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Research Article

An Obstacle Problem for Noncoercive Operators

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We study the obstacle problem for second order nonlinear equations whose model appears in the stationary diffusion-convection problem. We assume that the growth coefficient of the convection term lies in the Marcinkiewicz space weak-L^N.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with C^1-boundary, $N > 2$, and let $\mathscr{A} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a Caratheòdory function; that is,

$$\begin{aligned} &x \longrightarrow \mathscr{A}(x,\xi) \text{ is measurable for any } \xi \in \mathbb{R}^N; \\ &\xi \longrightarrow \mathscr{A}(x,\xi) \text{ is continuous for almost every } x \in \Omega. \end{aligned} \tag{1}$$

We assume that there exist $0 < \alpha < \beta$ such that for almost every $x \in \Omega$ we have

$$|\mathscr{A}(x,\xi)| \leqslant \beta|\xi| + \varphi(x) \quad \text{with } \varphi \in L^2(\Omega), \tag{2}$$

$$\alpha|\xi-\eta|^2 \leqslant \langle \mathscr{A}(x,\xi) - \mathscr{A}(x,\eta), \xi - \eta\rangle \quad \text{(strong monotonicity)} \tag{3}$$

for any vectors ξ and η in $\mathbb{R}^N$. Moreover, we assume that $\mathscr{B} : \Omega \times \mathbb{R} \to \mathbb{R}^N$ is a Caratheòdory function verifying the following properties.

(i) There exists a nonnegative function $b : \Omega \to \mathbb{R}_+$ in the Lorentz space $b \in L^{N,\infty}(\Omega)$ such that

$$|\mathscr{B}(x,s) - \mathscr{B}(x,t)| \leqslant b(x)|s-t|, \tag{4}$$

for almost every $x \in \Omega$ and for any $s, t \in \mathbb{R}$.

(ii) Consider

$$b_0(x) := \mathscr{B}(x,0) \in L^2(\Omega). \tag{5}$$

The space $L^{N,\infty}$ is also known as the Marcinkiewicz space weak-L^N.

Let $g \in W^{1,2}(\Omega)$ and let $\psi : \Omega \to [-\infty, +\infty]$. We define

$$\mathscr{K}_{\psi,g} = \left\{ v \in g + W_0^{1,2}(\Omega) : v \geqslant \psi \text{ a.e. in } \Omega \right\}. \tag{6}$$

Definition 1. Given $F \in L^2(\Omega, \mathbb{R}^N)$, one says that $u \in \mathscr{K}_{\psi,g}$ is a solution of the obstacle problem $\mathrm{OP}(F,\psi,g)$ if

$$\begin{aligned} &\int_\Omega \langle \mathscr{A}(x,\nabla u) + \mathscr{B}(x,u), \nabla(v-u)\rangle\, dx \\ &\quad \geqslant \int_\Omega \langle F, \nabla(v-u)\rangle\, dx \end{aligned} \tag{7}$$

for every $v \in \mathscr{K}_{\psi,g}$.

For a classical treatment of obstacle problem we refer to [1, 2]. See also [3, 4] and references therein.

Under assumptions (2) and (4) the left hand side of (7) is finite by the Sobolev embedding theorem.

We point out that assumptions (1)–(5) do not guarantee that the operator

$$\mathscr{A}(x,s,\xi) = \mathscr{A}(x,\xi) + \mathscr{B}(x,s) \tag{8}$$

for any $\xi \in \mathbb{R}^N$, $s \in \mathbb{R}$, and almost every $x \in \Omega$ is coercive and monotone.

The aim of this paper is to establish existence and uniqueness of solutions of $\mathrm{OP}(F,\psi,g)$ in the sense of Definition 1. Our first result is the following.

Theorem 2. *Assume that assumptions* (3) *and* (4) *are verified, and let*

$$b \in L^{N,\infty}(\Omega). \tag{9}$$

Then, there exists at most one solution $u \in \mathcal{K}_{\psi,g}$ *of problem* (7).

