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Research Article

Cyclic Growth and Global Stability of Economic Dynamics of Kaldor Type in Two Dimensions

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This article proposes nonlinear economic dynamics continuous in two dimensions of Kaldor type, the saving rate and the investment rate, which are functions of ecological origin verifying the nonwasting properties of the resources and economic assumption of Kaldor. The important results of this study contain the notions of bounded solutions, the existence of an attractive set, local and global stability of equilibrium, the system permanence, and the existence of a limit cycle.

1. Introduction

The nonlinear complex dynamics have been introduced in the analysis of the economic phenomena to explain on the one hand the fluctuations noticed in the study of the chronological series and on the other hand the economic crisis in the capitalist system. The economists Goodwin (1967) and Kaldor (1955-1956) used the dynamic samples to explain that the graphs of cyclic and chaotic evolution are endogeneous to the economic system itself.

To simplify things, a great number of these models have been elaborated with more restrictive assumption such as linearity. The challenges of structural reforms of dynamics justify the fact that mathematicians are interested in them. Our contribution will thus consist in proposing economic models inspired by the ecological models whose lessons may be important in terms of analyzing systems and their regulation in this context of climate protection (cf. [1, 2]).

The basic economic models that we use in this article are those of Kaldor proposed in the works of Hans-Walter Lorenz (cf. [3]).

Our study will consist first of modifying the models of Kaldor conferring them with the ecological properties adapted to economy. Therefore, we propose a dynamic model typical of Kaldor-Holling-2 and Leslie-Gower with some modifications. Next, we will study the qualitative comportment of the model at level 2. We set criteria for which we have on the one hand the marking out of the solutions and the existence of an attractive set and, on the overhand, the local stability of the equilibrium and the permanence of the system.

At last we study the global stability of one interior equilibrium through the construction of the Lyapunov function.

2. Dynamics of Kaldor Ecologic in Two Dimensions

2.1. Economic Dynamics of Type Kaldor with the Effective Growth Rate in Two Dimensions. Let us start by giving the notations and definitions of the rates of parameters and functions in applied economies.

(1) The productivity of capital is the quotient of the GDP Y by the capital K. We set that $\sigma = Y/K$.

(2) The rate of investment is the quotient of the investment I by the GDP. We note that $f = I/Y$.

(3) The saving rate of the GDP ϕ is the quotient of the saving S by the GDP. We have $\phi = S/Y$.

(4) The rate accumulation of the capital is denoted by $h = I/K = \sigma f$.

(5) We design, respectively, by ϱ and α the monetary depreciation rate and the monetary adjustment coefficient.

(6) The ratio saving-capital will be noted as $g = S/K = \sigma\phi$.

Let us consider the original model of Kaldor (cf. [3] page 44):

$$\begin{aligned}\dot{Y} &= \alpha\left[I(Y,K) - S(Y,K)\right],\\ \dot{K} &= I(Y,K) - \varrho K,\\ &\varrho, \alpha, Y(0), K(0) \in \mathbb{R}_+,\ Y, K \in C^1\left([0;+\infty[\,;\mathbb{R}_+\right).\end{aligned} \tag{1}$$

In order to establish a connection between the economic models of Kaldor and the ecological patterns, let us give up the coercion of monetary scales by writing investment and saving functions in relation to the investment rate, of capital and saving hoarding. Let us also replace the growth rate of the GDP of model (1) by its effective growth rate (cf. [3, 4]). Therefore we get the following assumption.

Assumption 1 (Kaldor with effective growth). (1) Effective growth $\dot{Y}/Y = \text{Trend}(Y) + \alpha((I-S)/Y)$ with the coefficient of monetary mending, $\alpha \geq 0$ and the tendency, $\text{Trend}(Y)$, such that $\text{Trend}(0) \geq 0$.

(2) The ratio saving-capital, $g(Y,K)$, is a function verifying $\partial g(Y,K)/\partial Y > 0$.

(3) The investment rate $f(Y,K)$ is a function verifying $\partial f(Y,K)/\partial Y > 0$ and there is a threshold $K_s \geq 0$ so that $\partial f(Y,K)/\partial K < 0$, $\forall K \geq K_s$.

(4) The accumulation rate of the capital $h(Y,K)$ is a function verifying $\partial h(Y,K)/\partial Y > 0$, $\partial h(Y,K)/\partial K < 0$.

With Assumption 1, we get below the dynamics of Kaldor with an effective growth rate whose $f(Y,K)$ and $g(Y,K)$ can have ecological properties.

$$\begin{aligned}\dot{Y} &= \text{Trend}(Y)\,Y + \alpha f(Y,K)\,Y - \alpha g(Y,K)\,K,\\ \dot{K} &= h(Y,K)\,K - \varrho K,\\ &\varrho, \alpha, Y(0), K(0) \in \mathbb{R}_+,\ Y, K \in C^1\left([0;+\infty[\,;\mathbb{R}_+\right).\end{aligned} \tag{2}$$

Interpretation. The dynamics of Kaldor with an effective growth rate (2) present some similarities to the classical ecological dynamics. The function $g_Y(Y,K) = \alpha[f(Y,K) - \phi(Y,K)]Y$ define the action of the capital K upon production Y. It stimulates the increase of the production when the investment rate f is superior to the saving rate ϕ and stops it in the end.

The function $g_K(Y,K) = h(Y,K)K$ defines in its part the action of the production over the capital. The economic model alone summarizes two types of ecological interaction such as mutualising type and the prey-predators type. Let us consider the economic assumption of ecological inspirations on investment rate f, the rate of capital accumulation h, and the ratio saving-capital g.

Assumption 2.

$$f(Y,K) = \widetilde{b}_1(K)\left(1 - \frac{m_2 K}{Y + C_2}\right) \quad \text{with } \widetilde{b}_1(K) = a_1 K + b_1, \tag{3}$$

$$h(Y,K) = \widetilde{b}_2(Y)\left(1 - \frac{m_2 K}{Y + C_2}\right) \quad \text{with } \widetilde{b}_2(Y) = a_2 Y + b_2, \tag{4}$$

$$g(Y,K) = \frac{m_1 Y}{Y + C_1}, \tag{5}$$

where $a_1, a_2, b_1, b_2, m_1, C_1 \in \mathbb{R}$.

Obviously, the investments are first funded by capitals. We can thus suppose that $f(Y,K) = \widetilde{b}_1(K) > 0$. Next, the investors adjust the rate in relation to the realities for they hate to invest in vain. However it is not necessary to invest more when net profit is beyond expectancies. Let us estimate the losses of the investment by using the function $LG(Y,K) = m_2 K/(Y + C_2)$, where m_2 is the maximum value of the losses of the investment rate and C_2 is the maximum value of the stock of capital. Then we get $f(Y,K) = \widetilde{b}_1(K) - LG(Y,K)\widetilde{b}_1(K) = \widetilde{b}_1(K)(1 - m_2K/(Y + C_2))$. For simplicity, we can take $\widetilde{b}_1(K) = a_1K + b_1$, where a_1 and b_1 are the constants depending on the economic policies of investments. Then, we get a rate of investment that verifies the economic requirements (5) of Assumption 1.

Concerning the accumulation of capital, it is known that $h(Y,K) = \sigma f(Y,K) = \widetilde{b}_2(Y)(1 - m_2K/(Y + C_2))$ so $\widetilde{b}_2(Y) = a_1Y + \sigma b_1$. If $\sigma = ((a_2 - a_1)Y + b_2)/b_1$ then $\widetilde{b}_2(Y) = a_2Y + b_2$ with a_2 the part of GDP converted in the stock of capital and $b_2 \in \mathbb{R}$. Therefore $h(Y,K)$ verifies the economic constraint (11) of Assumption 1.

Concerning the saving, let us take $S(Y,K) = g(Y)K$, where $g(Y) = m_1Y/(Y + C_1)$ is the ratio saving-capital. The function g verifies condition (4) of Assumption 1.

Suppose now that the tendency is linear and decreasing: $\text{Trend}(Y) = a_0(1 - Y/C_0)$. In fact, the tendency of the growth rate of GDP is at the start, a constant a_0 for a given period (forthcoming). But it faces some losses due, for example, to corruption, bribes, slush funds, tax haven, whitening of fraudulent funds, manipulations of accounts and media, and any other harmful activity to the growth of the GDP (cf. [5], pages 11, 15-16, and 35–65). Those losses are estimated to a_0Y/C_0 with C_0 the maximum value (monetary) of the GDP that we can get from this economy for the given period.

