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Research Article

On Certain Subclasses of Analytic Multivalent Functions Using Generalized Salagean Operator

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We introduce and study two subclasses of multivalent functions denoted by $\mathscr{M}_{p,n}^{m,\alpha,\beta,\sigma}(\lambda_1;\lambda_2)$ and $\mathscr{N}_{p,n}^{m,\alpha,\beta,\sigma}(\mu,\delta;\gamma)$. Further, by using the method of differential subordination, certain inclusion relations between the two subclasses aforementioned are given. Moreover, several consequences of the main results are also discussed.

1. Introduction

Let $\mathscr{A}_{(p,n)}$ denote the class of the functions f of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \quad (n, p \in \mathbb{N} = \{1, 2, 3, \ldots\}), \tag{1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and let denote $\mathscr{A} := \mathscr{A}_{(1,1)}$.

A function $f \in \mathscr{A}_{(p,n)}$ is said to be multivalent starlike functions of order α in $\mathbb{U}$, if it satisfies the following inequality:

$$\mathfrak{R}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad z \in \mathbb{U}, \ (0 \le \alpha < p, \ p \in \mathbb{N}), \tag{2}$$

and we denote this class by $S_{p,n}^{*}(\alpha)$.

A function $f \in \mathscr{A}_{(p,n)}$ is said to be multivalent convex functions of order α in $\mathbb{U}$, if it satisfies the following inequality:

$$\mathfrak{R}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad z \in \mathbb{U}, \ (0 \le \alpha < p, \ p \in \mathbb{N}), \tag{3}$$

and we denote this class by $C_{p,n}(\alpha)$.

For a function $f \in \mathscr{A}_{(p,n)}$, Goyal et al. [1] introduced the following generalized Salagean differential operator:

$$D_\sigma^0 f(z) = f(z), \tag{4}$$

$$D_\sigma^1 f(z) = D_\sigma f(z) = (1-\sigma) f(z) + \frac{\sigma}{p} z f'(z), \quad (\sigma \ge 0), \tag{5}$$

$$D_\sigma^m f(z) = D_\sigma\left(D_\sigma^{m-1} f(z)\right), \quad (m \in \mathbb{N}). \tag{6}$$

If f is given by (1), then from (5) and (6) we have

$$D_\sigma^m f(z) = z^p + \sum_{k=p+n}^{\infty}\left[1 + \left(\frac{k}{p} - 1\right)\sigma\right]^m a_k z^k. \tag{7}$$

Remark 1. For $\sigma = p = 1$, the differential operator $D_\sigma^m f(z)$ reduces to Salagean differential operator $D^m f(z)$ [2].

Definition 2. Let $\mathscr{M}_{p,n}^{m,\alpha,\beta,\sigma}(\lambda_1;\lambda_2)$ be the class of functions $f \in \mathscr{A}_{(p,n)}$ that satisfy the condition

$$\Re\left\{(1-\lambda_1)\frac{z\left(D_\sigma^{\alpha,\beta,m}f(z)\right)'}{D_\sigma^{\alpha,\beta,m}f(z)} + \lambda_1\left(1+\frac{z\left(D_\sigma^{\alpha,\beta,m}f(z)\right)''}{\left(D_\sigma^{\alpha,\beta,m}f(z)\right)'}\right)\right\} > \lambda_2, \quad z\in\mathbb{U}, \tag{8}$$

where ($0\le\sigma,\ 0\le\alpha<\beta\le 1,\ \lambda_1\in\mathbb{R},\ 0\le\lambda_2<p,\ m,p\in\mathbb{N}$), and let $\mathcal{N}_{p,n}^{m,\alpha,\beta,\sigma}(\mu,\delta;\gamma)$ be the class of functions $f\in\mathcal{A}_{(p,n)}$ that satisfy the conditions

$$\frac{\left(D_\sigma^{\alpha,\beta,m}f(z)\right)\left(D_\sigma^{\alpha,\beta,m}f(z)\right)'}{z^{2p-1}}\neq 0, \quad z\in\mathbb{U},$$
$$\Re\left\{\left(\frac{D_\sigma^{\alpha,\beta,m}f(z)}{z^p}\right)^{\mu}\left(\frac{\left(D_\sigma^{\alpha,\beta,m}f(z)\right)'}{z^{p-1}}\right)^{\delta}\right\} > \gamma, \quad z\in\mathbb{U}, \tag{9}$$
$$\left(0\le\sigma,\ \delta,\mu\in\mathbb{R},\ 0\le\alpha<\beta\le 1,\ 0\le\gamma<p^{\delta}(\alpha+\beta p)^{\delta+\mu},\ m,p\in\mathbb{N}\right),$$

where

$$D_\sigma^{\alpha,\beta,m}f(z)=\alpha D_\sigma^m f(z)+\beta z\left(D_\sigma^m f(z)\right)'. \tag{10}$$

Remark 3. By specifying different values, we have some well-known subclasses of the classes $A_{(p,n)}$ and $A_{(n)} = A_{(1,n)}$ appearing from the families of the classes $\mathcal{M}_{p,n}^{m,\alpha,\beta,\sigma}(\lambda_1;\lambda_2)$ and $\mathcal{N}_{p,n}^{m,\alpha,\beta,\sigma}(\mu,\delta;\gamma)$.

