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Research Article

Periodic Travelling Wave Solutions of Discrete Nonlinear Schrödinger Equations

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The existence of nonzero periodic travelling wave solutions for a general discrete nonlinear Schrödinger equation (DNLS) on one-dimensional lattices is proved. The DNLS features a general nonlinear term and variable range of interactions going beyond the usual nearest-neighbour interaction. The problem of the existence of travelling wave solutions is converted into a fixed point problem for an operator on some appropriate function space which is solved by means of Schauder's Fixed Point Theorem.

1. Introduction

Coherent structures arising in the form of travelling waves, solitons, and breathers in systems of coupled oscillators have attracted considerable interest not least due to the important role they play for applications in physics, biology, and chemistry (for reviews see [1–3]). In this context a variety of nonlinear lattice systems has been studied including Fermi-Pasta-Ulam systems, discrete nonlinear Klein-Gordon systems, phase oscillator lattices, Josephson junction systems, reaction-diffusion systems, and the discrete nonlinear Schrödinger equation. Some exact results concerning the existence, stability, and uniqueness of coherent structures in the above-mentioned systems have been obtained; see, for example, [4–27].

In particular with regard to the existence of periodic travelling waves (TWs) in nonlinear lattice systems various methods have been used. For instance, the existence of small amplitude waves in nonlinear discrete Klein-Gordon systems was proved with the usage of spatial dynamics and centre manifold reduction [15, 21]. For the Frenkel-Kontorova model the existence of TWs was shown by means of fixed point methods in [16]. Utilising a modified Lyapunov-Schmidt technique, existence of periodic TWs in Newton's cradle was proved in [25]. Systems with nonlocal interactions were considered in [19, 24, 28].

In [24] existence and bifurcation results for periodic TWs of a general infinite DNLS system as given in (1) in the next section were derived using variational methods. With the current study we present a (less involved) proof of the existence of periodic TWs for the same system but on finite lattices. To obtain our existence result, some appropriate function space is introduced on which the original problem is formulated as a fixed point problem for a corresponding operator. By exploiting Schauder's Fixed Point Theorem the existence of periodic TWs is established. The main advantage of considering finite lattices is that for periodic TW solutions the associated conserved norm (power), given in terms of a sum over the squares of the amplitudes at the lattice sites, is of finite value. Hereby the coercivity of the power can be used to define suitable subsets in function space in order to apply Schauder's Fixed Point Theorem. Nevertheless, imposing periodic boundary conditions an infinite lattice is constructed. The presented method can readily be applied to prove the existence of periodic TWs not only in general DNLS systems of higher dimensions but also in various other nonlinear lattice systems.

2. General DNLS Systems

In the current study we are interested in the existence of periodic TW solutions of the following general discrete nonlinear Schrödinger equation on finite one-dimensional lattices:

$$i\frac{d\psi_n}{dt} = \sum_{j=1}^{N_c}\kappa_j\left[\psi_{n+j} - 2\psi_n + \psi_{n-j}\right] + F\left(|\psi_n|^2\right)\psi_n, \qquad 1 \le n \le N, \tag{1}$$

with $\psi_n \in \mathbb{C}$.

The solutions satisfy periodicity conditions:

$$\psi_{N+m}(t) = \psi_m(t), \tag{2}$$

for $m \in \mathbb{Z}$; namely, we consider the DNLS on rings. Moreover, by means of the periodic boundary conditions an infinite one-dimensional lattice can be obtained. Each unit interacts with its N_c neighbouring oscillators to the left and right, respectively. $N_c = 1, \ldots, [(N-1)/2]$ determines the interaction radius, which ranges from nearest-neighbour interaction obtained for $N_c = 1$ to global coupling when $N_c = (N-1)/2$ and N is odd.

Assume the following condition on $F(|\psi_n|^2)$ holds.

(**A**) $F \in C(\mathbb{R}_+, \mathbb{R})$ for $\mathbb{R}_+ = [0, \infty)$, $F(0) = 0$. There are constants $a > 0$, $b > 0$ such that

$$|F(x)| < a\left(1 + x^b\right), \tag{3}$$

for any $x \ge 0$.