Then, model (2) becomes the economic dynamics:

$$\begin{aligned}\dot{Y} &= \left[a_0 Y\left(1-\frac{Y}{C_0}\right)\right]+\alpha\left(a_1 K+b_1\right)Y\left(1-\frac{m_2 K}{Y+C_2}\right)\\ &\quad -\left[\alpha\frac{m_1 Y}{Y+C_1}\right]K,\\ \dot{K} &= \left[\left(a_2 Y+b_2\right)\left(1-\frac{m_2 K}{Y+C_2}\right)\right]K-\varrho K,\end{aligned} \tag{6}$$

$$Y(0)>0,\ K(0)>0,\ Y,K\in C^1\left([0;+\infty[\,;\mathbb{R}_+\right),$$

with $(a_0, C_0, C_1, C_2, m_1, m_2, \alpha) \in (\mathbb{R}_+^*)^7$ and $(a_1, a_2, b_1, b_2, \varrho) \in (\mathbb{R}_+)^5$. Y indicates the product and K the stock of capital, and $\dot{Y}$ and $\dot{K}$ indicate, respectively, the increasing speed of the product and the stock of capital.

System (6) defined in this way is a more realistic system. It takes into account a great deal of economic observations of interactions between the product (GDP) and the stock of capital of an economy; namely,

(1) in the absence of the stock of capital, there will not be any explosion of the GDP because the increase of GDP becomes logistic so that, despite the technical progress, the economic production remains limited,

(2) in the absence of the production of the GDP, the economy will not be in short of capital if $b_2 - \varrho > 0$ because the evolution is as well logistic due to the diversification of the economy or the opportunities to convert a stock of physical capital in the stock of monetary capital,

(3) when the production (GDP) becomes abundant, there is a “saturation” of the ratio saving-capital and of the investment expressing the adoption of a non-wasting policy of the economic resources,

(4) when the production of the GDP is insufficient, the ratio saving-capital gets adapted and becomes proportional to the available GDP in order to avoid a shortage of production.

2.2. Presentation of Model (6) Reduced. In order to facilitate the qualitative study of system (6) that possesses 12 parameters $(a_0, a_1, a_2, b_1, b_2, C_0, C_1, C_2, m_1, m_2, \alpha, \varrho) \in (\mathbb{R}_+^*)^{12}$, let us change variables by reducing the number of parameters to 8.

Definition of New Variables

$$\begin{aligned}\tau &= a_0 t,\\ u(\tau) &= \frac{Y(t)}{C_0},\\ v(\tau) &= \frac{m_2}{C_0}K(t).\end{aligned} \tag{7}$$

Definition of the Parameters

$$\begin{aligned}\beta_1 &= \frac{\alpha a_1 C_0}{a_0 m_2},\\ \beta_2 &= \frac{a_2 C_0}{a_0},\\ \alpha_1 &= \frac{\alpha b_1}{a_0},\\ \alpha_2 &= \frac{b_2}{a_0},\\ d_1 &= \frac{C_1}{C_0},\\ d_2 &= \frac{C_2}{C_0};\\ \gamma &= \alpha\frac{m_1}{a_0 m_2},\\ \delta &= \frac{\varrho}{a_0}.\end{aligned} \tag{8}$$

System (6) becomes then the reduced system:

$$\begin{aligned}\dot{u}(\tau) &= [1-u(\tau)]\,u(\tau)\\ &\quad +\left(\beta_1 v(\tau)+\alpha_1\right)\left(1-\frac{v(\tau)}{u(\tau)+d_2}\right)u(\tau)\\ &\quad -\frac{\gamma u(\tau)v(\tau)}{u(\tau)+d_1},\\ \dot{v}(\tau) &= \left(\beta_2 u(\tau)+\alpha_2\right)\left(1-\frac{v(\tau)}{u(\tau)+d_2}\right)v(\tau)-\delta v,\end{aligned} \tag{9}$$

$$(u,v)\in C^1\left(\mathbb{R}_+,\mathbb{R}^2\right)\ u(0)>0,\ v(0)>0,$$

with $(d_1, d_2, \gamma) \in (\mathbb{R}_+^*)^3$ and $(\beta_1, \beta_2, \alpha_1, \alpha_2, \delta) \in (\mathbb{R}_+)^5$ so that $(\beta_2, \alpha_2) \neq (0,0)$.

3. Boundness of Model (9) and Existence of a Positively Invariant Attracting Set

In this section, we give the conditions of the boundness of the capital and the stock of capital justifying the fact that the economic resources are limited.

Lemma 3. *The interior* $\operatorname{int}(\mathbb{R}_+^2)$ *and the boundary* $\partial(\mathbb{R}_+^2)$ *of the positive quadrant are, respectively, unvarying for system (9).*

Proof. Given $(u(0), v(0)) \in \mathbb{R}_+^2$, if $0 \leq \tau < +\infty$, due to the continuity of u and v over the compact $[0;\tau]$ then we have

$$\begin{aligned} u(\tau) &= u(0)\exp\left\{\int_0^\tau \left[1 - u(t) + (\beta_1 v(t) + \alpha_1)\left(1 - \frac{v(t)}{u(t)+d_2}\right) - \frac{\gamma v(t)}{u(t)+d_1}\right]dt\right\}, \\ v(\tau) &= v(0)\exp\left\{\int_0^\tau \left[(\beta_2 u(t) + \alpha_2)\left(1 - \frac{v(t)}{u(t)+d_2}\right) - \delta\right]dt\right\}. \end{aligned} \tag{10}$$

□

So if $(u(0), v(0)) = (0,0)$ then $(u(\tau), v(\tau)) = (0;0)$.

If $u(0) > 0$ then $u(\tau) > 0$ and if $v(0) > 0$ then $v(\tau) > 0$, $\forall \tau \in \mathbb{R}_+$.

Therefore $(u(0), v(0)) \in \partial(\mathbb{R}_+^2) \Rightarrow (u(\tau), v(\tau)) \in \partial(\mathbb{R}_+^2)$, $\forall \tau \in \mathbb{R}_+$, and $(u(0), v(0)) \in \operatorname{int}(\mathbb{R}_+^2) \Rightarrow (u(\tau), v(\tau)) \in \operatorname{int}(\mathbb{R}_+^2)$, $\forall \tau \in \mathbb{R}_+$.

Lemma 4 (cf. [6]). *Given $(A, B) \in \mathbb{R}_+^2$ and ϕ continuous and derivable function so that there is $t_0 \geq 0$ verifying $\phi(t_0) > 0$, then, $\forall t \geq t_0$,*

$$\begin{aligned} &\frac{d\phi}{dt} \leq B - A\phi \Longrightarrow \\ &\phi(t) \leq \frac{B}{A} - \left[\frac{B}{A} - \phi(t_0)\right]e^{-A(t-t_0)}, \end{aligned} \tag{11}$$

$$\begin{aligned} &\frac{d\phi}{dt} \leq B - A\phi \Longrightarrow \\ &\phi(t) \leq \frac{B}{A}\left[1 + \left(\frac{A\phi(t_0)}{B} - 1\right)e^{-A(t-t_0)}\right]^{-1}, \end{aligned} \tag{12}$$

$$\begin{aligned} &\frac{d\phi}{dt} \leq \phi(B - A\phi) \Longrightarrow \\ &\phi(t) \leq \frac{B}{A}\left[1 + \left(\frac{B}{A\phi(t_0)} - 1\right)e^{-B(t-t_0)}\right]^{-1}, \end{aligned} \tag{13}$$

$$\begin{aligned} &\frac{d\phi}{dt} \geq \phi(B - A\phi) \Longrightarrow \\ &\phi(t) \geq \frac{B}{A}\left[1 + \left(\frac{B}{A\phi(t_0)} - 1\right)e^{-B(t-t_0)}\right]^{-1}. \end{aligned} \tag{14}$$

Definition 5 (see [7, 8]). A solution $(u, v) = \phi(t, t_0, u_0, v_0)$ of (9) is said to be a boundary in $\mathbb{R}_+^2$, if there is compact $\mathscr{A}$ of $\mathbb{R}_+^2$ and a time T ($T = T(t_0, u_0, v_0)$) so that $\forall (t_0, u_0, v_0) \in \mathbb{R} \times \mathbb{R}_+^2$, we have $(u, v) = \phi(t, t_0, u_0, v_0) \in \mathscr{A}$ for every $t \geq t_0$.

Theorem 6. *Let us suppose that $0 < \beta_1 < 4$. Let us set down*

$$\begin{aligned} M_u &= \frac{4d_2\beta_1 + (\beta_1 d_2 + 2\alpha_1)\beta_1 d_2 + \alpha_1^2}{\beta_1 d_2(4 - \beta_1)}, \\ L_1 &= \frac{(4 + (4 - \beta_1)M_u)^2}{16(4 - \beta_1)} \\ &\quad + \frac{(M_u + d_2)(\beta_2 M_u + 1 + \alpha_2 - \delta)^2}{4\alpha_2}. \end{aligned} \tag{15}$$

Let us consider the following set:

$$\mathscr{A} = \left\{(u, v) \in \mathbb{R}_+^2,\ 0 \leq u \leq M_u,\ 0 \leq u + v \leq L_1\right\}. \tag{16}$$

(1) $\limsup[u(t)] \leq M_u$.

(2) $\limsup[u(t) + v(t)] \leq L_1$.