(i) $\mathcal{M}_{p,n}^{0,1,0,\sigma}(0;\lambda_1)=\mathcal{N}_{p,n}^{0,1,0,\sigma}(-1,1,\lambda_1)=S_{p,n}^*$, ($0\le\lambda_1<p$) is the class of multivalent starlike functions of order λ_1.

(ii) $\mathcal{M}_{1,n}^{0,1,0,\sigma}(0;\lambda_1)=\mathcal{N}_{1,n}^{0,1,0,\sigma}(-1,1,\lambda_1)=S_{1,n}^*=S_n^*$, ($0\le\lambda_1<1$) is the class of starlike functions of order λ_1.

(iii) $\mathcal{M}_{p,n}^{1,0,1,1}(0;\lambda_1)=\mathcal{C}_{p,n}(\lambda_1)$, ($0\le\lambda_1<p$) is the class of multivalent convex functions of order λ_1.

(iv) $\mathcal{M}_{1,n}^{1,0,1,1}(0;\lambda_1)=\mathcal{C}_{1,n}(\lambda_1)=\mathcal{C}_n(\lambda_1)$, ($0\le\lambda_1<1$) is the class of convex functions of order γ.

(v) $\mathcal{N}_{1,n}^{1,1,0,1}(1,\delta;\lambda_1)=\mathcal{B}_n(\delta;\lambda_1)$, ($\delta\ge-1, 0\le\lambda_1<1$) is the subclass of Bazilević functions.

Let $\mathcal{H}[a,n]$ be denoted by the class

$$\mathcal{H}[a,n]=\left\{h\in\mathcal{H}(\mathbb{U}):h(z)=a+a_nz^n+\cdots,\ z\in\mathbb{U}\right\}. \tag{11}$$

In this investigation, we focus on certain inequalities consisting of the following differential operator $\mathcal{J}_{p,n}^{m,\alpha,\beta,\sigma}(\mu,\delta):\mathcal{A}_{(p,n)}\to\mathcal{H}[(\mu+\delta),n+p]$:

$$\mathcal{J}_{p,n}^{m,\alpha,\beta,\sigma}(\mu,\delta)f(z)=\mu\frac{z\left(D_\sigma^{\alpha,\beta,m}f(z)\right)'}{D_\sigma^{\alpha,\beta,m}f(z)}+\delta\left(1+\frac{z\left(D_\sigma^{\alpha,\beta,m}f(z)\right)''}{\left(D_\sigma^{\alpha,\beta,m}f(z)\right)'}\right) \tag{12}$$

that generalizes the expression used in the definition of class $\mathcal{M}_{p,n}^{m,\alpha,\beta,\sigma}(\lambda_1;\lambda_2)$ and we receive several properties of the expression

$$\left(\frac{D_\sigma^{\alpha,\beta,m}f(z)}{z^p}\right)^{\mu}\left(\frac{\left(D_\sigma^{\alpha,\beta,m}f(z)\right)'}{z^{p-1}}\right)^{\delta}, \quad (z\in\mathbb{U}), \tag{13}$$

including relations between classes $\mathcal{M}_{p,n}^{m,\alpha,\beta,\sigma}(\lambda_1;\lambda_2)$ and $\mathcal{N}_{p,n}^{m,\alpha,\beta,\sigma}(\mu,\delta;\gamma)$.

In order to prove our main results, we will need the following lemmas due to Miller and Mocanu [3].