We also prove the following.

Theorem 3. *Let assumptions* (1)–(5) *be verified and let* $\mathcal{K}_{\psi,g} \neq \emptyset$. *Assume that*

$$\mathrm{dist}_{\mathrm{L}^{\mathrm{N},\infty}}(\mathrm{b}, \mathrm{L}^{\infty}) < \frac{\alpha}{4\mathrm{S}_2}. \tag{10}$$

Then, for every $F \in L^2(\Omega, \mathbb{R}^N)$, *problem* (7) *admits a solution* $u \in \mathcal{K}_{\psi,g}$. *Here* S_2 *is the Sobolev constant.*

We remark that L^∞ is not dense in $L^{N,\infty}$. Moreover, condition (10) does not give any smallness control on the norm of b in $L^{N,\infty}$ (see Section 2.1). This fact is very relevant when we have to prove a priori estimates for the solutions of $\mathrm{OP}(F, \psi, g)$. Indeed, in order to prove our results we follow a classical approach. First, we construct a coercive and monotone operator. Then we reduce the existence to applying a fixed point theorem.

Theorem 3 is new also in case of equations. In [5–8], the authors considered operators with a lower order term having the growth coefficient b in spaces in which the bounded functions are dense.

A condition similar to (10) has been used in [9] for proving the existence of solutions to linear equations. In that paper, an example shows that, in general, condition (10) cannot be dropped in order to achieve existence of solutions. Regularity results for solutions have been obtained in [10].

2. Preliminary Results

2.1. Some Functional Spaces. Let Ω be a bounded domain in $\mathbb{R}^N$. For a measurable $E \subset \Omega$, we denote by $|E|$ its Lebesgue measure. For a measurable function $f : \Omega \to \mathbb{R}$ we denote by

$$\mu_f(\lambda) = |\{x \in \Omega : |f(x)| > \lambda\}|, \quad (\lambda \geqslant 0) \tag{11}$$

its distribution function and by

$$f^*(t) = \inf\{\lambda : \mu_f(\lambda) \leqslant t\} \tag{12}$$

its decreasing rearrangement; see [11]. Clearly, $f^*(t) = 0$ if $t > |\Omega|$. For $0 < p < \infty$ and $0 < q < \infty$, we consider the quantity

$$\|f\|_{p,q} = \left\{\int_0^\infty \left[t^{1/p} f^*(t)\right]^q \frac{dt}{t}\right\}^{1/q} \tag{13}$$

and for $q = \infty$ the obvious modification

$$\|f\|_{p,\infty} = \sup_{0<t<\infty}\left\{t^{1/p} f^*(t)\right\}. \tag{14}$$

The Lorentz space $L^{p,q} = L^{p,q}(\Omega)$ consists of all measurable functions f satisfying $\|f\|_{p,q} < \infty$. The space $L^{p,\infty}$ is also known as Marcinkiewicz space M^p or weak-L^p. The quantity $\|\ \|_{p,q}$ is equivalent to a norm which makes $L^{p,q}$ a Banach space; see [11, 12]. For $p = q$, the space $L^{p,p}$ coincides with the usual Lebesgue L^p space. Moreover,

$$\begin{gathered} 1 < p_1 < p_2 < \infty,\ 1 \leqslant q_1,\ q_2 \leqslant \infty \Longrightarrow L^{p_1,q_1} \supset L^{p_2,q_2}, \\ 1 < p < \infty,\ 1 \leqslant q_1 < q_2 \leqslant \infty \Longrightarrow L^{p,q_1} \subset L^{p,q_2} \end{gathered} \tag{15}$$

with continuous injections. In particular, if $1 < r < p < \infty$,

$$L^p \subset L^{p,\infty} \subset L^r. \tag{16}$$

The following Hölder-type inequality holds. For $1 < p_i < \infty$ and $1 \leqslant q_i \leqslant \infty$, $i = 1, \ldots, n$, if

$$\sum_{i=1}^n \frac{1}{p_i} = 1 = \sum_{i=1}^n \frac{1}{q_i}, \tag{17}$$

then

$$\int_\Omega \left|\prod_{i=1}^n f_i(x)\right| dx \leqslant \prod_{i=1}^n \|f_i\|_{p_i,q_i}. \tag{18}$$