(3) *$\mathscr{A}$ is unvarying for model (9).*

(4) *$\mathscr{A}$ is an attractive region for any solution of model (9) from the positive quadrant $\mathbb{R}_+^2$.*

Proof. Let us consider system (9). Let us set down

$$\begin{aligned} f_1(u, v) &= [1 - u(\tau)]u(\tau) \\ &\quad + (\beta_1 v(\tau) + \alpha_1)\left(1 - \frac{v(\tau)}{u(\tau) + d_2}\right)u(\tau) \\ &\quad - \frac{\gamma u(\tau) v(\tau)}{u(\tau) + d_1}, \\ f_2(u, v) &= (\beta_2 u(\tau) + \alpha_2)\left(1 - \frac{v(\tau)}{u(\tau) + d_2}\right)v(\tau) - \delta v. \end{aligned} \tag{17}$$

(1) Let us show that $\limsup[u(t)] \leq M_u = (4d_2\beta_1 + (\beta_1 d_2 + 2\alpha_1)\beta_1 d_2 + \alpha_1^2)/\beta_1 d_2(4 - \beta_1)$. We have $du/dt = (1-u)u + (\beta_1 v + \alpha_1)(1 - v/(u + d_2))u - \gamma uv/(u + d_1)$ and $\max_v\{(\beta_1 v + \alpha_1)(1 - v/(u + d_2))u\} = u[\beta_1(u + d_2) + \alpha_1]^2/4\beta_1(u + d_2)$. Then, $du/dt \leq (B_1 - A_1 u)u$ with $B_1 = (4d_2\beta_1 + (\beta_1 d_2 + 2\alpha_1)\beta_1 d_2 + \alpha_1^2)/4\beta_1 d_2$ and $A_1 = (4 - \beta_1)/4$. Then through the application property (13) of Lemma 4, we have $\forall t \geq 0$, $u(t) \leq M_u[1 + (B_1/A_1 u(0) - 1)e^{-B_1 t}]^{-1}$ with $M_u = B_1/A_1$. Therefore, $\forall \varepsilon > 0$, $\exists T_1 > 0$ so that $\forall t > T_1$, $u(t) \leq M_u + \varepsilon$.

$$\begin{aligned} \text{Then } \limsup[u(t)] &\leq M_u \\ &= \frac{4d_2\beta_1 + (\beta_1 d_2 + 2\alpha_1)\beta_1 d_2 + \alpha_1^2}{\beta_1 d_2(4 - \beta_1)}. \end{aligned} \tag{18}$$

So $\forall t > T_1, u(t) \leq M_u$.

(2) Let us prove that $\limsup[u(t) + v(t)] \leq L_1$. Let us set down $S(t) = u(t) + v(t)$. We have $du/dt \leq (B_1 - A_1 u)u$

and $dv/dt = [\beta_2 u + \alpha_2 - \delta - (\beta_2 u/(u+d_2))v - (\alpha_2/(u+d_2))v]v \le [(\beta_2 M_u + \alpha_2 - \delta) - (\alpha_2/(M_u + d_2))v]v$; then, we have $dS/dt + S(t) \le (1 + B_1 - A_1 u)u + [(1 + \beta_2 M_u + \alpha_2 - \delta) - (\alpha_2/(M_u + d_2))v]v$. Now $\forall a, b \in \mathbb{R}_+^*$, $\max_{x\ge 0}(b - ax)x = b^2/4a$. Then $dS/dt + S(t) \le L_1 = (4+(4-\beta_1)M_u)^2/16(4-\beta_1)+(M_u+d_2)(\beta_2 M_u+1+\alpha_2-\delta)^2/4\alpha_2$. Let us set down $A_2 = 1 > 0$ and $B_2 = L_1 > 0$ then, $dS/dt \le (B_2 - A_2 S)$. Now $S(0) > 0$; then, through the application property (12) of Lemma 4, we have $\forall t \ge 0$, $S(t) \le L_1[1 + (S(0)/L_1 - 1)e^{-t}]^{-1}$; then $\forall \varepsilon > 0$, $\exists T_2 > 0$ so that $\forall t > T_2$, $S(t) \le L_1 + \varepsilon$.

So $\limsup [S(t)] \le L_1$.

$$\text{Consequently } \forall t \ge T_2,\ u(t) + v(t) \le L_1. \tag{19}$$

(3) Let us show that $\mathscr{A}$ is unvarying for model (9). Given $(u(0), v(0)) \in \mathscr{A}$, from Lemma 3, of (18) and (19), $\forall t \ge T = \max(T_1, T_2)$, $(u(0), v(0)) \in \operatorname{int}(\mathscr{A}) \Rightarrow (u(t), v(t)) \in \mathscr{A}$, $(u(0), v(0)) \in \{0\} \times]0; L_1] \Rightarrow (u(t), v(t)) \in \{0\} \times]0; L_1] \subset \mathscr{A}$, and $(u(0), v(0)) \in]0; M_u] \times \{0\} \Rightarrow (u(t), v(t)) \in]0; M_u] \times \{0\} \subset \mathscr{A}$. So $\mathscr{A}$ is unvarying for model (9).

(4) Let us show that $\mathscr{A}$ is an attractive solution of model (9). Let us show that $(u(0), v(0)) \in \mathbb{R}_+^2$. We deduce from Lemma 3, (18), and (19) that $\forall t \ge T = \max(T_1, T_2)$, $(u(0), v(0)) \in \operatorname{int}(\mathbb{R}_+^2) \Rightarrow (u(t), v(t)) \in \mathscr{A}$, and $(u(0), v(0)) \in \partial(\mathbb{R}_+^2) \Rightarrow (u(t), v(t)) \in \mathscr{A}$, $\forall t \ge T$. Consequently $\mathscr{A}$ is an attractive region for any solution of model (9) from the positive quadrant $\mathbb{R}_+^2$.

□

4. Equilibrium of Model (9)

We are now going to give the conditions of a balanced growth (stationary) of the product and the stock of capital and the quantitative values of the parameters in equilibrium.

4.1. Case in Which $(\beta_1, \beta_2) \ne (0, 0)$

4.1.1. Trivial Equilibrium

Proposition 7. *(1) If $0 < \alpha_2 \le \delta$ then system (9) admits two trivial equilibriums:*

$$\begin{aligned} U_0^* &= (0; 0), \\ U_1^* &= (1 + \alpha_1; 0). \end{aligned} \tag{20}$$

(2) If $\alpha_2 > \delta > 0$ then system (9) admits three trivial equilibriums:

$$\begin{aligned} U_0^* &= (0; 0), \\ U_1^* &= (1 + \alpha_1; 0), \\ U_2^* &= \left(0; \frac{(\alpha_2 - \delta) d_2}{\alpha_2}\right). \end{aligned} \tag{21}$$

4.1.2. Interior Equilibrium

Theorem 8. *Let $P(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$ with*

$$\begin{aligned} a_4 &= -\beta_2^2, \\ a_3 &= (1 - d_1)\beta_2^2 - 2\beta_2\alpha_2 + \beta_2(\beta_1\delta - \gamma\beta_2), \\ a_2 &= 2\beta_2\alpha_2(1 - d_1) - \alpha_2^2 + d_1\beta_2^2 + \alpha_1\delta\beta_2 - \gamma\beta_2\alpha_2 \\ &\quad + \beta_1\beta_2\delta d_1 + (\beta_2 d_2 + \alpha_2 - \delta)(\beta_1\delta - \gamma\beta_2), \\ a_1 &= \alpha_1\delta(\beta_2 d_1 + \alpha_2) + (1 - d_1)\alpha_2^2 + 2\beta_2\alpha_2 d_1 \\ &\quad + (\beta_1\delta - \gamma\beta_2) d_2(\alpha_2 - \delta) \\ &\quad + (\beta_1\delta d_1 - \gamma\alpha_2)(\beta_2 d_2 + \alpha_2 - \delta), \\ a_0 &= d_1\alpha_2^2 + \alpha_1\alpha_2\delta d_1 + (\beta_1\delta d_1 - \gamma\alpha_2) d_2(\alpha_2 - \delta). \end{aligned} \tag{22}$$

(1) System (9) does not admit the interior equilibrium if $\beta_2 M_u + \alpha_2 - \delta < 0$.

(2) Any interior equilibrium $E^* = (u^*; v^*)$ of system (9) satisfies the following relations:

$$\begin{aligned} &P(u^*) = 0, \\ &v^* = \frac{(\beta_2 u^* + \alpha_2 - \delta)}{\beta_2 u^* + \alpha_2}(u^* + d_2) \\ &\qquad \text{with } \beta_2 u^* + \alpha_2 - \delta > 0. \end{aligned} \tag{23}$$

Proof. Let us consider model (9); then, $(\beta_2, \alpha_2) \ne (0, 0)$. Given $E^* = (u^*; v^*)$ an equilibrium of the model (9),

(1) if $\beta_2 M_{u^*} + \alpha_2 - \delta < 0$ then $\beta_2 u^* + \alpha_2 - \delta < 0$ so for any $u^* \ge 0$, we have $v^* < 0$; therefore the system does not admit any interior equilibrium.

