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Research Article

Resolvent for Non-Self-Adjoint Differential Operator with Block-Triangular Operator Potential

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A resolvent for a non-self-adjoint differential operator with a block-triangular operator potential, increasing at infinity, is constructed. Sufficient conditions under which the spectrum is real and discrete are obtained.

1. Introduction

The theory of non-self-adjoint singular differential operators, generated by scalar differential expressions, has been well studied. An overview on the theory of non-self-adjoint singular ordinary differential operators is provided in V. E. Lyantse's Appendix I to the monograph of Naimark [1]. In this regard the papers of Naimark [2], Lyantse [3], Marchenko [4], Rofe-Beketov [5], Schwartz [6], and Kato [7] should be noted. The questions regarding equations with non-Hermitian matrix or operator coefficients have been studied insufficiently. For a differential operator with a triangular matrix potential decreasing at infinity, which has a bounded first moment due to the inverse scattering problem, it is stated in [8, 9] that the discrete spectrum of the operator consists of a finite number of negative eigenvalues, and the essential spectrum covers the positive semiaxis. The questions regarding an operator with a block-triangular matrix potential that increases at infinity are considered in [10, 11]. In the future, by the author of this paper similar questions are considered for equations with block-triangular operator coefficients. In [11, 12] Green's function of a non-self-adjoint operator is constructed.

In this article we construct a resolvent for a non-self-adjoint differential operator, using which the structure of the operator spectrum is set.

2. Preliminary Notes

Let H_k, $k = 1, 2, \dots, r$, be finite-dimensional or infinite-dimensional separable Hilbert space with inner product $(\cdot, \cdot)$ and norm $|\cdot|$, $\dim H_k \le \infty$. Denote $\mathbf{H} = H_1 \oplus H_2 \oplus \cdots \oplus H_r$. Element $h \in \mathbf{H}$ will be written in the form $h = \operatorname{col}(h_1, h_2, \dots, h_r)$, where $h_k \in H_k$, $k = \overline{1, r}$, I_k, I are identity operators in H_k and $\mathbf{H}$ accordingly.

We denote by $L_2(\mathbf{H}, (0, \infty))$ the Hilbert space of vector-valued functions $y(x)$ with values in $\mathbf{H}$ with inner product

$$\langle y, z \rangle = \int_0^\infty (y(x), z(x))\, dx \tag{1}$$

and the corresponding norm $\|\cdot\|$.

Consider the equation with block-triangular operator potential

$$l[y] = -y'' + V(x)\, y = \lambda y, \quad 0 \le x < \infty, \tag{2}$$

where

$$V(x) = v(x) \cdot I + U(x),$$

$$U(x) = \begin{pmatrix} U_{11}(x) & U_{12}(x) & \cdots & U_{1r}(x) \\ 0 & U_{22}(x) & \cdots & U_{2r}(x) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & U_{rr}(x) \end{pmatrix}, \tag{3}$$

$v(x)$ is a real scalar function, and $0 < v(x) \to \infty$ monotonically, as $x \to \infty$, and it has monotone absolutely continuous derivative. Also, $U(x)$ is a relatively small perturbation; for example, $|U(x)| \cdot v^{-1}(x) \to 0$ as $x \to \infty$ or $|U| v^{-1} \in L^{\infty}(\mathbb{R}_+)$. The diagonal blocks $U_{kk}(x)$, $k = \overline{1, r}$, are assumed as bounded self-adjoint operators in H_k, $U_{kl} : H_l \to H_k$.

In case where

$$v(x) \geq C x^{2\alpha}, \quad C > 0, \ \alpha > 1, \tag{4}$$

we suppose that coefficients of (2) satisfy relations

$$\begin{gathered} \int_0^{\infty} |U(t)| \cdot v^{-1/2}(t)\, dt < \infty, \\ \int_0^{\infty} v'^2(t) \cdot v^{-5/2}(t)\, dt < \infty, \\ \int_0^{\infty} v''(t) \cdot v^{-3/2}(t)\, dt < \infty. \end{gathered} \tag{5}$$

Let us consider the functions

$$\begin{gathered} \gamma_0(x) = \frac{1}{\sqrt[4]{4v(x)}} \cdot \exp\left(-\int_0^x \sqrt{v(u)}\, du\right), \\ \gamma_{\infty}(x) = \frac{1}{\sqrt[4]{4v(x)}} \cdot \exp\left(\int_0^x \sqrt{v(u)}\, du\right). \end{gathered} \tag{6}$$