Lemma 4. *Let $\Omega\subset\mathbb{C}$ and suppose that the function $\psi:\mathbb{C}^2\times\mathbb{U}\to\mathbb{C}$ satisfies $\psi(Me^{i\theta},Ke^{i\theta};z)\notin\Omega$ for all $K\ge Mn$, $\theta\in\mathbb{R}$, and $z\in\mathbb{U}$. If $h(z)=a+h_nz^n+\cdots$ is analytic in $\mathbb{U}$ and $\psi(h(z),zh'(z);z)\in\Omega$ for all $z\in\mathbb{U}$, then $|h(z)|<M$, $z\in\mathbb{U}$.*

Lemma 5. *Let $\Omega\subset\mathbb{C}$ and suppose that the function $\psi:\mathbb{C}^2\times\mathbb{U}\to\mathbb{C}$ satisfies $\psi(ix,y;z)\notin\Omega$ for all $x\in\mathbb{R}$, $y\le-n(1+x^2)/2$, and $z\in\mathbb{U}$. If $h(z)=a+h_nz^n+\cdots$ is analytic in $\mathbb{U}$ and $\psi(h(z),zh'(z);z)\in\Omega$ for all $z\in\mathbb{U}$, then $\Re\{h(z)\}>0$, $z\in\mathbb{U}$.*

2. Main Results

Following the same techniques and procedure given by Goswami et al. [4], we have the following results.

Theorem 6. *Let $f(z)\in\mathcal{A}_{(p,n)}$ with $(D_\sigma^{\alpha,\beta,m}f(z))(D_\sigma^{\alpha,\beta,m}f(z))'/z^{2p-1}\neq 0$ for all $z\in\mathbb{U}$, where $D_\sigma^{\alpha,\beta,m}$ is given by (10), and also let $\mu,\delta\in\mathbb{R}$. If*

$$\Re\left\{\mathcal{J}_{p,n}^{m,\alpha,\beta,\sigma}(\mu,\delta)f(z)\right\} < p(\delta+\mu)+\frac{nM}{M+p^{\delta}(\alpha+\beta p)^{\delta+\mu}}, \quad (z\in\mathbb{U}), \tag{14}$$

where $p^{\delta}(\alpha+\beta p)^{\delta+\mu} \le M$, *then*

$$\left|\left(\frac{D_{\sigma}^{\alpha,\beta,m} f(z)}{z^{p}}\right)^{\mu}\left(\frac{\left(D_{\sigma}^{\alpha,\beta,m} f(z)\right)'}{z^{p-1}}\right)^{\delta} - p^{\delta}(\alpha+\beta p)^{\delta+\mu}\right| < M, \quad (z\in\mathbb{U}), \tag{15}$$

where the powers are the principal ones.

Proof. Let the function $h(z)$ be defined by

$$h(z) = \left(\frac{D_{\sigma}^{\alpha,\beta,m} f(z)}{z^{p}}\right)^{\mu}\left(\frac{\left(D_{\sigma}^{\alpha,\beta,m} f(z)\right)'}{z^{p-1}}\right)^{\delta} - p^{\delta}(\alpha+\beta p)^{\delta+\mu}. \tag{16}$$

From the assumptions $f \in \mathscr{A}_{(p,n)}$ with $(D_{\sigma}^{\alpha,\beta,m} f(z))(D_{\sigma}^{\alpha,\beta,m} f(z))'/z^{2p-1} \neq 0$ for all $z \in \mathbb{U}$, we have that $h \in \mathscr{H}[0,n]$. By a simple manipulation, we have

$$\mathscr{J}_{p,n}^{m,\alpha,\beta,\sigma}(\mu,\delta) f(z) = p(\delta+\mu) + \frac{zh'(z)}{h(z)+p^{\delta}(\alpha+\beta p)^{\delta+\mu}}. \tag{17}$$

Now letting

$$\psi(r,s,z) = p(\delta+\mu) + \frac{s}{r+p^{\delta}(\alpha+\beta p)^{\delta+\mu}},$$
$$\Omega = \left\{w\in\mathbb{C} : \mathfrak{R}(w) < p(\delta+\mu) + \frac{nM}{M+p^{\delta}(\alpha+\beta p)^{\delta+\mu}}\right\}, \tag{18}$$

we have from (17) and (14) that

$$\psi\left(h(z), zh'(z); z\right) = \mathscr{J}_{p,n}^{m,\alpha,\beta,\sigma}(\mu,\delta) f(z) \in \Omega \quad \forall z\in\mathbb{U}. \tag{19}$$

Further, for any $\theta \in \mathbb{R}$, $K \ge nM$, and $z \in \mathbb{U}$, since $M \ge p^{\delta}(\alpha+\beta p)^{\delta+\mu}$, we also have

$$\begin{aligned}\mathfrak{R}\left\{\psi\left(Me^{i\theta}, Ke^{i\theta}; z\right)\right\} &= p(\delta+\mu) + K\mathfrak{R}\left(\frac{1}{M+e^{-i\theta}p^{\delta}(\alpha+\beta p)^{\delta+\mu}}\right) \\ &\ge p(\delta+\mu) + \frac{nM}{M+p^{\delta}(\alpha+\beta p)^{\delta+\mu}}, \quad (z\in\mathbb{U}),\end{aligned} \tag{20}$$

which shows that $\psi(Me^{i\theta}, Ke^{i\theta}; z) \notin \Omega$ for all $\theta \in \mathbb{R}$, $K \ge nM$, and $z \in \mathbb{U}$. Therefore, according to Lemma 4, we obtain $|h(z)| < M$ $(z \in \mathbb{U})$. Hence, (15) is proven. □