See [9]. An elementary but often useful property is expressed by the equality

$$\left\||f|^\alpha\right\|_{p,q} = \||f|\|^\alpha_{\alpha p,\alpha q} \tag{19}$$

which holds for $\alpha > 0$.

We note the equality

$$\|\chi_E\|_{p,q} = \left(\frac{p}{q}\right)^{1/q} |E|^{1/p} \tag{20}$$

for every measurable $E \subset \Omega$. Here, for $q = \infty$ we assume $(p/q)^{1/q} = 1$.

We remark that, for any $p \in]1, \infty[$, L^∞ is not dense in $L^{p,\infty}$. We consider the distance of a given $f \in L^{p,\infty}$ to L^∞:

$$\mathrm{dist}_{L^{p,\infty}}(f, L^\infty) = \inf_{g \in L^\infty} \|f - g\|_{p,\infty}. \tag{21}$$

To find a formula for the distance, we consider the truncation operator. For $k > 0$, we set

$$T_k(y) = \frac{y}{|y|}\min\{|y|, k\}. \tag{22}$$

Then

$$\mathrm{dist}_{L^{p,\infty}}(f, L^\infty) = \lim_{k \to \infty} \|f - T_k f\|_{p,\infty}. \tag{23}$$

Indeed, $\forall g \in L^\infty$ and $\forall k \geqslant \|g\|_\infty$, we have, for almost every $x \in \Omega$,

$$|f(x) - g(x)| \geqslant |f(x) - T_k f(x)|. \tag{24}$$

For other comments on the distance to L^∞ and some applications, we refer to [13].

Example 4. Let Ω be the unit ball of $\mathbb{R}^N$ and $p \in]1, \infty[$. The function

$$f(x) = |x|^{-N/p} \tag{25}$$

belongs to $L^{p,\infty}$. Setting $\omega_N = |\Omega|$, for $k > 0$ and $\lambda > 0$, we compute

$$\mu_{f-T_k f}(\lambda) = \omega_N(\lambda + k)^{-p}, \tag{26}$$

$$(f - T_k f) * (t) = \begin{cases} \left(\dfrac{t}{\omega_N}\right)^{-1/p} - k, & 0 < t < \omega_N\, k^{-p} \\ 0, & t \geqslant \omega_N\, k^{-p}. \end{cases} \tag{27}$$

Hence

$$\|f - T_k f\|_{p,\infty} = \omega_N^{1/p} \tag{28}$$

does not depend on k.

On the contrary, for all $1 \leqslant q < \infty$, starting with the definition of $\|\,\|_{p,q}$, a simple application of Lebesgue dominated convergence theorem shows that L^∞ is dense in $L^{p,q}$. Hence, for $1 \leqslant q < \infty$, $L^{p,q}$, and in particular the Lebesgue space L^p, is contained in the closure of L^∞ in $L^{p,\infty}$. The closure of L^∞ coincides with the closure of C_0^∞. The elements of the closure can be characterized by the condition of having absolutely continuous norm; see [11, Section 1.3].

Fundamental to us will be the Sobolev embedding theorem in Lorentz spaces (see [12]; see also [14, 15]).

Theorem 5. *Let one assume that $1 < p < N$, $1 \leqslant q \leqslant p$; then every function $g \in W_0^{1,1}(\Omega)$ verifying $|\nabla g| \in L^{p,q}$ actually belongs to $L^{p^*,q}$, where $p^* = Np/(N-p)$, and*

$$\|g\|_{p^*,q} \leqslant S_p \|\nabla g\|_{p,q}, \tag{29}$$

where $S_p = c(N)(p/(N-p))$.