It is easy to see that $\gamma_0(x) \to 0$, $\gamma_{\infty}(x) \to \infty$ as $x \to \infty$. These solutions constitute a fundamental system of solutions of the scalar differential equation

$$-z'' + (v(x) + q(x))\, z = 0, \tag{7}$$

where $q(x)$ is determined by a formula (cf. with the monograph [13])

$$q(x) = \frac{5}{16}\left(\frac{v'(x)}{v(x)}\right)^2 - \frac{1}{4}\frac{v''(x)}{v(x)}. \tag{8}$$

In such a way for all $x \in [0, \infty)$ one has

$$W(\gamma_0, \gamma_{\infty}) \coloneqq \gamma_0(x) \cdot \gamma'_{\infty}(x) - \gamma'_0(x) \cdot \gamma_{\infty}(x) = 1. \tag{9}$$

In case of $v(x) = x^{2\alpha}$, $0 < \alpha \leq 1$, we suppose that the coefficients of (2) satisfy the relation

$$\int_a^{\infty} |U(t)| \cdot t^{-\alpha} dt < \infty, \quad a > 0. \tag{10}$$

Now functions $\gamma_0(x, \lambda)$ and $\gamma_{\infty}(x, \lambda)$ are defined as follows:

$$\begin{gathered} \gamma_0(x, \lambda) = \frac{1}{\sqrt[4]{4(x^{2\alpha} - \lambda)}} \cdot \exp\left(-\int_a^x \sqrt{u^{2\alpha} - \lambda}\, du\right), \\ \gamma_{\infty}(x, \lambda) = \frac{1}{\sqrt[4]{4(x^{2\alpha} - \lambda)}} \cdot \exp\left(\int_a^x \sqrt{u^{2\alpha} - \lambda}\, du\right). \end{gathered} \tag{11}$$

These functions also form a fundamental system of solutions of the scalar differential equation, which is obtained by replacing $v(x)$ with $v(x) - \lambda$ in formulas (7) and (8).

In [10] the asymptotic behavior of the functions $\gamma_0(x, \lambda)$ and $\gamma_{\infty}(x, \lambda)$ was established as $x \to \infty$. If $(\alpha+1)/2\alpha = n \in \mathbb{N}$, that is, $\alpha = 1/(2n-1)$, then functions $\gamma_0(x, \lambda)$ and $\gamma_{\infty}(x, \lambda)$ as $x \to \infty$ will have the following asymptotic behavior:

$$\begin{aligned} \gamma_0(x, \lambda) = {} & c \cdot \exp\left(-\frac{x^{1+\alpha}}{1+\alpha} + \frac{\lambda}{2} \cdot \frac{x^{1-\alpha}}{1-\alpha} \right. \\ & \left. + \sum_{k=2}^{n-1} \frac{1 \cdot 3 \cdot \ldots \cdot (2k-3)}{k!} \cdot \left(\frac{\lambda}{2}\right)^k \cdot \frac{x^{1-(2k-1)\alpha}}{1-(2k-1)\alpha}\right) \\ & \cdot x^{((1\cdot 3 \cdot \ldots \cdot (2n-3))/n!)\cdot(\lambda/2)^n - \alpha/2} \cdot (1 + o(1)), \\ \gamma_{\infty}(x, \lambda) = {} & c \cdot \exp\left(\frac{x^{1+\alpha}}{1+\alpha} - \frac{\lambda}{2} \cdot \frac{x^{1-\alpha}}{1-\alpha} \right. \\ & \left. - \sum_{k=2}^{n-1} \frac{1 \cdot 3 \cdot \ldots \cdot (2k-3)}{k!} \cdot \left(\frac{\lambda}{2}\right)^k \cdot \frac{x^{1-(2k-1)\alpha}}{1-(2k-1)\alpha}\right) \\ & \cdot x^{-(((1\cdot 3 \cdot \ldots \cdot (2n-3))/n!)\cdot(\lambda/2)^n + \alpha/2)} \cdot (1 + o(1)). \end{aligned} \tag{12}$$

In particular, with $\alpha = 1$ $(n = 1)$ one has

$$\begin{gathered} \gamma_0(x, \lambda) = c \cdot x^{(\lambda-1)/2} \cdot \exp\left(-\frac{x^2}{2}\right)(1 + o(1)), \\ \gamma_{\infty}(x, \lambda) = c \cdot x^{-(\lambda+1)/2} \cdot \exp\left(\frac{x^2}{2}\right)(1 + o(1)). \end{gathered} \tag{13}$$