The standard DNLS, arising for $F(|\psi_n|^2) = |\psi_n|^2$ and $\kappa_1 \neq 0$, $\kappa_{j\neq 1} = 0$ in (1), is known to support periodic travelling wave solutions (see, e.g., [29]). As mentioned above the existence of periodic TWs in system (1) on the lattice $\mathbb{Z}$ was given in [24] using variational methods. Here we present a proof of the existence of periodic travelling wave solutions of (1) on finite lattices. Nevertheless, by imposing periodic boundary conditions an infinite lattice is constructed. For the existence statement we introduce some appropriate function space on which the original problem is converted into a fixed point problem for a corresponding operator. By means of Schauder's Fixed Point Theorem the existence of periodic TWs is established.

System (1) possesses two conserved quantities: the energy

$$\mathcal{H} = \sum_{n=1}^{N}\left[G\left(|\psi_n|^2\right) - \sum_{m\neq n}\kappa_m|\psi_m - \psi_n|^2\right], \qquad G(x) = \int_0^x f(x)\,dx, \tag{4}$$

and the power

$$\mathcal{P} = \sum_{n=1}^{N}|\psi_n|^2. \tag{5}$$

We consider travelling wave solutions of the form

$$\psi_n(t) = \Psi(kn - \omega t), \tag{6}$$

with a 2π-periodic function $\Psi(u)$, $u = kn - \omega t$, where $k \in (-\pi, \pi)$ and $\omega \in \mathbb{R}\setminus\{0\}$ are the wave parameters.

In order for a travelling wave solution to satisfy the periodicity conditions in (2) we adopt the lattice size accordingly. This means that for a given wavenumber $|k| = \pi q$ with rational $q = r/s$ and two relatively prime integers $r, s \in \mathbb{Z}_+\setminus\{0\}$, $r < s$, the number of sites of the lattice, N, is supposed to be an appropriate multiple of the minimal spatial period of the associated periodic travelling wave, determined by $L = s/r$, so that the periodicity conditions in (2) are fulfilled.

3. Statement of the Existence Problem

Regarding the existence of periodic travelling wave solutions we state the following.

Theorem 1. *Let* (**A**) *hold. Then for any rational number $q \in \mathbb{Q}\cap(0,1)$ there exists nonzero periodic travelling wave solution $\psi_n(t) = \Psi(kn - \omega t) \equiv \Psi(u)$ of (1) with $\Psi \in C^1(\mathbb{R}, \mathbb{C})$, such that*

$$\Psi(u + 2\pi) = \Psi(u), \quad \forall u \in \mathbb{R}, \tag{7}$$

provided that

$$|\omega| \ge \mathcal{R}\left(1 + p\frac{a\left(1+\mathcal{P}^b\right)}{\mathcal{R}(1+q) + 4\overline{\kappa} + a\left(1+\mathcal{P}^b\right)}\right), \tag{8}$$

where

$$\overline{\kappa} = \sum_{j=1}^{N_c}\kappa_j, \tag{9}$$

$$p = \tilde{q} + \frac{1}{1-q}, \tag{10}$$

$$\tilde{q} = \begin{cases} \dfrac{1}{q} & \text{for } 0 < q < \dfrac{1}{2} \\ \dfrac{1}{1-q} & \text{for } \dfrac{1}{2} \le q < 1, \end{cases} \tag{11}$$

and $\mathcal{R}$ determines the range $[-\mathcal{R}, \mathcal{R}]$ of the function $g \in C(\mathbb{R}\setminus\{0\}, \mathbb{R})$ given by

$$g(x) = \frac{2}{x}\sum_{j=1}^{N_c}\kappa_j\sin^2(jx). \tag{12}$$

In the following we reformulate the original problem as a fixed point problem in a Banach space in a similar vein to the approach in [16].

4. Proof of the Existence Theorem

To prove the assertions of the theorem we utilise Schauder's Fixed Point Theorem (see, e.g., in [30]): let M be a closed convex subset of a Banach space X. Suppose $T: M \to M$ is continuous mapping such that $T(M)$ is a relatively compact subset of M. Then T has a fixed point.