2.2. Monotone Operators. Let X be a reflexive Banach space with dual X^*. Let $\langle\cdot,\cdot\rangle$ denote the pairing between X^* and X. Let $\mathbb{K} \subset X$ be a closed convex set.

Definition 6. A mapping $A : \mathbb{K} \to X^*$ is called monotone if

$$\langle Au - Av, u - v\rangle \geqslant 0 \quad \forall u, v \in \mathbb{K}. \tag{30}$$

The monotone mapping A is called strictly monotone if

$$\langle Au - Av, u - v\rangle = 0 \quad \text{implies } u \equiv v. \tag{31}$$

Definition 7. $A : \mathbb{K} \to X^*$ is called coercive on $\mathbb{K}$ if there exists an element $\varphi \in \mathbb{K}$ such that

$$\frac{\langle Au - A\varphi, u - \varphi\rangle}{\|u - \varphi\|} \longrightarrow +\infty \quad \text{as } \|u\| \longrightarrow +\infty \text{ for any } u \in \mathbb{K}. \tag{32}$$

The following existence and uniqueness result is contained in [1] (see [1], Cap. III, Theorem 1.7 and Corollary 1.8).

Theorem 8. *Let $\mathbb{K} \neq \emptyset$ and let $A : \mathbb{K} \to X^*$ be strictly monotone, coercive, and continuous on finite dimensional subspaces. Then, there exists*

$$u \in \mathbb{K} : \langle Au, v - u\rangle \geqslant 0 \quad \textit{for any } v \in \mathbb{K}. \tag{33}$$

Such a solution is unique.

2.3. The Leray-Schauder Theorem. We will use the well-known Leray-Schauder fixed point theorem in the following form (see [16, Theorem 11.3, page 280]).

A continuous mapping between two Banach spaces is called compact if the images of bounded sets are precompact.

Theorem 9. *Let $\mathcal{F}$ be a compact mapping of a Banach space X into itself, and suppose there exists a constant K such that $\|x\|_X < K$ for all $x \in X$ and $t \in [0,1]$ satisfying $x = t\mathcal{F}(x)$. Then, $\mathcal{F}$ has a fixed point.*

3. Uniqueness of Solutions: Proof of Theorem 2

Proof of Theorem 2. Suppose that $u_1, u_2 \in \mathcal{K}_{\psi,g}$ verify (7); that is, suppose that

$$\begin{aligned} &\int_\Omega \langle \mathcal{A}(x, \nabla u_1) + \mathcal{B}(x, u_1), \nabla(v - u_1)\rangle\, dx \\ &\quad \geqslant \int_\Omega \langle F, \nabla(v - u_1)\rangle\, dx, \end{aligned} \tag{34}$$

$$\begin{aligned} &\int_\Omega \langle \mathcal{A}(x, \nabla u_2) + \mathcal{B}(x, u_2), \nabla(v - u_2)\rangle\, dx \\ &\quad \geqslant \int_\Omega \langle F, \nabla(v - u_2)\rangle\, dx \end{aligned} \tag{35}$$

$\forall v \in \mathcal{K}_{\psi,g}$. We will prove that $u = u_1 - u_2 \equiv 0$ a.e. in Ω. To this aim we use as test functions $v_\varepsilon = T_\varepsilon(u_2 - u_1) + u_1$ in (34) and $w_\varepsilon = T_\varepsilon(u_1 - u_2) + u_2$ in (35) for a number $\varepsilon > 0$. Those functions are admissible since v_ε and w_ε belong to $g + W_0^{1,2}(\Omega)$ and $v_\varepsilon \geqslant \psi$, $w_\varepsilon \geqslant \psi$ a.e. on Ω. Observing that $\nabla T_\varepsilon(u_1 - u_2) = -\nabla T_\varepsilon(u_2 - u_1)$, we obtain

$$\begin{aligned} &\int_\Omega \langle \mathcal{A}(x, \nabla u_1) - \mathcal{A}(x, \nabla u_2) + \mathcal{B}(x, u_1) - \mathcal{B}(x, u_2), \\ &\quad \nabla T_\varepsilon(u_1 - u_2)\rangle\, dx \leqslant 0. \end{aligned} \tag{36}$$