Theorem 7. *Let* $f(z) \in \mathscr{A}_{(p,n)}$ *with* $(D_{\sigma}^{\alpha,\beta,m}f(z))(D_{\sigma}^{\alpha,\beta,m}f(z))'/z^{2p-1} \neq 0$ *for all* $z \in \mathbb{U}$, *where* $D_{\sigma}^{\alpha,\beta,m}$ *is given by (10), and also let* $\mu, \delta \in \mathbb{R}$. *If*

$$\mathfrak{R}\left\{\mathscr{J}_{p,n}^{m,\alpha,\beta,\sigma}(\mu,\delta) f(z)\right\} > k(\mu,\delta,\alpha,\beta;\gamma), \quad (z\in\mathbb{U}), \tag{21}$$

where $\gamma \in [0, p^{\delta}(\alpha+\beta p)^{\delta+\mu})$ *and*

$$k(\mu,\delta,\alpha,\beta;\gamma) = \begin{cases} p(\delta+\mu) - \dfrac{n\gamma}{2\left[p^{\delta}(\alpha+\beta p)^{\delta+\mu}-\gamma\right]}, & \text{if } \gamma \in \left[0, \dfrac{p^{\delta}(\alpha+\beta p)^{\delta+\mu}}{2}\right] \\ p(\delta+\mu) - \dfrac{n\left[p^{\delta}(\alpha+\beta p)^{\delta+\mu}-\gamma\right]}{2\gamma}, & \text{if } \gamma \in \left[\dfrac{p^{\delta}(\alpha+\beta p)^{\delta+\mu}}{2}, p^{\delta}(\alpha+\beta p)^{\delta+\mu}\right), \end{cases} \tag{22}$$

then $f \in \mathscr{N}_{p,n}^{m,\sigma,\alpha,\beta}(\mu,\delta;\gamma)$.

Proof. Suppose that

$$h(z) = \frac{1}{p^{\delta}(\alpha+\beta p)^{\delta+\mu}-\gamma}\left[\left(\frac{D_{\sigma}^{\alpha,\beta,m} f(z)}{z^{p}}\right)^{\mu} \cdot \left(\frac{\left(D_{\sigma}^{\alpha,\beta,m} f(z)\right)'}{z^{p-1}}\right)^{\delta} - \gamma\right]. \tag{23}$$

Then, $h(z) = 1 + h_n z^n + \cdots$ is analytic in $\mathbb{U}$. It is easily seen from (23) that

$$\mathcal{J}_{p,n}^{m,\alpha,\beta,\sigma}(\mu,\delta) f(z) = p(\delta+\mu) + \frac{\left(p^{\delta}(\alpha+\beta p)^{\delta+\mu}-\gamma\right) zh'(z)}{\left(p^{\delta}(\alpha+\beta p)^{\delta+\mu}-\gamma\right) h(z)+\gamma}. \quad (24)$$

Further, since

$$\psi(r,s;z) = p(\delta+\mu) + \frac{\left(p^{\delta}(\alpha+\beta p)^{\delta+\mu}-\gamma\right) s}{\left(p^{\delta}(\alpha+\beta p)^{\delta+\mu}-\gamma\right) r+\gamma}, \quad \Omega = \{w\in\mathbb{C} : \Re(w) > k(\mu,\delta,\alpha,\beta;\gamma)\}, \quad (25)$$

it leads to

$$\psi\left(h(z), zh'(z); z\right) = \mathcal{J}_{p,n}^{m,\alpha,\beta,\sigma}(\mu,\delta) f(z) \in \Omega \quad \forall z\in\mathbb{U}. \quad (26)$$