Proof. Travelling wave solutions Ψ satisfy the advance-delay equation

$$-i\omega\Psi'(u) = \sum_{j=1}^{N_c}\kappa_j\Delta_j\Psi(u) + F\left(|\Psi(u)|^2\right)\Psi(u), \quad (13)$$

where $\Delta_j\Psi(u) = \Psi(u+j) - 2\Psi(u) + \Psi(u-j)$ and $\Psi(u+2\pi) = \Psi(u)$, $\forall u \in \mathbb{R}$, so that according to the Bloch-Floquet Theorem a solution must be of the form

$$\Psi(u) = \exp(iqu)\,\Phi(u), \quad (14)$$

where $q \in \mathbb{Q} \cap (0,1)$ and

$$\Phi(u+2\pi) = \Phi(u), \quad \forall u \in \mathbb{R}. \quad (15)$$

Substituting (14) into (13) one obtains

$$\begin{aligned} -i\omega\Phi'(u) + \omega q\Phi(u) &= \sum_{j=1}^{N_c}\kappa_j\widetilde{\Delta}_j\Phi(u) \\ &\quad + F\left(|\Phi(u)|^2\right)\Phi(u), \end{aligned} \quad (16)$$

with

$$\begin{aligned} \widetilde{\Delta}_j\Phi(u) &= \Phi(u+j)\exp(iqj) - 2\Phi(u) \\ &\quad + \Phi(u-j)\exp(-iqj). \end{aligned} \quad (17)$$

Thus, the task amounts to finding 2π-periodic functions $\Phi \in C^1(\mathbb{R},\mathbb{C})$ satisfying (16).

For the forthcoming discussion (16) is suitably rearranged as follows:

$$\begin{aligned} &-i\omega\Phi'(u) + \omega q\Phi(u) - \sum_{j=1}^{N_c}\kappa_j\widetilde{\Delta}_j\Phi(u) \\ &\quad = F\left(|\Phi(u)|^2\right)\Phi(u). \end{aligned} \quad (18)$$

Note that terms nonlinear in Φ feature only on the r.h.s. of (18).

Let $q \in \mathbb{Q} \cap (0,1)$ be fixed. We identify $\mathbb{C}$ with $\mathbb{R}^2$. Denote by X_q^h the real Banach spaces

$$X_q^h = \left\{\Theta \in C_{2\pi}^h(\mathbb{R},\mathbb{C})\right\}, \quad h = 0,1, \quad (19)$$

where $C_\pi^h(\mathbb{R},\mathbb{C})$ is the Banach space of 2π-periodic and C^h functions $\Theta : \mathbb{R} \to \mathbb{C}$ equipped with norms given by

$$\begin{aligned} \|\Theta\|_{C_{2\pi}^0} &= \max_{u\in[0,2\pi]}|\Theta(u)|, \quad \Theta \in C_{2\pi}^0(\mathbb{R},\mathbb{C}), \\ \|\Theta\|_{C_{2\pi}^1} &= \max_{u\in[0,2\pi]}|\Theta(u)| + \max_{u\in(0,2\pi)}\left|\Theta'(u)\right|, \\ &\qquad \Theta \in C_{2\pi}^1(\mathbb{R},\mathbb{C}), \end{aligned} \quad (20)$$

respectively. X_q^1 is compactly embedded in X_q^0 ($X_q^1 \Subset X_q^0$).

We decompose functions $\Theta \in X_q^1$ in a Fourier series

$$\Theta(u) = \sum_{l\in\mathbb{Z}}\Theta_l\exp(ilu). \quad (21)$$

Related to the l.h.s. of (18) we consider the linear mapping: $M_q : X_q^1 \to X_q^0$:

$$M_q(\Theta) = -i\omega\Theta'(u) + \omega q\Theta(u) - \sum_{j=1}^{N_c}\kappa_j\widetilde{\Delta}_j\Theta(u). \quad (22)$$

We demonstrate that this mapping is invertible and get an upper bound for the norm of its inverse.