(2) Given $E^* = (u^*; v^*) \in (\mathbb{R}_+^*)^2$ an interior equilibrium, then we have

$$\begin{aligned} &(1 - u^*) + (\beta_1 v^* + \alpha_1)\left(1 - \frac{v^*}{u^* + d_2}\right) - \frac{\gamma v^*}{u^* + d_1} \\ &\quad = 0, \\ &(\beta_2 u^* + \alpha_2)\left(1 - \frac{v^*}{u^* + d_2}\right) = \delta. \end{aligned} \tag{24}$$

Therefore,

$$\frac{(1-u^*)(\beta_2 u^*+\alpha_2)^2(u^*+d_1)+\alpha_1\delta(\beta_2 u^*+\alpha_2)(u^*+d_1)+[\beta_1\delta(u^*+d_1)-\gamma(\beta_2 u^*+\alpha_2)](\beta_2 u^*+\alpha_2-\delta)(u^*+d_2)}{(\beta_2 u^*+\alpha_2)^2(u^*+d_1)} = 0,$$
$$v^* = \frac{(\beta_2 u^*+\alpha_2-\delta)}{\beta_2 u^*+\alpha_2}(u^*+d_2). \quad (25)$$

Posing

$$\begin{aligned} a_4 &= -\beta_2^2, \\ a_3 &= (1-d_1)\beta_2^2 - 2\beta_2\alpha_2 + \beta_2(\beta_1\delta-\gamma\beta_2), \\ a_2 &= 2\beta_2\alpha_2(1-d_1) - \alpha_2^2 + d_1\beta_2^2 + \alpha_1\delta\beta_2 - \gamma\beta_2\alpha_2 \\ &\quad + \beta_1\beta_2\delta d_1 + (\beta_2 d_2+\alpha_2-\delta)(\beta_1\delta-\gamma\beta_2), \\ a_1 &= \alpha_1\delta(\beta_2 d_1+\alpha_2) + (1-d_1)\alpha_2^2 + 2\beta_2\alpha_2 d_1 \\ &\quad + (\beta_1\delta-\gamma\beta_2)d_2(\alpha_2-\delta) \\ &\quad + (\beta_1\delta d_1-\gamma\alpha_2)(\beta_2 d_2+\alpha_2-\delta), \\ a_0 &= d_1\alpha_2^2 + \alpha_1\alpha_2\delta d_1 + (\beta_1\delta d_1-\gamma\alpha_2)d_2(\alpha_2-\delta) \end{aligned} \quad (26)$$

and $P(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, we obtain

$$E^* = (u^*;v^*) \in (\mathbb{R}_+^*)^2 \Longleftrightarrow \begin{cases} P(u^*) = 0, \\ v^* = \dfrac{(\beta_2 u^*+\alpha_2-\delta)}{\beta_2 u^*+\alpha_2}(u^*+d_2) \quad \text{with } \beta_2 u^*+\alpha_2-\delta > 0. \end{cases} \quad (27)$$

□

Corollary 9. *Let us suppose that $\alpha_2-\delta>0$ and considering the polynomial $P(x)$ defined in Theorem 8, let us set down $p = a_4a_2 - 3a_3^2/8$, $q = a_3^3/8 - a_4a_3a_2/2 + a_4^2a_1$, $r = a_3^3/8 - a_4a_3a_2/4 + a_4a_3^2a_2/16 - 3a_3^3/4^4$, $A = 4r + p^2/3$, $B = -q^2 - 2p^3/27 + 8pr/3$, and $\Delta = 27B^2 - 4A^3$.*

System (9) admits a unique interior equilibrium $E^* = (u^*, v^*)$ such that $P(u^*) = 0$ and $v^* = ((\beta_2u^*+\alpha_2-\delta)/(\beta_2u^*+\alpha_2))(u^*+d_2)$ in each of the following cases:

(a) $\Delta > 0$, $a_0 > 0$.

(b) $\Delta = 0$ *and one of the following conditions is verified:*

(i) $a_0 > 0$, $p < 0$, $q > 0$ *and* $a_3 < 0$.

(ii) $a_0 < 0$, $p > 0$, $q > 0$ *and* $a_3 > 0$.

(iii) $a_0 < 0$, $p > 0$, $q < 0$, $a_3 < 0$, *and* $a_3^2/4 < -(2\sqrt[3]{B/2} + 2p/3)$.

(iv) $a_0 > 0$, $p < 0$, $q < 0$, $a_3 < 0$ *and* $-(2\sqrt[3]{B/2} + 2p/3) < a_3^2/4$.

(v) $a_0 < 0$, $p > 0$, $q < 0$, $a_3 > 0$ *and* $-(2\sqrt[3]{B/2} + 2p/3) < a_3^2/4$.

Proof. Given $P(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ the polynomial is defined in Theorem 8 and $E^* = (u^*, v^*) \in (\mathbb{R}_+^*)^2$ such that $P(u^*) = 0$ and $v^* = ((\beta_2u^*+\alpha_2-\delta)/(\beta_2u^*+\alpha_2))(u^*+d_2)$.

Let us pose $x = (1/a_4)(y - a_3/4)$; then, the equation $P(x) = 0$ will be reduced to (E_1) : $y^4 + py^2 + qy + r = 0$ with $p = a_4a_2 - 3a_3^2/8$, $q = a_3^3/8 - a_4a_3a_2/2 + a_4^2a_1$, $r = a_3^3/8 - a_4a_3a_2/4 + a_4a_3^2a_2/16 - 3a_3^3/4^4$. The characteristic equation of (E_1) is (E_2) : $u^3 - 2pu^2 + (p^2-4r)u + q^2 = 0$. Let us pose $A = 4r + p^2/3$, $B = -q^2 - 2p^3/27 + 8pr/3$, $\Delta = 27B^2 - 4A^3$. Given, u_1, u_2, u_3 solutions of (E_2) certifying $\sqrt{-u_1} \times \sqrt{-u_2} \times \sqrt{-u_3} = -q$, then, y_1, y_2, y_3, and y_4, the solutions in $\mathbb{C}$ of (E_1), are $y_1 = (1/2)(\sqrt{-u_1} + \sqrt{-u_2} + \sqrt{-u_3})$, $y_2 = (1/2)(\sqrt{-u_1} - \sqrt{-u_2} - \sqrt{-u_3})$, $y_3 = (1/2)(-\sqrt{-u_1} + \sqrt{-u_2} - \sqrt{-u_3})$, $y_4 = (1/2)(-\sqrt{-u_1} - \sqrt{-u_2} + \sqrt{-u_3})$.

So, the roots in $\mathbb{C}$ of $P(x)$ are $x_i = (1/a_4)(y_i - a_3/4)$ for $i \in \{1,2,3,4\}$.

The roots x_1, x_2, x_3, and x_4 in $\mathbb{C}$ of $P(x)$ prove the following system: $x_1x_2x_3x_4 = a_0/a_4$ and $x_1 + x_2 + x_3 + x_4 = -a_3/a_4$.

By examining the number of positive roots of $P(x)$ and knowing that $P(u^*) = 0$ and $v^* = ((\beta_2u^*+\alpha_2-\delta)/(\beta_2u^*+\alpha_2))(u^*+d_2) > 0$ because $\alpha_2 - \delta > 0$, we get Corollary 9. □

4.2. Case in Which $(\beta_1, \beta_2) = (0, 0)$. Considering the conditions $(\beta_1, \beta_2) = (0, 0)$ and $\alpha_1 \neq 0, \alpha_2 \neq 0$ in system (9), we get the following system:

$$\begin{aligned} \dot{u}(\tau) &= [1 - u(\tau)]\, u(\tau) + \alpha_1 \left(1 - \frac{v(\tau)}{u(\tau) + d_2}\right) u(\tau) \\ &\quad - \frac{\gamma u(\tau) v(\tau)}{u(\tau) + d_1}, \\ \dot{v}(\tau) &= \alpha_2 \left(1 - \frac{v(\tau)}{u(\tau) + d_2}\right) v(\tau) - \delta v, \\ &(u, v) \in C^1\left(\mathbb{R}_+, \mathbb{R}^2\right),\ u(0) > 0,\ v(0) > 0, \end{aligned} \tag{28}$$

with $(d_1, d_2, \alpha_1, \alpha_2, \gamma) \in (\mathbb{R}_+^*)^5$ and $\delta \in \mathbb{R}_+$.