In the case $(\alpha+1)/2\alpha \notin \mathbb{N}$, set $n = [(\alpha+1)/2\alpha] + 1$, with $[\beta]$ being the integral part of β, to obtain the following asymptotic behavior for $\gamma_0(x, \lambda)$ and $\gamma_{\infty}(x)$ at infinity:

$$\begin{aligned} \gamma_0(x, \lambda) = {} & c \cdot x^{-\alpha/2} \exp\left(-\frac{x^{1+\alpha}}{1+\alpha} + \frac{\lambda}{2} \cdot \frac{x^{1-\alpha}}{1-\alpha} \right. \\ & \left. + \sum_{k=2}^{n-1} \frac{1 \cdot 3 \cdot \ldots \cdot (2k-3)}{k!} \cdot \left(\frac{\lambda}{2}\right)^k \cdot \frac{x^{1-(2k-1)\alpha}}{1-(2k-1)\alpha}\right) \\ & \cdot \exp\left(-\frac{1 \cdot 3 \cdot \ldots \cdot (2n-3)}{n!} \cdot \left(\frac{\lambda}{2}\right)^n \cdot \frac{x^{-\alpha}}{\alpha}\right) \cdot (1 \\ & + o(x^{-\alpha})), \\ \gamma_{\infty}(x, \lambda) = {} & c \cdot x^{-\alpha/2} \exp\left(\frac{x^{1+\alpha}}{1+\alpha} - \frac{\lambda}{2} \cdot \frac{x^{1-\alpha}}{1-\alpha} \right. \\ & \left. - \sum_{k=2}^{n-1} \frac{1 \cdot 3 \cdot \ldots \cdot (2k-3)}{k!} \cdot \left(\frac{\lambda}{2}\right)^k \cdot \frac{x^{1-(2k-1)\alpha}}{1-(2k-1)\alpha}\right) \\ & \cdot \exp\left(\frac{1 \cdot 3 \cdot \ldots \cdot (2n-3)}{n!} \cdot \left(\frac{\lambda}{2}\right)^n \cdot \frac{x^{-\alpha}}{\alpha}\right) \cdot (1 \\ & + o(x^{-\alpha})). \end{aligned} \tag{14}$$

In [10] for an equation with matrix coefficients, and in the furtherance for equations with operator coefficients, the following theorem is proved.

Theorem 1. *If for (2) conditions (4)-(5) are satisfied for $\alpha > 1$ or condition (10) for $0 < \alpha \leq 1$, then the equation has a unique decreasing at infinity operator solution $\Phi(x,\lambda)$, satisfying the conditions*

$$\lim_{x\to\infty}\frac{\Phi(x,\lambda)}{\gamma_0(x,\lambda)} = I, \qquad \lim_{x\to\infty}\frac{\Phi'(x,\lambda)}{\gamma_0'(x,\lambda)} = I. \tag{15}$$

Also, there exists increasing at infinity operator solution $\Psi(x,\lambda)$, satisfying the conditions

$$\lim_{x\to\infty}\frac{\Psi(x,\lambda)}{\gamma_\infty(x,\lambda)} = I, \qquad \lim_{x\to\infty}\frac{\Psi'(x,\lambda)}{\gamma_\infty'(x,\lambda)} = I. \tag{16}$$

Corollary 2. *If $\alpha = 1$, that is, $v(x) = x^2$, then, under condition (10), the solutions $\Phi(x,\lambda)$ and $\Psi(x,\lambda)$ have common (known) asymptotic behavior, as in the quality $\gamma_0(x,\lambda)$ and $\gamma_\infty(x,\lambda)$ you can take the following functions:*

$$\gamma_0(x,\lambda) = x^{(\lambda-1)/2}\cdot\exp\left(-\frac{x^2}{2}\right), \qquad \gamma_\infty(x,\lambda) = x^{-(\lambda+1)/2}\cdot\exp\left(\frac{x^2}{2}\right). \tag{17}$$