Also, for any $x\in\mathbb{R}$, $y \le -n(1+x^2)/2$ and $z\in\mathbb{U}$, we have

$$\begin{aligned}\Re\{\psi(ix,y;z)\} &= p(\delta+\mu) + \frac{\gamma\left(p^{\delta}(\alpha+\beta p)^{\delta+\mu}-\gamma\right) y}{\left[p^{\delta}(\alpha+\beta p)^{\delta+\mu}-\gamma\right]^2 x^2+\gamma^2} \\ &\le p(\delta+\mu) - \frac{n\gamma\left[p^{\delta}(\alpha+\beta p)^{\delta+\mu}-\gamma\right]}{2}\frac{1+x^2}{\left[p^{\delta}(\alpha+\beta p)^{\delta+\mu}-\gamma\right]^2 x^2+\gamma^2} \\ &\equiv q(z) \le k(\mu,\delta,\alpha,\beta;\gamma) \\ &= \begin{cases}\lim_{x\to\infty} q(z), & \text{if } \gamma\in\left[0, \dfrac{p^{\delta}(\alpha+\beta p)^{\delta+\mu}}{2}\right] \\ q(0), & \text{if } \gamma\in\left[\dfrac{p^{\delta}(\alpha+\beta p)^{\delta+\mu}}{2}, p^{\delta}(\alpha+\beta p)^{\delta+\mu}\right);\end{cases}\end{aligned} \quad (27)$$

that is, $\psi(ix,y;z)\notin\Omega$. Finally, by Lemma 5, we obtain that $Re(h(z)) > 0$. The proof of Theorem 7 is complete. □

3. Corollaries and Consequences

We will discuss some interesting consequences of the main theorems that extend some previous results obtained in ([4, 5]).

Putting $\alpha = 1$, $\beta = 0$ in Theorems 6 and 7, we get the following corollaries.

Corollary 8. *Let $f(z)\in\mathcal{A}_{(p,n)}$ with $(D_\sigma^m f(z))(D_\sigma^m f(z))'/z^{2p-1} \ne 0$ for all $z\in\mathbb{U}$, where D_σ^m is given by (7), and also let $\mu,\delta\in\mathbb{R}$. If*

$$\Re\left\{\mu\frac{z(D_\sigma^m f(z))'}{D_\sigma^m f(z)} + \delta\left(1+\frac{z(D_\sigma^m f(z))''}{(D_\sigma^m f(z))'}\right)\right\} < p(\delta+\mu) + \frac{nM}{M+p^{\delta}}, \quad (z\in\mathbb{U}), \quad (28)$$

where $p^{\delta}\le M$, then

$$\left|\left(\frac{D_\sigma^m f(z)}{z^p}\right)^{\mu}\left(\frac{(D_\sigma^m f(z))'}{z^{p-1}}\right)^{\delta} - p^{\delta}\right| < M, \quad (z\in\mathbb{U}), \quad (29)$$

where the powers are the principal ones.

Corollary 9. *Let $f(z)\in\mathcal{A}_{(p,n)}$ with $(D_\sigma^m f(z))(D_\sigma^m f(z))'/z^{2p-1} \ne 0$ for all $z\in\mathbb{U}$, where D_σ^m is given by (7), and also let $\mu,\delta\in\mathbb{R}$. If*

$$\Re\left\{\mu\frac{z(D_\sigma^m f(z))'}{D_\sigma^m f(z)} + \delta\left(1+\frac{z(D_\sigma^m f(z))''}{(D_\sigma^m f(z))'}\right)\right\} > \varphi(\mu,\delta;\gamma), \quad (z\in\mathbb{U}), \quad (30)$$

where $\gamma\in[0,p^{\delta})$ and

$$\varphi(\mu,\delta;\gamma) = k(\mu,\delta,1,0;\gamma) = \begin{cases} p(\delta+\mu) - \dfrac{n\gamma}{2[p^{\delta}-\gamma]}, & \text{if } \gamma\in\left[0,\dfrac{p^{\delta}}{2}\right] \\ p(\delta+\mu) - \dfrac{n[p^{\delta}-\gamma]}{2\gamma}, & \text{if } \gamma\in\left[\dfrac{p^{\delta}}{2}, p^{\delta}\right), \end{cases} \quad (31)$$

then

$$\Re\left\{\left(\frac{D_\sigma^m f(z)}{z^p}\right)^{\mu}\left(\frac{(D_\sigma^m f(z))'}{z^{p-1}}\right)^{\delta}\right\} > \gamma, \quad (z\in\mathbb{U}), \quad (32)$$

where the powers are the principal ones.

Taking $\mu = 1-\lambda_1$ and $\delta = \lambda_1$ in Corollaries 8 and 9, respectively, we obtain the following special cases.