Applying the operator M_q to the Fourier elements $\exp(ilu)$ in (21) results in

$$M_q\exp(ilu) = \nu_l(q)\exp(ilu), \quad (23)$$

where

$$\nu_l(q) = \omega(q+l) + 4\sum_{j=1}^{N_c}\kappa_j\sin^2\left(\frac{q+l}{2}j\right). \quad (24)$$

By the assumption (8) one has $\nu_l(q) \neq 0$, $\forall l \in \mathbb{Z}$, so that the mapping M_q possesses an inverse obeying $M_q^{-1}\exp(ilu) = (1/\nu_l)\exp(ilu)$. For the bounded linear operator $M_q^{-1} : X_q^0 \to X_q^1$ one derives:

$$\begin{aligned} \left\|M_q^{-1}\right\|_{X_q^0,X_q^1} &\equiv \left\|M_q^{-1}\right\| = \sup_{0\neq\Theta\in X_q^0}\frac{\left\|M_q^{-1}\Theta\right\|_{X_q^1}}{\|\Theta\|_{X_q^0}} = \sup_{0\neq\Theta\in X_q^0}\frac{\left\|\sum_{l\in\mathbb{Z}}(1/\nu_l)\,\Theta_l\exp(ilu)\right\|_{X_q^1}}{\|\Theta\|_{X_q^0}} \\ &= \sup_{0\neq\Theta\in X_q^0}\frac{\left(\sup_{u\in[0,2\pi]}\left|\sum_{l\in\mathbb{Z}}(1/\nu_l)\,\Theta_l\exp(ilu)\right| + \sup_{u\in[0,2\pi]}\left|\left(\sum_{l\in\mathbb{Z}}(1/\nu_l)\,\Theta_l\exp(ilu)\right)'\right|\right)}{\|\Theta\|_{X_q^0}} \\ &\leq \sup_{l\in\mathbb{Z}}\frac{1+|l|}{|\nu_l|}\sup_{0\neq\Theta\in X_q^0}\frac{\sup_{u\in[0,2\pi]}\left|\sum_{l\in\mathbb{Z}}\Theta_l\exp(ilu)\right|}{\|\Theta\|_{X_q^0}} = \sup_{l\in\mathbb{Z}}\frac{1+|l|}{|\nu_l|} \end{aligned}$$

$$= \sup_{l\in\mathbb{Z}} \frac{1+|l|}{\left|(q+l)\left(\omega + (4/(q+l))\sum_{j=1}^{N_c}\kappa_j \sin^2(((q+l)/2)\,j)\right)\right|} \le \left(\tilde{q} + \frac{1}{1-q}\right)\frac{1}{|\omega| - \mathcal{R}}$$
$$\le \frac{(1+q)\,\mathcal{R} + 4\overline{\kappa} + a\left(1+\mathcal{P}^b\right)}{a\left(1+\mathcal{P}^b\right)\mathcal{R}}, \tag{25}$$

where $\tilde{q}$ is given in (11).

For periodic travelling wave solutions $\Phi \in C^1_{2\pi}(\mathbb{R},\mathbb{C})$ one derives, using (3), (5), (8), and (16), the bounds

$$\max_{u\in[0,2\pi]} |\Phi(u)| \le \mathcal{P}^{1/2},$$
$$\max_{u\in(0,2\pi)} \left|\Phi'(u)\right| \le \left(q + \frac{4\overline{\kappa} + a\left(1+\mathcal{P}^b\right)}{\mathcal{R}}\right)\mathcal{P}^{1/2}. \tag{26}$$

We consider then the closed and convex subsets of X_q^0 and X_q^1 determined by

$$Y_q^0 = \left\{\Theta \in X_q^0 : \|\Theta\|_{C^0_{2\pi}} \le \mathcal{P}^{1/2}\right\},$$
$$Y_q^1 = \left\{\Theta \in X_q^1 : \|\Theta\|_{C^1_{2\pi}} \le \left(1 + q + \frac{4\overline{\kappa} + a\left(1+\mathcal{P}^b\right)}{\mathcal{R}}\right)\mathcal{P}^{1/2}\right\}, \tag{27}$$

respectively. Y_q^1 is compactly embedded in Y_q^0 ($Y_q^1 \Subset Y_q^0$).