3. Resolvent of the Non-Self-Adjoint Operator

Let the following boundary condition be given at $x = 0$:

$$\cos A\cdot y'(0) - \sin A\cdot y(0) = 0, \tag{18}$$

where A is block-triangular operator of the same structure as the potential $V(x)$ (3) of the differential equation (2), and A_{kk}, $k = \overline{1,r}$, are the bounded self-adjoint operators in H_k, which satisfy the conditions

$$-\frac{\pi}{2}I_k \ll A_{kk} \leq \frac{\pi}{2}I_k. \tag{19}$$

Together with problem (2) and (18) we consider the separated system

$$l_k[y_k] = -y_k'' + (v(x)I_k + U_{kk}(x))y_k = \lambda y_k, \qquad k = \overline{1,r} \tag{20}$$

with the boundary conditions

$$\cos A_{kk}\cdot y_k'(0) - \sin A_{kk}\cdot y_k(0) = 0, \quad k = \overline{1,r}. \tag{21}$$

Let L' denote the minimal differential operator generated by differential expression $l[y]$ and the boundary condition (18), and let L_k', $k = \overline{1,r}$, denote the minimal differential operator on $L_2(\mathbf{H},(0,\infty))$ generated by differential expression $l_k[y_k]$ and the boundary conditions (21). Taking into account the conditions on coefficients, as well as sufficient smallness of perturbations $U_{kk}(x)$, and conditions (19), we conclude that, for every symmetric operator L_k', $k = \overline{1,r}$, there is a case of limit point at infinity. Hence their self-adjoint extensions L_k are the closures of operators L_k', respectively. The operators L_k are semibounded below, and their spectra are discrete.

Let L denote the operator extensions L', by requiring that $L_2(\mathbf{H},(0,\infty))$ be the domain of operator L.

The following theorem is proved in [10].

Theorem 3. *Suppose that for (2) conditions (4)-(5) are satisfied for $\alpha > 1$ or condition (10) for $0 < \alpha \leq 1$. Then the discrete spectrum of the operator L is real and coincides with the union of spectra of the self-adjoint operators L_k, $k = \overline{1,r}$; that is,*

$$\sigma_d(L) = \bigcup_{k=1}^{r}\sigma(L_k). \tag{22}$$

Comment 4. Note that this theorem contains a statement of the discrete spectrum of the non-self-adjoint operator L only and no allegations of its continuous and residual spectrum.

Along with (2) we consider the equation

$$l_1[y] = -y'' + V^*(x)y = \lambda y \tag{23}$$

($V^*(x)$ is adjoint to the operator $V(x)$). If the space $\mathbf{H}$ is finite-dimensional, then (23) can be rewritten as

$$\tilde{l}[\tilde{y}] = -\tilde{y}'' + \tilde{y}V(x) = \lambda\tilde{y}, \tag{24}$$

where $\tilde{y} = (\tilde{y}_1\ \ \tilde{y}_2\dots\tilde{y}_r)$ and the equation is called the left.

For operator functions $Y(x,\lambda), Z(x,\lambda) \in B(\mathbf{H})$ let

$$W\{Z^*,Y\} = Z^{*\prime}(x,\overline{\lambda})Y(x,\lambda) - Z^*(x,\overline{\lambda})Y'(x,\lambda). \tag{25}$$

If $Y(x,\lambda)$ is operator solution of (2) and $Z(x,\lambda)$ is operator solution of (23), the Wronskian does not depend on x.

Now we denote $Y(x,\lambda)$ and $Y_1(x,\lambda)$ as the solutions of (2) and (23), respectively, satisfying the initial conditions

$$\begin{aligned} Y(0,\lambda) &= \cos A,\\ Y'(0,\lambda) &= \sin A,\\ Y_1(0,\lambda) &= (\cos A)^*,\\ Y_1'(0,\lambda) &= (\sin A)^*,\\ &\lambda\in\mathbb{C}. \end{aligned} \tag{26}$$

Because the operator function $Y_1^*(x,\overline{\lambda})$ satisfies equation

$$-Y_1^{*\prime\prime}(x,\overline{\lambda}) + Y_1^*(x,\overline{\lambda})\cdot V(x) = \lambda Y_1^*(x,\overline{\lambda}), \tag{27}$$

the operator function $\widetilde{Y}(x,\lambda) =: Y_1^*(x,\overline{\lambda})$ is a solution to the left of the equation

$$-\widetilde{Y}''(x,\lambda) + \widetilde{Y}(x,\lambda)\cdot V(x) = \lambda\widetilde{Y}(x,\lambda) \tag{28}$$

and satisfies the initial conditions $\widetilde{Y}(0,\lambda) = \cos A$, $\widetilde{Y}'(0,\lambda) = \sin A$, $\lambda\in\mathbb{C}$.