Corollary 10. *Let $f(z)\in\mathcal{A}_{(p,n)}$ with $(D_\sigma^m f(z))(D_\sigma^m f(z))'/z^{2p-1} \ne 0$ for all $z\in\mathbb{U}$, where D_σ^m is given by (7), and also let $\lambda_1\in\mathbb{R}$. If*

$$\Re\left\{(1-\lambda_1)\frac{z(D_\sigma^m f(z))'}{D_\sigma^m f(z)} + \lambda_1\left(1+\frac{z(D_\sigma^m f(z))''}{(D_\sigma^m f(z))'}\right)\right\} < p + \frac{nM}{M+p^{\lambda_1}}, \quad (z\in\mathbb{U}), \quad (33)$$

where $p^{\lambda_1}\le M$, then

$$\left|\left(\frac{D_\sigma^m f(z)}{z^p}\right)^{1-\lambda_1}\left(\frac{(D_\sigma^m f(z))'}{z^{p-1}}\right)^{\lambda_1} - p^{\lambda_1}\right| < M, \quad (z\in\mathbb{U}), \quad (34)$$

where the powers are the principal ones.

Corollary 11. *Let* $f(z) \in \mathscr{A}_{(p,n)}$ *with* $(D_\sigma^m f(z))(D_\sigma^m f(z))'/z^{2p-1} \neq 0$ *for all* $z \in \mathbb{U}$, *where* D_σ^m *is given by (7), and also let* $\lambda_1 \in \mathbb{R}$. *If*

$$\Re\left\{(1-\lambda_1)\frac{z(D_\sigma^m f(z))'}{D_\sigma^m f(z)} + \lambda_1\left(1+\frac{z(D_\sigma^m f(z))''}{(D_\sigma^m f(z))'}\right)\right\} > \chi(\lambda_1;\gamma), \quad (z \in \mathbb{U}), \tag{35}$$

where $\gamma \in [0, p^{\lambda_1})$ *and*

$$\chi(\lambda_1;\gamma) = k(1-\lambda_1, \lambda_1, 1, 0; \gamma) = \begin{cases} p - \dfrac{n\gamma}{2[p^{\lambda_1}-\gamma]}, & \text{if } \gamma \in \left[0, \dfrac{p^{\lambda_1}}{2}\right] \\ p - \dfrac{n[p^{\lambda_1}-\gamma]}{2\gamma}, & \text{if } \gamma \in \left[\dfrac{p\lambda_1}{2}, p^{\lambda_1}\right), \end{cases} \tag{36}$$

then

$$\Re\left\{\left(\frac{D_\sigma^m f(z)}{z^p}\right)^{1-\lambda_1}\left(\frac{(D_\sigma^m f(z))'}{z^{p-1}}\right)^{\lambda_1}\right\} > \gamma, \quad (z \in \mathbb{U}), \tag{37}$$

where the powers are the principal ones.

Next, upon taking $\alpha = 0$, $\beta = 1$ in Theorems 6 and 7, we obtain the following results.

Corollary 12. *Let* $f(z) \in \mathscr{A}_{(p,n)}$ *with* $(D_\sigma^m f(z))'[(D_\sigma^m f(z))' + z(D_\sigma^m f(z))'']/z^{2(p-1)} \neq 0$ *for all* $z \in \mathbb{U}$, *where* D_σ^m *is given by (7), and also let* $\mu, \delta \in \mathbb{R}$. *If*

$$\Re\left\{\mathscr{I}_{p,n}^{m,0,1,\sigma}(\mu,\delta) f(z)\right\} < p(\delta+\mu) + \frac{nM}{M+p^{2\delta+\mu}}, \quad (z \in \mathbb{U}), \tag{38}$$

where $p^{2\delta+\mu} \le M$, *then*

$$\left|\left(\frac{(D_\sigma^m f(z))'}{z^{p-1}}\right)^{\mu}\left(\frac{(D_\sigma^m f(z))' + z(D_\sigma^m f(z))''}{z^{p-1}}\right)^{\delta} - p^{2\delta+\mu}\right| < M, \quad (z \in \mathbb{U}), \tag{39}$$

where the powers are the principal ones.

Corollary 13. *Let* $f(z) \in \mathscr{A}_{(p,n)}$ *with* $(D_\sigma^m f(z))'[(D_\sigma^m f(z))' + z(D_\sigma^m f(z))'']/z^{2(p-1)} \neq 0$ *for all* $z \in \mathbb{U}$, *where* D_σ^m *is given by (7), and also let* $\mu, \delta \in \mathbb{R}$. *If*

$$\Re\left\{\mathscr{I}_{p,n}^{m,0,1,\sigma}(\mu,\delta) f(z)\right\} > \phi(\mu,\delta;\gamma), \quad (z \in \mathbb{U}), \tag{40}$$