Furthermore associated with the r.h.s. of (18) we introduce the nonlinear operator $N_q : Y_q^0 \to Y_q^0$, as

$$N_q(\Theta) = F\left(|\Theta|^2\right)\Theta. \tag{28}$$

Clearly, the operator N_q is uniformly continuous on Y_q^0. The range is contained in a bounded ball in Y_q^0, since

$$\left\|N_q(\Theta)\right\|_{Y_q^0} = \left\|F\left(|\Theta|^2\right)\Theta\right\|_{C^0_{2\pi}}$$
$$= \max_{u\in[0,2\pi]} \left|F\left(|\Theta(u)|^2\right)\Theta(u)\right| \tag{29}$$
$$\le a\left(1+\mathcal{P}^b\right)\mathcal{P}^{1/2}.$$

Finally, we express problem (18) as a fixed point equation in terms of a mapping $Y_q^0 \to Y_q^1 \Subset Y_q^0$:

$$\Phi = M_q^{-1} \circ N_q(\Phi) \equiv T_q(\Phi). \tag{30}$$

We get

$$\left\|T_q(\Phi)\right\|_{Y_q^1} = \left\|M_q^{-1}\left(N_q(\Phi)\right)\right\| \le \left\|M_q^{-1}\right\| \left\|N_q(\Phi)\right\|_{Y_q^0}$$
$$\le \left(1 + q + \frac{4\overline{\kappa} + a\left(1+\mathcal{P}^b\right)}{\mathcal{R}}\right)\mathcal{P}^{1/2}, \tag{31}$$

verifying that indeed

$$T_q\left(Y_q^0\right) \subseteq Y_q^1. \tag{32}$$

Hence T_q maps bounded subsets Y_q^0 of X_q^0 into relatively compact subsets Y_q^1 of Y_q^0.

It remains to prove that T_q is continuous on Y_q^0. As N is uniformly continuous on Y_q^0, one has $\forall t \in [0,2\pi]$ and $\forall \Phi_1, \Phi_2 \in Y_q^0$ that for a fixed arbitrary $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left\|N_q(\Phi_1)(t) - N_q(\Phi_2)(t)\right\|_{Y_q^0}$$
$$< \frac{a\left(1+\mathcal{P}^b\right)\mathcal{R}}{(1+q)\,\mathcal{R} + 4\overline{\kappa} + a\left(1+\mathcal{P}^b\right)}\epsilon \tag{33}$$

if $\|\Phi_1 - \Phi_2\|_{Y_q^0} < \delta$. Hence, for arbitrary $\Phi_1, \Phi_2 \in Y_q^0$, we have

$$\left\|M_q^{-1}\left(N_q(\Phi_1)\right) - M_q^{-1}\left(N_q(\Phi_2)\right)\right\|_{Y_q^1 \Subset Y_q^0}$$
$$\le \left\|M_q^{-1}\right\| \left\|N_q(\Phi_1) - N_q(\Phi_2)\right\|_{Y_q^0} < \epsilon, \tag{34}$$

verifying that $T_q(\Phi)$ is continuous on Y_q^0. Schauder's Fixed Point Theorem implies then that the fixed point equation $\Phi = T_q(\Phi)$ has at least one solution.

Furthermore, the spectrum of linear plane wave solutions (phonons) arising for zero nonlinear term, determined by the r.h.s. of system (18), forms a continuous band with values in the interval $[-\mathcal{R}, \mathcal{R}]$. However, since by hypothesis (8) the values of the frequency of oscillations ω lie outside the range of the linear (phonon) band, the corresponding orbits are anharmonic. This necessitates amplitude-depending tuning of the frequency so that the latter comes to lie outside of the phonon spectrum. Thus it must hold that $\|N_q(\Phi)\|_{Y_q^0} = \|F(|\Phi|^2)\Phi\|_{C^0_{2\pi}} \not\equiv 0$ which is fulfilled only if $\Phi \not\equiv 0$. That is, the fixed point equation (30) possesses only nonzero solutions and the proof is finished. □

5. Summary

To summarise, we have proven the existence of nonzero periodic travelling wave solutions for a general DNLS (including as a special case the standard DNLS) on finite one-dimensional lattices. To this end the existence problem has been reformulated as a fixed point problem for an operator on a function space which is solved with the help of Schauder's Fixed Point Theorem. Our method can be straightforwardly extended not only to treating the general DNLS on lattices of higher dimension but also to other types of nonlinear lattice systems such as nonlinear discrete Klein-Gordon systems.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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