Now we set

$$\Omega_\varepsilon = \{x \in \Omega : |u| > \varepsilon\}. \tag{37}$$

We have, using (3), (36), and (4),

$$\begin{aligned}\alpha \int_\Omega \left|\nabla T_\varepsilon(u)\right|^2 dx &= \alpha \int_{\Omega\setminus\Omega_\varepsilon} \left|\nabla(u_1-u_2)\right|^2 dx \\ &\leqslant \int_\Omega \langle \mathscr{A}(x,\nabla u_1) - \mathscr{A}(x,\nabla u_2), \nabla T_\varepsilon(u_1-u_2)\rangle dx \\ &\leqslant \int_\Omega b(x)\left|u_1-u_2\right|\left|\nabla T_\varepsilon(u_1-u_2)\right| dx \\ &\leqslant \varepsilon \int_{0<|u|\leqslant\varepsilon} b(x)\left|\nabla T_\varepsilon(u)\right| dx \\ &\leqslant \varepsilon\left(\int_{0<|u|\leqslant\varepsilon} |b(x)|^2 dx\right)^{1/2}\left(\int_\Omega \left|\nabla T_\varepsilon(u)\right|^2 dx\right)^{1/2}.\end{aligned} \tag{38}$$

Then we have

$$\alpha^2 \|\nabla T_\varepsilon(u)\|_2^2 \leqslant \varepsilon^2 \int_{0<|u|\leqslant\varepsilon} |b(x)|^2 dx. \tag{39}$$

Now, let $0<\varepsilon<\eta$, so that

$$\varepsilon^2\left|\Omega_\eta\right| = \int_{|u|>\eta} \left|T_\varepsilon(u)\right|^2 dx \leqslant c\int_\Omega \left|\nabla T_\varepsilon(u)\right|^2 dx, \tag{40}$$

where $c = c(N)$. Combining (39) and the last inequality we obtain

$$\alpha^2\left|\Omega_\eta\right| \leqslant c\int_{0<|u|\leqslant\varepsilon} |b(x)|^2 dx. \tag{41}$$

Letting $\varepsilon \to 0^+$ we obtain $|\Omega_\eta| = 0$, and then, by the arbitrariness of $\eta > 0$, we can conclude that $u(x) = u_1(x) - u_2(x) \equiv 0$ for almost every $x \in \Omega$. □

4. Existence of Solutions: Proof of Theorem 3

As $\mathscr{K}_{\psi,g} \neq \emptyset$, it is not restrictive to assume $g \geqslant \psi$ a.e. in Ω. Moreover, let us observe that if assumption (10) holds true then by (23) there exists a positive constant $M = M(\alpha, b, N)$ such that

$$\left\|b - T_M b\right\|_{N,\infty} < \frac{\alpha}{4S_2}. \tag{42}$$

Let us fix such a value of M.

Here below we denote

$$\vartheta(x) = \frac{T_M b(x)}{b(x)}, \tag{43}$$

where as above T_M is the truncation operator at level M.