where $\gamma \in [0, p^{2\delta+\mu})$ *and*

$$\phi(\mu,\delta;\gamma) = k(\mu,\delta,0,1;\gamma) = \begin{cases} p(\delta+\mu) - \dfrac{n\gamma}{2[p^{2\delta+\mu}-\gamma]}, & \text{if } \gamma \in \left[0, \dfrac{p^{2\delta+\mu}}{2}\right] \\ p(\delta+\mu) - \dfrac{n[p^{2\delta+\mu}-\gamma]}{2\gamma}, & \text{if } \gamma \in \left[\dfrac{p^{2\delta+\mu}}{2}, p^{2\delta+\mu}\right), \end{cases} \tag{41}$$

then

$$\Re\left\{\left(\frac{(D_\sigma^m f(z))'}{z^{p-1}}\right)^{\mu} \cdot \left(\frac{(D_\sigma^m f(z))' + z(D_\sigma^m f(z))''}{z^{p-1}}\right)^{\delta}\right\} > \gamma, \quad (z \in \mathbb{U}), \tag{42}$$

where the powers are the principal ones.

Taking $\mu = 1-\lambda_1$ and $\delta = \lambda_1$ in Corollaries 12 and 13, respectively, we obtain the following special cases.

Corollary 14. *Let* $f(z) \in \mathscr{A}_{(p,n)}$ *with* $(D_\sigma^m f(z))'[(D_\sigma^m f(z))' + z(D_\sigma^m f(z))'']/z^{2(p-1)} \neq 0$ *for all* $z \in \mathbb{U}$, *where* D_σ^m *is given by (7), and also let* $\lambda_1 \in \mathbb{R}$. *If*

$$\Re\left\{\mathscr{I}_{p,n}^{m,0,1,\sigma}(1-\lambda_1,\lambda_1) f(z)\right\} < p + \frac{nM}{M+p^{\lambda_1+1}}, \quad (z \in \mathbb{U}), \tag{43}$$

where $p^{\lambda_1+1} \le M$, *then*

$$\left|\left(\frac{(D_\sigma^m f(z))'}{z^{p-1}}\right)^{1-\lambda_1} \cdot \left(\frac{(D_\sigma^m f(z))' + z(D_\sigma^m f(z))''}{z^{p-1}}\right)^{\lambda_1} - p^{\lambda_1+1}\right| < M, \quad (z \in \mathbb{U}), \tag{44}$$

where the powers are the principal ones.

Corollary 15. *Let* $f(z) \in \mathscr{A}_{(p,n)}$ *with* $(D_\sigma^m f(z))'[(D_\sigma^m f(z))' + z(D_\sigma^m f(z))'']/z^{2(p-1)} \neq 0$ *for all* $z \in \mathbb{U}$, *where* D_σ^m *is given by (7), and also let* $\lambda_1 \in \mathbb{R}$. *If*

$$\Re\left\{\mathscr{I}_{p,n}^{m,0,1,\sigma}(1-\lambda_1,\lambda_1) f(z)\right\} > \psi(\lambda_1;\gamma), \quad (z \in \mathbb{U}), \tag{45}$$

where $\gamma \in [0, p^{\lambda_1+1})$ *and*

$$\psi(\lambda_1;\gamma) = k(1-\lambda_1,\lambda_1,0,1;\gamma) = \begin{cases} p(\delta+\mu) - \dfrac{n\gamma}{2[p^{\lambda_1+1}-\gamma]}, & \text{if } \gamma \in \left[0, \dfrac{p^{\lambda_1+1}}{2}\right] \\ p(\delta+\mu) - \dfrac{n[p^{2\delta+\mu}-\gamma]}{2\gamma}, & \text{if } \gamma \in \left[\dfrac{p^{\lambda_1+1}}{2}, p^{\lambda_1+1}\right), \end{cases} \tag{46}$$

then

$$\Re\left\{\left(\frac{(D_\sigma^m f(z))'}{z^{p-1}}\right)^{1-\lambda_1} \cdot \left(\frac{(D_\sigma^m f(z))' + z(D_\sigma^m f(z))''}{z^{p-1}}\right)^{\lambda_1}\right\} > \gamma, \quad (z\in\mathbb{U}), \tag{47}$$

where the powers are the principal ones.

In the next result, we will find the relation between $\mathscr{M}_{p,n}^{m,\alpha,\beta,\sigma}(\lambda_1;\gamma)$ and $\mathscr{M}_{p,n}^{m,\alpha,\beta,\sigma}(1-\lambda_1,\lambda_1;\gamma)$. For this purpose, taking $\mu = 1-\lambda_1$ and $\delta = \lambda_1$ in Theorem 7, we obtain the following result.