Operator solutions of (23) decreasing and increasing at infinity will be denoted by $\Phi_1(x,\lambda)$, $\Psi_1(x,\lambda)$, and the corresponding solutions of (28) are denoted by $\widetilde{\Phi}(x,\lambda)$ and $\widetilde{\Psi}(x,\lambda)$. The system operator solutions $Y(x,\lambda), \widetilde{\Phi}(x,\lambda) \in B(\mathbf{H})$ of (2) and (28), respectively, will take the form of Wronskian $W\{\widetilde{\Phi}, Y\} = \widetilde{\Phi}'(x,\lambda)Y(x,\lambda) - \widetilde{\Phi}(x,\lambda)Y'(x,\lambda)$.

Let us designate

$$G(x,t,\lambda) = \begin{cases} Y(x,\lambda)\left(W\left\{\widetilde{\Phi},Y\right\}\right)^{-1}\widetilde{\Phi}(t,\lambda) & 0 \le x \le t \\ -\Phi(x,\lambda)\left(W\left\{\widetilde{Y},\Phi\right\}\right)^{-1}\widetilde{Y}(t,\lambda) & x \ge t. \end{cases} \tag{29}$$

It is proved in [12] that the operator function $G(x,t,\lambda)$ is Green's function of the differential operator L; that is, it possesses all the classical properties of Green's function. In particular, for a fixed t the function $G(x,t,\lambda)$ of the variable x is an operator solution of (2) on each of the intervals $[0,t)$, (t,∞), and it satisfies the boundary condition (18), and at a fixed x, the function $G(x,t,\lambda)$ satisfies (28) in the variable t on each of the intervals $[0,x)$, (x,∞), and it satisfies the boundary condition $(\cos A)^* \cdot y'(0) - (\sin A)^* \cdot y(0) = 0$.

By definition (28), function $G(x,t,\lambda)$ is meromorphic by parameter λ with the poles coinciding with the eigenvalues of the operator L.

We consider the operator R_λ defined in $L_2(\mathbf{H},(0,\infty))$ by the relation

$$\begin{aligned}(R_\lambda f)(x) &= \int_0^\infty G(x,t,\lambda) f(t)\,dt \\ &= -\int_0^x \Phi(x,\lambda)\left(W\left\{\widetilde{Y},\Phi\right\}\right)^{-1}\widetilde{Y}(t,\lambda) f(t)\,dt \\ &\quad + \int_x^\infty Y(x,\lambda)\left(W\left\{\widetilde{\Phi},Y\right\}\right)^{-1}\widetilde{\Phi}(t,\lambda) f(t)\,dt.\end{aligned} \tag{30}$$

Theorem 5. *The operator R_λ is the resolvent of the operator L.*

4. Proof of Theorem 5

One can directly verify that, for any function $f(x) \in L_2(\mathbf{H},(0,\infty))$, the vector-function $y(x,\lambda) = (R_\lambda f)(x)$ is a solution of the equation $l[y] - \lambda y = f$ whenever $\lambda \notin \sigma(L)$. We will prove that $y(x,\lambda) \in L_2(\mathbf{H},(0,\infty))$.

Since operator solutions $\Phi(x,\lambda)$ and $\Psi(x,\lambda)$ form a fundamental system of solutions of (2), the operator solution $Y(x,\lambda)$ of (2) satisfying the initial conditions (26) can be written as $Y(x,\lambda) = \Phi(x,\lambda)A(\lambda)+\Psi(x,\lambda)B(\lambda)$, where $A(\lambda) = W\{\widetilde{\Psi},Y\}$, $B(\lambda) = -W\{\widetilde{\Phi},Y\}$; that is,

$$Y(x,\lambda) = \Phi(x,\lambda)W\left\{\widetilde{\Psi},Y\right\} - \Psi(x,\lambda)W\left\{\widetilde{\Phi},Y\right\}. \tag{31}$$

Similarly, the operator solution $\widetilde{Y}(x,\lambda)$ of (28) can be represented in the following form:

$$\widetilde{Y}(x,\lambda) = W\left\{\widetilde{Y},\Phi\right\}\widetilde{\Psi}(x,\lambda) - W\left\{\widetilde{Y},\Psi\right\}\widetilde{\Phi}(x,\lambda). \tag{32}$$