4.2.1. Trivial Equilibrium

Proposition 10. *(1) If $\alpha_2 \leq \delta$, then, system (28) admits two trivial equilibriums:*

$$\begin{aligned} U_0^* &= (0; 0), \\ U_1^* &= (1 + \alpha_1; 0). \end{aligned} \tag{29}$$

(2) If $\alpha_2 > \delta$, then, system (28) admits three trivial equilibriums:

$$\begin{aligned} U_0^* &= (0; 0), \\ U_1^* &= (1 + \alpha_1; 0), \\ U_2^* &= \left(0; \frac{(\alpha_2 - \delta) d_2}{\alpha_2}\right). \end{aligned} \tag{30}$$

4.2.2. Interior Equilibrium

Theorem 11. *Given that $P(x) = a_2 x^2 + a_1 x + a_0$ with*

$$\begin{aligned} a_2 &= -\alpha_2^2, \\ a_1 &= \alpha_2^2 (1 - d_1) + \delta\alpha_1\alpha_2 - \gamma\alpha_2 (\alpha_2 - \delta), \\ a_0 &= \alpha_2^2 d_1 + \delta\alpha_1\alpha_2 d_1 - \gamma\alpha_2 d_2 (\alpha_2 - \delta) \end{aligned} \tag{31}$$

(1) *if $\alpha_2 - \delta \leq 0$ then system (28) does not admit any interior equilibriums,*
(2) *if $\alpha_2 - \delta > 0$ then any interior equilibrium $E^* = (u^*; v^*)$ of system (28) verifies the following system:*

$$\begin{aligned} P(u^*) &= 0, \\ v^* &= \frac{(\alpha_2 - \delta)}{\alpha_2} (u^* + d_2). \end{aligned} \tag{32}$$

Corollary 12. *Let us suppose that $\alpha_2 - \delta > 0$ and consider the polynomial $P(x)$ of Theorem 11. System (28) admits a unique interior equilibrium $E^* = (u^*, v^*)$ so that $u^* = (a_1 + \sqrt{a_1^2 + 4\alpha_2^2 a_0})/2\alpha_2^2$ and $v^* = ((\alpha_2 - \delta)/\alpha_2)(u^* + d_2)$ if one of the following conditions is verified:*

(i) $\delta > \alpha_2(\gamma d_2 - d_1)/(\alpha_1 d_1 + \gamma d_2)$.
(ii) $\delta < \alpha_2(\gamma d_2 - d_1)/(\alpha_1 d_1 + \gamma d_2)$, $a_1^2 + 4\alpha_2^2 a_0 = 0$, *and* $a_1 > 0$.
(iii) $\delta = \alpha_2(\gamma d_2 - d_1)/(\alpha_1 d_1 + \gamma d_2)$ *and* $a_1 > 0$.

Proof. If $E^* = (u^*, v^*)$ is an interior equilibrium of (28) and $P(x) = -\alpha_2^2 x^2 + a_1 x + a_0$, the polynomial, a_0 and a_1 are stipulated in Theorem 11. Then, $P(u^*) = 0$ and $v^* = ((\alpha_2 - \delta)/\alpha_2)(u^* + d_2)$.

We get $a_0 = \alpha_2^2 d_1 + \delta\alpha_1\alpha_2 d_1 - \gamma\alpha_2 d_2(\alpha_2 - \delta) = 0 \Leftrightarrow \delta = \alpha_2(\gamma d_2 - d_1)/(\alpha_1 d_1 + \gamma d_2)$.

By examining the number of positive roots of $P(x)$ in which $a_0 = 0$ and $a_0 \neq 0$, we obtain researched results. □

5. Local Stability and Permanence of Model (9)

In this section, we first define the conditions in which this balanced growth of the product and the stock of capital of the economy are stable or unstable. Then, let us examine the possibility of having permanently those two parameters of the economy (sustainable development). This permanence of the product and the stock of capital of the economy are noticed either through the convergence (of both parameters) towards a stable equilibrium or through a fluctuation of both parameters around an unstable equilibrium, that is, the convergence towards a limited cycle.

5.1. Local Stability of Model (9). In system (9), we pose the following:

$$\begin{aligned} f_1(u, v) &= [1 - u(\tau)]\, u(\tau) \\ &\quad + (\beta_1 v(\tau) + \alpha_1) \left(1 - \frac{v(\tau)}{u(\tau) + d_2}\right) u(\tau) \\ &\quad - \frac{\gamma u(\tau) v(\tau)}{u(\tau) + d_1}, \\ f_2(u, v) &= (\beta_2 u(\tau) + \alpha_2) \left(1 - \frac{v(\tau)}{u(\tau) + d_2}\right) v(\tau) \\ &\quad - \delta v. \end{aligned} \tag{33}$$

We have $\dot{u}(\tau) = f_1(u, v)$ and $\dot{v}(\tau) = f_2(u, v)$.

Let us note that $J(E)$, the Jacobian matrix of the system, is linear around $E = (u, v)$. Then, we have

$$J(E) = \begin{pmatrix} \dfrac{\partial f_1}{\partial u}(E) & \dfrac{\partial f_1}{\partial v}(E) \\ \dfrac{\partial f_2}{\partial u}(E) & \dfrac{\partial f_2}{\partial v}(E) \end{pmatrix} = \begin{pmatrix} J_{11}(E) & J_{12}(E) \\ J_{21}(E) & J_{22}(E) \end{pmatrix} \tag{34}$$

with

$$\begin{aligned} \frac{\partial f_1}{\partial u} &= 1 - 2u + (\beta_1 v + \alpha_1)\left(1 - \frac{v}{u + d_2}\right) \\ &\quad + \frac{(\beta_1 v + \alpha_1)\, uv}{(u + d_2)^2} - \frac{\gamma d_1 v}{(u + d_1)^2}, \end{aligned}$$

$$\frac{\partial f_1}{\partial v} = \left[\beta_1 - \frac{2\beta_1}{u+d_2}v - \frac{\alpha_1}{u+d_2} - \frac{\gamma}{u+d_1}\right]u,$$
$$\frac{\partial f_2}{\partial u} = \left[\beta_2\left(1-\frac{v}{u+d_2}\right) + \frac{(\beta_2 u+\alpha_2)v}{(u+d_2)^2}\right]v,$$
$$\frac{\partial f_2}{\partial v} = (\beta_2 u+\alpha_2)\left(1-\frac{2v}{u+d_2}\right) - \delta. \quad (35)$$

Theorem 13 (local stability). *(1) Stability of $U_0^* = (0;0)$:*

(a) U_0^* *is an unstable node if* $\alpha_2 > \delta$.

(b) U_0^* *is an unstable saddle point if* $\alpha_2 < \delta$,

 (i) *repulsive along the direction* u,

 (ii) *attractive along the direction* v.

(2) Stability of $U_1^* = (1+\alpha_1;0)$:

(a) U_1^* *is stable if* $\alpha_2 + (1+\alpha_1)\beta_2 < \delta$.

(b) U_1^* *is an unstable saddle point if* $\alpha_2 + (1+\alpha_1)\beta_2 > \delta$,

 (i) *attractive along the direction* u,

 (ii) *repulsive along the direction* w_1.

(3) Stability of $U_2^* = (0;(\alpha_2-\delta)d_2/\alpha_2)$ for $\alpha_2 > \delta$:

(a) U_2^* *is stable if* $Q(\delta) < 0$.

(b) U_2^* *is an unstable saddle point if* $Q(\delta) > 0$, *repulsive along the direction* w_2 *and attractive along the direction* v.

(4) Given $E^* = (u^*;v^*)$, an interior equilibrium of (9) verifying system (23) of Theorem 8, and J its associate Jacobian matrix,

(a) E^* *is stable (node or a focus) if* $\det(J) > 0$ *and* $\mathrm{Tr}(J) < 0$,

(b) E^* *is marginal or a center if* $\det(J) \geq 0$ *and* $\mathrm{Tr}(J) = 0$,

(c) E^* *is unstable if* $\det(J) < 0$ *or* $\det(J) > 0$ *and* $\mathrm{Tr}(J) > 0$; *more precisely,*

 (i) E^* *is a node or a focus if* $\det(J) > 0$ *and* $\mathrm{Tr}(J) > 0$,

 (ii) E^* *is a unstable saddle if* $\det(J) < 0$.

With $\mathrm{Tr}(J)$ *the trace and* $\det(J)$ *the determinant of* J, *the polynomial* $Q(x) = -d_2\beta_1 d_1 x^2 + [\gamma\alpha_2 d_2 + d_2 d_1\beta_1\alpha_2 + d_1\alpha_1\alpha_2]x + \alpha_2^2[d_1 - \gamma d_2]$, *the vectors* $w_1 = ((1+\alpha_1)[\beta_1 - \alpha_1/(1+\alpha_1+d_2) - \gamma/(1+\alpha_1+d_1)];\ (1+\alpha_1)(\beta_2+1)+\alpha_2-\delta)$, *and* $w_2 = (Q(\delta)/d_1\alpha_2^2 + \alpha_2 - \delta;\ [\beta_2 d_2\delta + \alpha_2(\alpha_2-\delta)](\alpha_2-\delta)/\alpha_2^2)$.