Corollary 16. *Let $f(z)\in\mathscr{A}_{(p,n)}$ with $(D_\sigma^{\alpha,\beta,m}f(z))(D_\sigma^{\alpha,\beta,m}f(z))'/z^{2p-1} \neq 0$ for all $z\in\mathbb{U}$, where $D_\sigma^{\alpha,\beta,m}$ is given by (10), and also let $\lambda_1\in\mathbb{R}$. If*

$$f(z) \in \mathscr{M}_{p,n}^{m,\alpha,\beta,\sigma}(\lambda_1;\varrho(\lambda_1,\alpha,\beta;\gamma)), \tag{48}$$

where $\gamma\in[0,p^{\lambda_1}(\alpha+\beta p))$ and

$$\varrho(\lambda_1,\alpha,\beta;\gamma) = k(1-\lambda_1,\lambda_1,\alpha,\beta;\gamma) = \begin{cases} p - \dfrac{n\gamma}{2[p^{\lambda_1}(\alpha+\beta p)-\gamma]}, & \text{if } \gamma\in\left[0,\dfrac{p^{\lambda_1}(\alpha+\beta p)}{2}\right] \\ p - \dfrac{n[p^{\lambda_1}(\alpha+\beta p)-\gamma]}{2\gamma}, & \text{if } \gamma\in\left[\dfrac{p^{\lambda_1}(\alpha+\beta p)}{2}, p^{\lambda_1}[(\alpha+\beta p)\right), \end{cases} \tag{49}$$

then $f(z)\in\mathscr{N}_{p,n}^{m,\alpha,\beta,\sigma}(1-\lambda_1,\lambda_1;\gamma)$.

Taking $\lambda_1 = 0$ and $n=1$ in the above corollary, we get the next special result.

Corollary 17. *Let $f(z)\in\mathscr{A}_{(p)}$ with $(D_\sigma^{\alpha,\beta,m}f(z))(D_\sigma^{\alpha,\beta,m}f(z))'/z^{2p-1}\neq 0$ for all $z\in\mathbb{U}$, where $D_\sigma^{\alpha,\beta,m}$ is given by (10), and also let $\lambda_1\in\mathbb{R}$. If*

$$f(z)\in\mathscr{M}_p^{m,\alpha,\beta,\sigma}(\varrho(\alpha,\beta;\gamma)), \tag{50}$$

where $\gamma\in[0,\alpha+\beta p)$ and

$$\varrho(\alpha,\beta;\gamma) = k(1,0,\alpha,\beta;\gamma) = \begin{cases} p - \dfrac{\gamma}{2[(\alpha+\beta p)-\gamma]}, & \text{if } \gamma\in\left[0,\dfrac{\alpha+\beta p}{2}\right] \\ p - \dfrac{[(\alpha+\beta p)-\gamma]}{2\gamma}, & \text{if } \gamma\in\left[\dfrac{\alpha+\beta p}{2},(\alpha+\beta p)\right), \end{cases} \tag{51}$$

then $f(z)\in\mathscr{N}_p^{m,\alpha,\beta,\sigma}(1,0;\gamma)$.

Again, for the special cases of μ and δ, Theorems 6 and 7 reduce at once to some results obtained by [4, 5].

Remark 18. Taking $p=1$ and $m=0$ in (7) and $\alpha=1$ and $\beta=0$ in (10), we get a known result obtained by Irmak et al. [5].

Remark 19. Taking $m=0$ in (7) and $\alpha=1-\beta$ in (10), we get a known result obtained by Goswami et al. [4].

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

Both authors agreed with the contents of the paper.

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References

[1] S. P. Goyal, O. Singh, and P. Goswami, "Some relations between certain classes of analytic multivalent functions involving generalized salagean operator," *Sohag Journal of Mathematics*, vol. 1, no. 1, pp. 27–32, 2014.

[2] G. S. Salagean, "Subclasses of univalent functions," in *Complex Analysis—Fifth Romanian-Finnish Seminar, Part 1: Proceedings of the Seminar Held in Bucharest 1981*, vol. 1013 of *Lecture Notes in Mathematics*, pp. 362–372, Springer, Berlin, Germany, 1983.

[3] S. S. Miller and P. T. Mocanu, *Differential Subordinations. Theory and Applications,*, vol. 225 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 2000.

[4] P. Goswami, T. Bulboacă, and S. Bansal, "Some relations between certain classes of analytic functions," *Journal of Classical Analysis*, vol. 1, no. 2, pp. 157–173, 2012.

[5] H. Irmak, T. Bulboacă, and N. Tuneski, "Some relations between certain classes consisting of α-convex type and Bazilević type functions," *Applied Mathematics Letters*, vol. 24, no. 12, pp. 2010–2014, 2011.