By using formulas (31) and (32), we can rewrite relation (30) as follows:

$$\begin{aligned}(R_\lambda f)(x) &= -\int_0^a \Phi(x,\lambda)\left(W\left\{\widetilde{Y},\Phi\right\}\right)^{-1}\widetilde{Y}(t,\lambda) f(t)\,dt \\ &\quad + \chi_1(x,\lambda) - \chi_2(x,\lambda) + \chi_3(x,\lambda) - \chi_4(x,\lambda),\end{aligned} \tag{33}$$

where $a > 0$ and

$$\begin{aligned}\chi_1(x,\lambda) &= \Phi(x,\lambda)\left(W\left\{\widetilde{Y},\Phi\right\}\right)^{-1}W\left\{\widetilde{Y},\Psi\right\} \cdot \int_a^x \widetilde{\Phi}(t,\lambda) f(t)\,dt, \\ \chi_2(x,\lambda) &= \Phi(x,\lambda)\int_a^x \widetilde{\Psi}(t,\lambda) f(t)\,dt, \\ \chi_3(x,\lambda) &= \Phi(x,\lambda)W\left\{\widetilde{\Psi},Y\right\}\left(W\left\{\widetilde{\Phi},Y\right\}\right)^{-1} \cdot \int_x^\infty \widetilde{\Phi}(t,\lambda) f(t)\,dt, \\ \chi_4(x,\lambda) &= \Psi(x,\lambda)\int_x^\infty \widetilde{\Phi}(t,\lambda) f(t)\,dt.\end{aligned} \tag{34}$$

Let us show that each of these vector-functions $\chi_1(x,\lambda)$, $\chi_2(x,\lambda)$, $\chi_3(x,\lambda)$, and $\chi_4(x,\lambda)$ belongs to $L_2(\mathbf{H},(0,\infty))$. Since the operator solution $\Phi(x,\lambda)$ decays fairly quickly as $x \to \infty$, then $|\Phi(x,\lambda)| \in L_2(0,\infty)$. It follows that

$$\begin{aligned}\left|\chi_1(x,\lambda)\right| &\le c(\lambda)\cdot|\Phi(x,\lambda)|\cdot\int_a^x \left|\widetilde{\Phi}(t,\lambda)\right|\cdot\left|f(t)\right|dt \\ &\le c(\lambda)\cdot|\Phi(x,\lambda)|\cdot\left(\int_a^x\left|\widetilde{\Phi}(t,\lambda)\right|dt\right)^{1/2} \cdot\left(\int_a^x\left|f(t)\right|dt\right)^{1/2} \\ &< c(\lambda)\cdot|\Phi(x,\lambda)|\cdot\left(\int_a^\infty\left|\widetilde{\Phi}(t,\lambda)\right|dt\right)^{1/2} \cdot\left(\int_a^\infty\left|f(t)\right|dt\right)^{1/2} \le c_1(\lambda)\cdot|\Phi(x,\lambda)|,\end{aligned} \tag{35}$$

and therefore $\chi_1(x,\lambda) \in L_2(\mathbf{H},(0,\infty))$. Similarly we get that $\chi_3(x,\lambda) \in L_2(\mathbf{H},(0,\infty))$. First we prove the assertion for the function $\chi_2(x,\lambda)$, when $\alpha > 1$ and the coefficients of (2) satisfy the conditions (4)-(5). In this case, we have

$$\left|\chi_2(x,\lambda)\right| \le |\Phi(x,\lambda)|\int_a^x\left|\widetilde{\Psi}(t,\lambda)\right|\left|f(t)\right|dt. \tag{36}$$

By virtue of the asymptotic formulas for the operator solutions $\Phi(x,\lambda)$ and $\Psi(x,\lambda)$ we obtain that

$$\left|\chi_2(x,\lambda)\right| \le c_1(\lambda)\gamma_0(x,\lambda)\int_a^x \gamma_\infty(t,\lambda)\left|f(t)\right|dt. \tag{37}$$

Let us rewrite this relation in the following form:

$$\begin{aligned}&\left|\chi_2(x,\lambda)\right|\\&\quad\leq c_1(\lambda)\gamma_0(x,\lambda)\gamma_\infty(x,\lambda)\int_a^x\frac{\gamma_\infty(t,\lambda)}{\gamma_\infty(x,\lambda)}\left|f(t)\right|dt.\end{aligned}\tag{38}$$

