, $0 \leq x \leq 1$.

Exact Solution. $y = 1 + \ln((2 + x)/(2 - x))$ with $h = 5/100$.

Source (see [12]). See Table 2.

Problem 6. $f(x, y, y') = y'$, $y(0) = 0$, $y'(1) = -1$, $0 \leq x \leq 1$.

Exact Solution. $y = 1 - e^x$ with $h = 1/100$.

Source (see [13]). See Table 3.

Problem 7 (a system of nonlinear IVP). $y_1'' = 1 - \cos(x) + \sin(y_2') + \cos(y_2')$, $y_1(0) = 0$, $y_1'(0) = 0$, $0 \leq x \leq 1$.

Table 3: Comparison of the proposed method with [13].

X value	Exact solution	Computed solution for $r = 1/5$; $s = 1/3$; $t = 1/2$	Error in new method	Error for [13]
0.100	−0.105170918075647710	−0.105170918075647660	5.551115$E(-17)$	2.2360$E(-13)$
0.200	−0.221402758160169850	−0.221402758160169990	1.387779$E(-16)$	1.6425$E(-12)$
0.300	−0.349858807576003180	−0.349858807576003520	3.330669$E(-16)$	3.4625$E(-12)$
0.400	−0.491824697641270570	−0.491824697641271070	4.996004$E(-16)$	8.6628$E(-12)$
0.500	−0.648721270700128640	−0.648721270700129420	7.771561$E(-16)$	1.1338$E(-11)$
0.600	−0.822118800390509550	−0.822118800390510880	1.332268$E(-15)$	2.0317$E(-11)$
0.700	−1.013752707470477500	−1.013752707470479300	1.776357$E(-15)$	3.2476$E(-11)$
0.800	−1.225540928492468800	−1.225540928492471600	2.886580$E(-15)$	4.5463$E(-11)$
0.900	−1.459603111156951200	−1.459603111156955000	3.774758$E(-15)$	6.1781$E(-11)$
1.000	−1.718281828459047300	−1.718281828459052400	5.107026$E(-15)$	8.2113$E(-11)$

Table 4: Exact and approximate solutions for solving y_1 using the developed method.

X value	Exact solution	Approximate solution	Error in developed method	Error in [14]
0.200	0.980066577841241630	0.980066580773730210	2.93$E(-09)$	3.331$E(-08)$
0.400	0.921060994002885100	0.921060786088783830	2.08$E(-07)$	7.27$E(-07)$
0.600	0.825335614909678330	0.825334546366523920	1.07$E(-06)$	3.09$E(-06)$
0.800	0.696706709347165500	0.696703745729634890	2.96$E(-06)$	8.01$E(-06)$
1.000	0.540302305868139770	0.540296131232123940	6.17$E(-06)$	1.60$E(-05)$

Table 5: Exact and approximate solutions for solving y_2 using the developed method.

X value	Exact solution	Approximate solution	Error in developed method	Error in [14]
0.200	0.408407044966673130	0.628318530716756810	1.20$E(-12)$	5.74$E(-12)$
0.400	1.036725575684631900	1.256637061492416500	5.65$E(-10)$	3.40$E(-10)$
0.600	1.665044106402590500	1.884955593121642800	9.68$E(-10)$	3.48$E(-09)$
0.800	2.293362637120548900	2.513274128242366100	5.37$E(-09)$	1.66$E(-08)$
1.000	2.921681167838507500	3.141592671767204200	1.82$E(-08)$	5.24$E(-08)$

Table 6: Comparison of the proposed method with [15].

X value	Exact solution	Approximate solution	Error in developed method	Error in [15]
π	−0.200326851873144250	−0.200326851873131260	1.30$E(-14)$	2.77$E(-6)$
2π	0.200027330586423440	0.200027330586373150	5.03$E(-14)$	2.97$E(-8)$
4π	0.198830853474466220	0.198830853474288970	1.77$E(-13)$	1.16$E(-7)$
6π	0.196842430954904000	0.196842430954587670	3.16$E(-13)$	2.53$E(-7)$
8π	0.194070581011836470	0.194070581011452750	3.84$E(-13)$	4.28$E(-7)$
10π	0.190527147620306290	0.190527147620002170	3.04$E(-13)$	6.29$E(-7)$

$y_2'' = 1/(4 + y_1^2) - 1/(5 - (\sin(x))^2)$, $y_2(0) = 0$, $y_2'(0) = \pi$, $0 \le x \le 1$.

Exact Solutions. $y_1 = \cos(x)$, $y_2 = \pi x$, with $h = (1/10)(r = 1/10, s = 3/10, t = 3/5)$; see Tables 4 and 5.

Problem 8 (the nonlinear undamped Duffing equation). $y'' + y + y^3 = Q\cos(\omega x)$, $y(0) = k_0$, $y'(0) = 0$, $0 \le x \le 10\pi$.

Exact Solutions. $y(x) = k_1\cos(\omega x) + k_2\cos(3\omega x) + k_3\cos(5\omega x) + k_4\cos(7\omega x)$, where $\omega = 1.01$, $Q = 0.002$, $k_0 = 0.200426728069$, $k_1 = 0.200179477536$, $k_2 = 0.246946143e(-3)$, $k_3 = 0.304016e(-6)$, $k_4 = 0.374e(-9)$, and $h = \pi/5$. ($r = 1/4$, $s = 1/2$, $t = 3/4$); see Table 6.

5. Conclusion

A general three-hybrid one-step block method of order 10 has been proposed for the direct solution of general second-order ODEs. The developed method is tested on five different problems. Numerical analysis shows that the developed method is consistent and zero stable which conclude its convergence. The computed results are then compared with the results of existing methods in terms of error by considering different

values of r, s, and t. The new method is found to have superiority over them as shown in Tables 1–6.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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