Proof. Given J_k, the Jacobian matrix of the system, is linear around the equilibrium U_k^* for $k \in \{0,1,2\}$ and J of $E^* = (u^*, v^*)$, we have the following:

(1) Stability of $U_0^* = (0;0)$:

We have

$$U_0^* = (0;0) \Longrightarrow J_0 = \begin{pmatrix} 1+\alpha_1 & 0 \\ 0 & \alpha_2-\delta \end{pmatrix}. \quad (36)$$

The numbers $\lambda_1 = 1+\alpha_1 > 0$ and $\lambda_2 = \alpha_2 - \delta$ are the eigenvalues of J_0. Eigenspace associated with λ_1 is $E_{\lambda_1}^{(0)} = \langle(1;0)\rangle$, where $\langle\{r_k\}_{1\leq k\leq N}\rangle$ indicates the vectorial subspace generated by the family: $\{r_k\}_{1\leq k\leq N^*}$ with $N \in \mathbb{N}$. The eigenspace associated with λ_2 is $E_{\lambda_2}^{(0)} = \langle(0;1)\rangle$. We have the following:

(a) If $\alpha_2 > \delta$ then $\lambda_1 > 0$ and $\lambda_2 > 0$; then U_0^* is an unstable node and its unstable manifold is $E^u = \langle(1;0),(0;1)\rangle = \mathbb{R}^2$.

(b) If $\alpha_2 < \delta$ then $\lambda_1 > 0$ and $\lambda_2 < 0$; then U_0^* is an unstable saddle point, the unstable manifold of which is $E^u = \langle(1;0)\rangle$ and the stable manifold is $E^s = \langle(0;1)\rangle$.

(c) If $\alpha_2 = \delta$ then $\lambda_1 > 0$ and $\lambda_2 = 0$; therefore U_0^* is an equilibrium the unstable manifold of which is $E^u = \langle(1;0)\rangle$ and of central manifold is $E^c = \langle(0;1)\rangle$.

(2) Stability of $U_1^* = (1+\alpha_1;0)$:

We have

$$J_1 = \begin{pmatrix} -(1+\alpha_1) & (1+\alpha_1)\left[\beta_1 - \dfrac{\alpha_1}{1+\alpha_1+d_2} - \dfrac{\gamma}{1+\alpha_1+d_1}\right] \\ 0 & (1+\alpha_1)\beta_2 + \alpha_2 - \delta \end{pmatrix}. \quad (37)$$

So $\lambda_1 = -(1+\alpha_1) < 0$ and $\lambda_2 = (1+\alpha_1)\beta_2 + \alpha_2 - \delta$ are the eigenvalues of J_1.

The eigenspace associated with λ_1 is $E_{\lambda_1}^{(1)} = \langle(1;0)\rangle$.

Note that $w_1 = ((1+\alpha_1)[\beta_1 - \alpha_1/(1+\alpha_1+d_2) - \gamma/(1+\alpha_1+d_1)];\ (1+\alpha_1)(\beta_2+1)+\alpha_2-\delta)$.

The eigenspace associated with λ_2 is $E_{\lambda_2}^{(1)} = \langle w_1\rangle$.

We get the following:

(a) If $\alpha_2 + (1+\alpha_1)\beta_2 < \delta$ then $\lambda_1 < 0$ and $\lambda_2 < 0$; then U_1^* is stable and its stable manifold is $E^u = \mathbb{R}^2$.

(b) If $\alpha_2 + (1+\alpha_1)\beta_2 > \delta$ then $\lambda_1 < 0$ and $\lambda_2 > 0$; then U_1^* is an unstable saddle point of which the stable manifold is $E^s = \langle(1;0)\rangle$ and the unstable manifold is $E^u = \langle w_1\rangle$.

(c) If $\alpha_2 + (1+\alpha_1)\beta_2 = \delta$ then $\lambda_1 < 0$ and $\lambda_2 = 0$; consequently U_1^* is an equilibrium the stable manifold of which is $E^s = \langle(1;0)\rangle$ and the central manifold is $E^c = \langle w_1\rangle$.

(3) Stability of $U_2^* = (0;(\alpha_2-\delta)d_2/\alpha_2)$:

We have

$$J_2 = \begin{pmatrix} \dfrac{Q(\delta)}{d_1\alpha_2^2} & 0 \\ \dfrac{[\beta_2 d_2\delta + \alpha_2(\alpha_2-\delta)](\alpha_2-\delta)}{\alpha_2^2} & -(\alpha_2-\delta) \end{pmatrix}. \quad (38)$$

So $\lambda_1 = Q(\delta)/d_1\alpha_2^2$ and $\lambda_2 = -(\alpha_2-\delta)$ are the eigenvalues of J_2.

Let us note that $w_2 = (Q(\delta)/d_1\alpha_2^2 + \alpha_2 - \delta; [\beta_2 d_2\delta + \alpha_2(\alpha_2-\delta)](\alpha_2-\delta)/\alpha_2^2)$.

The eigenspace associated with λ_1 is $E^{(2)}_{\lambda_1} = \langle w_2\rangle$.

The eigenspace associated with λ_2 is $E^{(2)}_{\lambda_2} = \langle (0,1)\rangle$. We get the following:

(a) If $Q(\delta) < 0$ then $\lambda_1 < 0$ and $\lambda_2 < 0$; then, U_2^* is stable and its stable manifold is $E^s = \mathbb{R}^2$.
(b) If $Q(\delta) \geq 0$ then $\lambda_1 < 0$ and $\lambda_2 \geq 0$; then U_2^* is an unstable saddle the unstable manifold of which is $E^u = \langle w_2\rangle$ and the stable manifold of which is $E^s = \langle (0,1)\rangle$.

(4) Stability of $E^* = (u^*; v^*)$:

Let λ_1^* and λ_2^* be the eigenvalues of J. Thus, let us note $\mathrm{Tr}(J) = \lambda_1^* + \lambda_2^*$ its trace and $\det(J) = \lambda_1^*\lambda_2^*$ its determinant. The eigenspace associated with λ_1^* is $E^*_{\lambda_1} = \langle (J_{12}; \lambda_1^* - J_{11})\rangle$ and the eigenspace associated with λ_2^* is $E^*_{\lambda_2} = \langle (J_{21}; \lambda_2^* - J_{22})\rangle$.

(a) If $\det(J) > 0$ and $\mathrm{Tr}(J) < 0$ then $\mathrm{Re}[\lambda_1^*] < 0$ and $\mathrm{Re}[\lambda_2^*] < 0$. So, E^* is stable (stable node or stable focus).
(b) If $\det(J) \geq 0$ and $\mathrm{Tr}(J) = 0$ then $\lambda_1^*, \lambda_2^* \in i\mathbb{R}$. Therefore E^* is a marginal or a center.
(c) If $\det(J) < 0$ or $\det(J) > 0$ and $\mathrm{Tr}(J) > 0$ then E^* is unstable. In fact,
   (i) if $\det(J) > 0$ and $\mathrm{Tr}(J) > 0$ then $\mathrm{Re}[\lambda_1^*] > 0$ and $\mathrm{Re}[\lambda_2^*] > 0$ or λ_1^* and λ_2^* are conjugated complexes. So, E^* is node or an unstable center,
   (ii) if $\det(J) < 0$ then E^* is an unstable saddle.

□

5.2. Permanence of Model (9)

Definition 14 (see [8]). Given solution $u = (x_1, x_2, \ldots, x_n)$ of a differential system

$$\dot{u} = f(t, u), \quad (39)$$

(1) a component x_i of the solution u of (39) is said to be *weakly persistent* if $\limsup[x_i(t)] > 0$,
(2) a component x_i of the solution u of (39) is said to be *highly persistent* if $\liminf[x_i(t)] > 0$,
(3) a component x_i of the solution u of (39) is said to be *uniformly persistent* if there is ε such that $\liminf[x_i(t)] \geq \varepsilon > 0$,
(4) System (39) is said to be *dissipative* as for any component x_i of the solution there is a constant $M_i > 0$ such that $\limsup[x_i(t)] \geq M_i > 0$,
(5) System (39) is said to be *permanent* if it is *uniformly persistent* and *dissipative*.

Let $\overline{\Omega}$ be complete metric space and Ω for an open set such that $\overline{\Omega} = \Omega \cup \partial\Omega$. Further, we shall take $\Omega = \mathrm{int}(\mathbb{R}^2_+)$.

Definition 15 (see [8]). A flow or semiflow on $\overline{\Omega}$ under which Ω and $\partial\Omega$ are forward invariant is said to be permanent if it is dissipative and if there is a number $\varepsilon > 0$ such that any trajectory starting in Ω will be at least at a distance ε from $\partial\Omega$ for all sufficiently large t.

Definition 16 (see [8]). (1) The ω-limit set $\omega(\partial\Omega)$ is said to be isolated if it has a covering $M = \bigcup_{k=1}^N M_k$ of pairwise disjoint sets M_k which are isolated and invariant with respect to the flow or the semiflow both on $\partial\Omega$ and on $\overline{\Omega} = \Omega \cup \partial\Omega$.