By using the definition of the functions $\gamma_0(x,\lambda)$ and $\gamma_\infty(x,\lambda)$ (see (6)) and by applying the Cauchy- Bunyakovsky inequality we obtain

$$\begin{aligned}\left|\chi_2(x,\lambda)\right|&\leq\frac{1}{2}c_1(\lambda)\frac{1}{\sqrt{v(x)}}\left(\int_a^x\sqrt{\frac{v(x)}{v(t)}}\right.\\&\quad\cdot\left.\exp\left(-2\int_t^x\sqrt{v(u)}du\right)dt\right)^{1/2}\\&\quad\cdot\left(\int_0^\infty\left|f(t)\right|^2dt\right)^{1/2}.\end{aligned}\tag{39}$$

Since $t\leq x$, we get $\exp(-2\int_t^x\sqrt{v(u)}du)\leq 1$, and then the latter estimate for $\chi_2(x,\lambda)$ can be rewritten as follows:

$$\begin{aligned}\left|\chi_2(x,\lambda)\right|&\leq c_2(\lambda)\frac{1}{\sqrt[4]{v(x)}}\left(\int_a^x\frac{1}{\sqrt{v(t)}}dt\right)^{1/2}\\&\leq c_2(\lambda)\frac{1}{\sqrt[4]{v(x)}}\left(\int_a^\infty\frac{1}{\sqrt{v(t)}}dt\right)^{1/2}.\end{aligned}\tag{40}$$

By formula (4), we get

$$\left|\chi_2(x,\lambda)\right|\leq\frac{c_3(\lambda)}{\sqrt[4]{v(x)}},\tag{41}$$

and hence if $\alpha>1$ and the coefficients of (2) satisfy the conditions (4) and (5), we have $\chi_2(x,\lambda)\in L_2(\mathbf{H},(0,\infty))$. In the case of $v(x)=x^{2\alpha}$, $0<\alpha\leq 1$, the assertion can be proved similarly.

For the function $\chi_4(x,\lambda)$ we will conduct the proof for the case when $v(x)=x^{2\alpha}$, $0<\alpha\leq 1$, and the coefficients of (2) satisfy condition (10). As in (37) we have

$$\left|\chi_4(x,\lambda)\right|\leq c_1(\lambda)\gamma_\infty(x,\lambda)\int_x^\infty\gamma_0(t,\lambda)\left|f(t)\right|dt,\tag{42}$$

which can be rewritten as follows:

$$\begin{aligned}&\left|\chi_4(x,\lambda)\right|\\&\quad\leq c_1(\lambda)\gamma_0(x,\lambda)\gamma_\infty(x,\lambda)\int_x^\infty\frac{\gamma_0(t,\lambda)}{\gamma_0(x,\lambda)}\left|f(t)\right|dt.\end{aligned}\tag{43}$$

Let us use the asymptotic behavior of the functions $\gamma_0(x,\lambda)$ and $\gamma_\infty(x,\lambda)$, for example, in the case $(\alpha+1)/2\alpha=n\in N$, that is, $\alpha=1/(2n-1)$ (see (12)). Setting $a(\alpha,\lambda)=((1\cdot 3\cdot\ldots\cdot(2n-3))/n!)\cdot(\lambda/2)^n$, we obtain

$$\begin{aligned}\left|\chi_4(x,\lambda)\right|&\leq c_2(\lambda)x^{-\alpha}\int_x^\infty\frac{\gamma_0(t,\lambda)}{\gamma_0(x,\lambda)}\left|f(t)\right|dt\leq c_2(\lambda)\\&\quad\cdot x^{-\alpha}\left(\int_a^x\left(\frac{\gamma_0(t,\lambda)}{\gamma_0(x,\lambda)}\right)^2dt\right)^{1/2}\left(\int_0^\infty\left|f(t)\right|^2dt\right)^{1/2},\\\left|\chi_4(x,\lambda)\right|&\leq c_3(\lambda)x^{-\alpha}\left(\int_x^\infty\left(\frac{t}{x}\right)^{2a(\alpha,\lambda)-\alpha}\right.\\&\quad\cdot\left.\exp\frac{-2x^{\alpha+1}\left((t/x)^{\alpha+1}-1\right)}{1+\alpha}dt\right)^{1/2}.\end{aligned}\tag{44}$$

Replacing variables $t=xu$, we get

$$\begin{aligned}\left|\chi_4(x,\lambda)\right|&\leq c_3(\lambda)x^{-\alpha+1/2}\left(\int_1^\infty u^{2a(\alpha,\lambda)-\alpha}\right.\\&\quad\cdot\left.\exp\frac{-2x^{\alpha+1}\left(u^{\alpha+1}-1\right)}{1+\alpha}du\right)^{1/2}.\end{aligned}\tag{45}$$

Since the inequality $\exp(-x^{\alpha+1}(u^{\alpha+1}-1)/(1+\alpha))\leq x^{-2}$ holds for all $\alpha\in(0,1]$ and $u\in[1,\infty)$ with sufficiently large x, we have

$$\begin{aligned}\left|\chi_4(x,\lambda)\right|&\leq c_3(\lambda)x^{-\alpha-1/2}\left(\int_1^\infty u^{2a(\alpha,\lambda)-\alpha}\right.\\&\quad\cdot\left.\exp\frac{-x^{\alpha+1}\left(u^{\alpha+1}-1\right)}{1+\alpha}du\right)^{1/2}.\end{aligned}\tag{46}$$

Hence it follows that $|\chi_4(x,\lambda)|\leq c_4(\alpha,\lambda)x^{-\alpha-1/2}$, and therefore $\chi_4(x,\lambda)\in L_2(\mathbf{H},(0,\infty))$. In case, where $0<\alpha\leq 1$ and $(\alpha+1)/2\alpha\notin N$, and where $\alpha>1$, the proof is similar.

Thus, $R_\lambda f\in L_2(\mathbf{H},(0,\infty))$ for any function $f\in L_2(\mathbf{H},(0,\infty))$. This completes the proof.

Since the resolvent R_λ is a meromorphic function of λ, the poles of which coincide with the eigenvalues of the operator L, the statement of Theorem 3 can be refined.

Theorem 6. *If conditions (4)-(5) where $\alpha>1$ or condition (10) where $0<\alpha\leq 1$ is satisfied for (2), then the spectrum of the operator L is real and discrete and coincides with the union of spectra of self-adjoint operators L_k, $k=\overline{1,m}$; that is,*

$$\sigma(L)=\bigcup_{k=1}^r\sigma(L_k).\tag{47}$$

5. Application

Here we consider (2) with matrix coefficients and use the same notation as in Section 3 (note that could be considered second-order equation with block-triangular coefficients of a more general form [14]). Suppose that every symmetric operator L'_k is lower semibounded. Let L be an arbitrary extension of the operator L', defined boundary condition at infinity, and L_k an arbitrary self-adjoint extension of the operator L'_k. If the conditions at infinity determine the Friedrichs extension L^0_k of the semibounded symmetric operator L'_k, the corresponding extension of L' will be denoted L^0. Besides, let us assume that coefficients of (2) for the problem of semiaxis are such that discrete spectrum of L operator coincides with the union of discrete spectra of L_k operators, $k = \overline{1,r}$, (sufficient conditions are specified above in Theorem 6).

Denote by $\mathrm{nul}_a T$ the algebraic multiplicity of 0 as an eigenvalue of T.

Denote by $N^0_a(\lambda)$ the number of eigenvalues $\lambda^0_n < \lambda < \lambda_e(L^0)$ of the operator L^0 counted according to their algebraic multiplicities. Here $\lambda_e(L^0)$ stands for the lower bound of the essential spectrum of the operator L^0.

In [14] is set oscillation theorem of Sturm for equations with block-triangular matrix potential.

Theorem 7. *Suppose the operator L^0 is generated by the differential expression $l[y]$ with matrix block-triangular potential, the boundary condition at 0 (18), and such boundary conditions at the infinity that one gets Friedrichs extensions for semibounded symmetric operators L'_k. Then for $\lambda < \lambda_e(L^0)$ one has*

$$\sum_{x\in(0,\infty)} \mathrm{nul}_a Y(x,\lambda) = N^0_a(\lambda) \tag{48}$$

(the sum is in those $x \in (0,\infty)$ for which $\mathrm{nul}_a Y(x,\lambda) \neq 0$).

In the same article a theorem about the connection between spectral and oscillation properties for any extension of the minimal operator is also proved. These theorems are generalizations for non-self-adjoint operators of the classical Sturm type oscillation theorems and this problem was considered for the first time.

6. Conclusion

In this work a resolvent is constructed for the Sturm-Liouville operator with a block-triangular operator potential increasing at infinite. The structure of the spectrum of such an operator is obtained.

Competing Interests

The author declared that no competing interests exist.

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