(2) The set $\omega(\partial\Omega)$ is said to be acyclic if there exists an isolated covering $M = \bigcup_{k=1}^N M_k$ such that no subset of M_k is a cycle.

Lemma 17 (see [8]). *Suppose that a semiflow on $\overline{\Omega}$ leaves both Ω and $\partial\Omega$ forward invariant, maps bounded sets in $\overline{\Omega}$ to precompact set for $t > 0$, and it is dissipative. If in addition*

(1) *$\omega(\partial\Omega)$ is isolated and acyclic,*

(2) *$E^s[M_k] \cap \Omega = \emptyset$ for all $k \in \{1; 2; \ldots; N\}$, where $M = \bigcup_{k=1}^N M_k$ is the isolated covering used in the definition of acyclicity of $\omega(\partial\Omega)$ and E^s denotes the stable manifold,*

then the semiflow is permanent.

Theorem 18. *Let us assume that $0 < \beta_1 < 4$ and $\alpha_2 > \delta$; then we pose*

$$M_u = \frac{4d_2\beta_1 + (\beta_1 d_2 + 2\alpha_1)\beta_1 d_2 + \alpha_1^2}{\beta_1 d_2(4-\beta_1)};$$

$$L_1 = \frac{(4 + (4-\beta_1)M_u)^2}{16(4-\beta_1)} + \frac{(M_u + d_2)(\beta_2 M_u + 1 + \alpha_2 - \delta)^2}{4\alpha_2};$$

$$m_u = -\frac{\beta_1}{d_2}L_1^2 - \left(\frac{\alpha_1}{d_2} + \frac{\gamma}{d_1}\right)L_1 + 1 + \alpha_1;$$

$$m_v = \frac{(\alpha_2-\delta)d_2}{\beta_2 d_2 + \alpha_2};$$

$$Q(x) = -d_2\beta_1 d_1 x^2 + [\gamma\alpha_2 d_2 + d_2 d_1\beta_1\alpha_2 + d_1\alpha_1\alpha_2]\, x + \alpha_2^2 [d_1 - \gamma d_2]. \tag{40}$$

Let us consider the following assumptions:

$$(H_1)\quad -\frac{\beta_1}{d_2}L_1^2 - \left(\frac{\alpha_1}{d_2} + \frac{\gamma}{d_1}\right)L_1 + 1 + \alpha_1 > 0;$$
$$(H_2)\quad Q(\delta) > 0. \tag{41}$$

Under the assumptions (H_1) *and* (H_2), *model (9) is permanent and any positive solution* (u, v) *of (9) verifies*

$$m_u \le u \le M_u; \quad m_v \le v \le L_1 - m_u. \tag{42}$$

Proof. Given $\Omega = \operatorname{int}(\mathbb{R}_+^2)$ and $\partial\Omega = \langle(1;0)\rangle \cup \langle(0;1)\rangle$, its frontier, and $\overline{\Omega} = \Omega \cup \partial\Omega$, we know that Ω and $\partial\Omega$ are invariants for model (9) (cf. Lemma 3) and that $\mathscr{A}$ is attractive bounded for any trajectory from $\overline{\Omega} = \mathbb{R}_+^2$ (cf. Theorem 6). Let us assume that $0 \le \beta_1 < 4$ and $\alpha_2 > \delta$ and let us apply Lemma 17:

(1) Let us justify the fact that model (9) is dissipative over $\overline{\Omega} = \mathbb{R}_+^2$:

(a) Let us show that $\liminf[u(t)] \ge m_u = -(\beta_1/d_2)L_1^2 - (\alpha_1/d_2 + \gamma/d_1)L_1 + 1 + \alpha_1$. Posing $A_1 = 1$ and $B_1 = 1 + \alpha_1 - L_1(\beta_1 L_1 + \alpha_1)/d_2 - \gamma L_1/d_1$, we have $du/dt \ge (B_1 - A_1 u)u$ and $u(0) > 0$. So, from property (14) of Lemma 4, we have $\liminf[u(t)] \ge m_u = -(\beta_1/d_2)L_1^2 - (\alpha_1/d_2 + \gamma/d_1)L_1 + 1 + \alpha_1$.

(b) Let us show that $\liminf[v(t)] \ge m_v = (\alpha_2 - \delta)d_2/(\beta_2 d_2 + \alpha_2)$. Posing $B_3 = (\alpha_2 - \delta)$ and $A_3 = (\beta_2 + \alpha_2/d_2)$, then, we have $dv/dt \ge (B_3 - A_3 v)v$ and $v(0) > 0$. We get $\liminf[v(t)] \ge m_v = (\alpha_2 - \delta)d_2/(\beta_2 d_2 + \alpha_2) > 0$ for $\alpha_2 > \delta$.

(c) We deduce that model (9) is dissipative over $\overline{\Omega} = \mathbb{R}_+^2$ as soon as the assumption (H_1) is verified.

(2) Let us prove that $\omega(\partial\Omega)$ is isolated and acyclic. We have $\partial\Omega = \langle(1;0)\rangle \cup \langle(0;1)\rangle$. On the one hand, $\omega[\langle(1;0)\rangle] = \{U_0^*; U_1^*\} \subset \langle(1;0)\rangle$; now the stable manifold U_1^* is $E^s[U_1^*] = \langle(1,0)\rangle$ if $\alpha_2 + (1+\alpha_1)\beta_2 > \delta$ and U_0^* is unstable if $\alpha_2 \ge \delta$. Then any trajectory from $\langle(1;0)\rangle$ other than U_0^* approaches U_1^* if $\alpha_2 \ge \delta$ for $\alpha_2 \ge \delta \Rightarrow \alpha_2 + (1 + \alpha_1)\beta_2 > \delta$. On the other hand, $\omega[\langle(0;1)\rangle] = \{U_0; U_2\} \subset \langle(0;1)\rangle$; now the stable manifold of U_2^* is $E^s[U_2^*] = \langle(0,1)\rangle$ if $Q(\delta) > 0$ and U_0^* is unstable if $\alpha_2 \ge \delta$; then any trajectory from $\langle(0;1)\rangle$ other than U_0^* approaches U_2^* if $\alpha_2 \ge \delta$ and $Q(\delta) > 0$. Given that $\alpha_2 > \delta$ then we deduce that $\omega[\partial\Omega] = \{U_0; U_1; U_2\} \subset \partial\Omega$ is isolated and acyclic if $Q(\delta) > 0$ (if the assumption (H_2) is verified).

(3) Let us justify that $E^s[\omega(\partial\Omega)] \cap \Omega = \emptyset$. We have $\omega[\partial\Omega] = \{U_0; U_1; U_2\}$ and $\Omega = \operatorname{int}(\mathbb{R}_+^2)$. If $\alpha_2 \ge \delta$, $\alpha_2 + (1+\alpha_1)\beta_2 > \delta$, and $Q(\delta) > 0$ then $E^s[\omega(\partial\Omega)] \cap \Omega = \emptyset$; now $\alpha_2 > \delta$. So $E^s[\omega(\partial\Omega)] \cap \Omega = \emptyset$ if $Q(\delta) > 0$.

Definitively, system (9) is permanent. □

5.3. Limit Cycle of Model (9). The theorem below presents the conditions for a cyclic growth of the product and the stock of capital of the economy.

Theorem 19. *Let us recall the notations of Theorem 13. Let us suppose that the assumptions of Theorem 18 are verified and that model (9) admits a unique interior equilibrium* $E^* = (u^*; v^*)$. *If* $\det(J) > 0$ *and* $\operatorname{Tr}(J) > 0$ *then model (9) admits a limit cycle contained in the attractive region* $\mathscr{A}$.

Proof. Under the assumption of Theorem 18, model (9) is permanent and if $\det(J) > 0$ and $\operatorname{Tr}(J) > 0$ then the unique interior equilibrium $E^* = (u^*; v^*)$ is unstable so model (9) admits a limit cycle contained in the compact and bounded region $\mathscr{A}$ (from Poincaré-Beddicton's theorem). □

6. Global Stability of Model (9)

We now define the conditions in which stability of the product and the stock of capital of the economy are global; that is, they do not depend on the quantities produced and the level of the stock at the initial period. For this study, we define appropriate Lyapunov function.

Theorem 20. *Posing* $M_u = (4d_2\beta_1 + (\beta_1 d_2 + 2\alpha_1)\beta_1 d_2 + \alpha_1^2)/\beta_1 d_2(4 - \beta_1)$ *and* $L_1 = (4 + (4 - \beta_1)M_u)^2/16(4 - \beta_1) + (M_u + d_2)(\beta_2 M_u + 1 + \alpha_2 - \delta)^2/4\alpha_2$, *let us consider the following assumptions:*

$$0 < \beta_1 < 4, \quad \alpha_2 > \delta. \tag{43}$$

(9) admits a single point of interior equilibrium $E^* = (u^*; v^*)$, (44)

$$\frac{\beta_1 L_1^2 + \alpha_1 L_1}{d_2^2} + \frac{\gamma L_1}{d_1^2} < 1. \tag{45}$$

$$0 < \beta_1 < \frac{\alpha_1}{M_u + d_2} + \frac{\gamma}{M_u + d_1}. \tag{46}$$

Under assumptions (43)–(46), the unique interior equilibrium of model (9) is globally and asymptotically stable.

Proof. Let us consider system (9). Let us suppose that assumption (44) is verified; then, model (9) admits a unique interior equilibrium $E^* = (x^*, y^*)$.

Let us note that $\lambda = \beta_1(u + d_2)/2(\beta_2 u + \alpha_2)$. We have $V_1 : \mathbb{R}^2 \to \mathbb{R}$ and $V_2 : \mathbb{R}^2 \to \mathbb{R}$ such that

$$V_1(u,v) = \left[u - u^* - \ln\left(\frac{u}{u^*}\right)\right] = \int_{u^*}^{u}\left[1 - \frac{u^*}{x}\right]dx,$$
$$V_2(u,v) = \lambda\left[v - v^* - \ln\left(\frac{v}{v^*}\right)\right] = \lambda\int_{v^*}^{v}\left[1 - \frac{v^*}{x}\right]dx. \quad (47)$$

We have the Lyapunov function $V : \mathbb{R}^2 \to \mathbb{R}$ such that $V(u,v) = V_1(u,v) + V_2(u,v)$.

Then, $dV/dt = dV_1/dt + dV_2/dt = (dV_1/du)\dot{u} + (dV_2/dv)\dot{v} = (u - u^*)(\dot{u}/u) + \lambda(v - v^*)(\dot{v}/v)$. Now

$$\frac{\dot{u}}{u} = \left[1 - u + (\beta_1 v + \alpha_1) - \frac{v(\beta_1 v + \alpha_1)}{u + d_2} - \frac{\gamma v}{u + d_1}\right],$$
$$\frac{\dot{v}}{v} = \left[(\beta_2 u + \alpha_2 - \delta) - \frac{(\beta_2 u + \alpha_2)}{u + d_2}v\right], \quad (48)$$

and $E^* = (u^*; v^*) \in (\mathbb{R}_+^*)^2$, a unique interior equilibrium of (9); then,

$$1 = u^* - (\beta_1 v^* + \alpha_1)\left(1 - \frac{v^*}{u^* + d_2}\right) + \frac{\gamma v^*}{u^* + d_1};$$
$$\alpha_2 - \delta = -\beta_2 u^* + \frac{(\beta_2 u^* + \alpha_2)v^*}{u^* + d_2}. \quad (49)$$

Therefore,

$$\frac{\dot{u}}{u} = (u - u^*)\left[-1 + \frac{\beta_1 v^2 + \alpha_1 v}{(u^* + d_2)(u + d_2)} + \frac{\gamma v}{(u^* + d_1)(u + d_1)}\right] + (v - v^*)\left[\beta_1 - \frac{\beta_1(v + v^*) + \alpha_1}{(u^* + d_2)} - \frac{\gamma}{(u^* + d_1)}\right];$$
$$\frac{\dot{v}}{v} = (u - u^*)\left[-\beta_2 - \frac{\beta_2 v^*}{u^* + d_2} + \frac{(\beta_2 u + \alpha_2)v}{(u^* + d_2)(u + d_2)}\right] - \frac{(\beta_2 u + \alpha_2)}{u^* + d_2}(v - v^*). \quad (50)$$

Let us pose the following:

$$g(u,v) = -1 + \frac{\beta_1 v^2 + \alpha_1 v}{(u^* + d_2)(u + d_2)} + \frac{\gamma v}{(u^* + d_1)(u + d_1)};$$
$$h(u,v) = \frac{1}{2}\left[\beta_1 - \frac{\beta_1(v + v^*) + \alpha_1}{(u^* + d_2)} - \frac{\gamma}{(u^* + d_1)}\right] + \frac{\lambda}{2}\left[-\beta_2 - \frac{\beta_2 v^*}{u^* + d_2} + \frac{(\beta_2 u + \alpha_2)v}{(u^* + d_2)(u + d_2)}\right]. \quad (51)$$

Consequently, $dV/dt = g(u,v)(u-u^*)^2 + 2h(u,v)(u-u^*)(v-v^*) - \lambda((\beta_2 u+\alpha_2)/(u^*+d_2))(v-v^*)^2$. Then, $dV/dt \le [g(u,v)+h(u,v)](u-u^*)^2 + [h(u,v) - \lambda((\beta_2 u+\alpha_2)/(u^*+d_2))](v-v^*)^2$. Thus, $dV/dt < 0$ if $g(u,v) < 0$ and $h(u,v) < 0$, $\forall(u,v) \in \mathbb{R}^2$.

Let us determine the conditions on the control parameters of model (9) such that $g(u,v) < 0$ and $h(u,v) < 0$, $\forall(u,v) \in \mathbb{R}^2$ (cf. [9], pages 110-111).

(1) Let us overestimate $g(u,v)$. We have $g(u,v) = -1 + (\beta_1 v^2 + \alpha_1 v)/(u^* + d_2)(u + d_2) + \gamma v/(u^* + d_1)(u + d_1)$. Then $g(u,v) \le -1 + (\beta_1 L_1^2 + \alpha_1 L_1)/d_2^2 + \gamma L_1/d_1^2$. Therefore $g(u,v) < 0$, $\forall(u,v) \in \mathbb{R}^2$ if $-1 + (\beta_1 L_1^2 + \alpha_1 L_1)/d_2^2 + \gamma L_1/d_1^2 < 0$.

(2) Let us overestimate $h(u,v)$. We have $\partial h/\partial v = (-(u + d_2)\beta_1 + \lambda(\beta_2 u + \alpha_2))/2(u^* + d_2)(u + d_2)$. Now $\lambda = \beta_1(u+d_2)/2(\beta_2 u+\alpha_2)$; then, $\partial h/\partial v = -\beta_1/4(u^*+d_2) < 0$. Consequently, $h(u,v) \le h(u,0)$, $\forall v \ge 0$. Then, $h(u,v) \le (1/2)[\beta_1 - \alpha_1/(u^* + d_2) - \gamma/(u^* + d_1)] + (\lambda/2)[-\beta_2 - \beta_2 v^*/(u^* + d_2)]$. Therefore, $h(u,v) \le 0$, $\forall(u,v) \in \mathbb{R}^2$, if $\beta_1 < \alpha_1/(M_u + d_2) + \gamma/(M_u + d_1)$.

(3) Let us deduce that $dV/dt < 0$. We know that $(\beta_1 L_1^2 + \alpha_1 L_1)/d_2^2 + \gamma L_1/d_1^2 < 1 \Rightarrow g(u,v) < 0$, $\forall(u,v) \in \mathbb{R}^2$, and $\beta_1 < \alpha_1/(M_u + d_2) + \gamma/(M_u + d_1) \Rightarrow h(u,v) \le 0$, $\forall(u,v) \in \mathbb{R}^2$. Now $dV/dt < 0$, $\forall(u,v) \in \mathbb{R}^2$, if $g(u,v) < 0$ and $h(u,v) < 0$, $\forall(u,v) \in \mathbb{R}^2$. Then, $dV/dt < 0$, $\forall(u,v) \in \mathbb{R}^2$, if assumptions (43)–(46) are verified. Therefore the unique interior equilibrium of model (9) is globally and asymptotically stable if assumptions (43)–(46) are verified.

□

7. Conclusion

Our work used the Kaldor model as basic economic model. By including it at the level of the investment rate and saving rate compatible ecological functions, we encourage the economic actors to adopt a behaviour permitting very rapidly entering a stability area (attracting set $\mathscr{A}$). This stability can be noticed on the one hand in the form of stationary growth of the stock of capital and the product (stable interior equilibrium) and on the other hand in the form of cyclic growth of the capital and the product (limit cycle). We therefore guarantee, under certain conditions, the permanence of the stock of capital, K, and that of the product, Y, in the economy avoiding, in that way, a shortage of the stock of capital or the production in the long term. Under certain conditions, this stability of the financial system (in relation to the capital and the product) is global; that is, it does depend on the level of the stock of capital and the level of production at the initial period.

In the first consideration, the model can be applied to a state, regional organisation, or to an enterprise. In the case of an enterprise, the product Y refers to the monetary value of the production. We can then infuse the existing production functions such as Cobb-Douglas, Leontief, and CES. In that case, we can substitute the saving with the quantity of work and the tendency will show a technical progress.

Secondly, the model can also be applied as an ecological model of two species where one of the species (e.g., man) "cultivate" the other species for its survival or to prevent the loss of that species through a culture rate (investment rate) which is nonnull.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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