Let $\mathscr{A} : \Omega\times\mathbb{R}^N \to \mathbb{R}^N$ and $\mathscr{B} : \Omega\times\mathbb{R} \to \mathbb{R}^N$ be Carathèodory functions satisfying (1)–(4). Let us consider the operator $\mathscr{A} : W^{1,2}(\Omega) \to (W^{1,2}(\Omega))^*$ defined by

$$\langle \mathscr{A}u, v\rangle = \int_\Omega \langle \mathscr{A}(x,\nabla u) + (1-\theta(x))\mathscr{B}(x,u), \nabla v\rangle dx, \qquad u, v \in W^{1,2}. \tag{44}$$

The operator $\mathscr{A}$ is strictly monotone and coercive on $\mathscr{K}_{\psi,g}(\Omega)$. In fact for $u, v \in \mathscr{K}_{\psi,g}(\Omega)$ we have

$$\begin{aligned}\langle \mathscr{A}u - \mathscr{A}v, u-v\rangle &= \int_\Omega \langle \mathscr{A}(x,\nabla u) - \mathscr{A}(x,\nabla v) + (1-\theta(x))(\mathscr{B}(x,u) \\ &\qquad - \mathscr{B}(x,v)), \nabla(u-v)\rangle dx \\ &\geqslant \alpha\int_\Omega |\nabla u - \nabla v|^2 dx \\ &\quad - \int_\Omega \left|b(x) - T_M b(x)\right| |u-v| \left|\nabla(u-v)\right| dx \\ &\geqslant \alpha\|\nabla(u-v)\|_2^2 - S_2\left\|b - T_M b\right\|_{L^{N,\infty}}\|\nabla(u-v)\|_2^2 \\ &= \left(\alpha - S_2\left\|b - T_M b\right\|_{L^{N,\infty}}\right)\|\nabla(u-v)\|_2^2.\end{aligned} \tag{45}$$

Then, by (42), we have $(\alpha - S_2\|b - T_M b\|_{L^{N,\infty}}) \geqslant \alpha/2 > 0$.

The following technical lemma will be useful in the sequel. We shall follow closely the proof of Lemma 4.1 in [7]. We include some details for the sake of completeness.

Lemma 10. *Let one assume* (1)–(5), $F \in L^2(\Omega, \mathbb{R}^n)$, $g \in W^{1,2}(\Omega)$, $\psi \leqslant g$, *and let* $0 < t \leqslant 1$. *Assume that* $u \in W^{1,2}$ *is such that* $u/t \in \mathscr{K}_{\psi,g}(\Omega)$ *and verifies*

$$\begin{aligned}&\int_\Omega \left\langle \mathscr{A}\left(x, \frac{\nabla u}{t}\right) + (1-\theta(x))\mathscr{B}\left(x, \frac{u}{t}\right), \nabla\left(v - \frac{u}{t}\right)\right\rangle dx \\ &\quad\geqslant \int_\Omega \left\langle F - \theta(x)\mathscr{B}(x,u), \nabla\left(v - \frac{u}{t}\right)\right\rangle dx\end{aligned} \tag{46}$$

for all $v \in \mathscr{K}_{\psi,g}$. *Then,*

$$\begin{aligned}&\left\|\log\left(1 + \left|\frac{u}{t} - g\right|\right)\right\|_{2^*} \\ &\quad\leqslant C\left(\|F\|_2 + \left\|b_0\right\|_2 + \|b\|_2 + \|b\|_{N,\infty}\|g\|_{2^*,2} + \|\nabla g\|_2 + \|\varphi\|_2\right),\end{aligned} \tag{47}$$

where C *depends only on* α, β, *and* N.

Proof. Taking $v = u/t - (u/t - g)/(1 + |u/t - g|) \in \mathscr{K}_{\psi,g}$ as a test function in (46), we obtain

$$\begin{aligned}&\int_\Omega \left\langle \mathscr{A}\left(x, \frac{\nabla u}{t}\right) + (1-\theta(x))\mathscr{B}\left(x, \frac{u}{t}\right), \right. \\ &\qquad \left. \nabla\left(\frac{u/t - g}{1 + |u/t - g|}\right)\right\rangle dx \\ &\quad\leqslant \int_\Omega \left\langle F - \theta(x)\mathscr{B}(x,u), \nabla\left(\frac{u/t - g}{1 + |u/t - g|}\right)\right\rangle dx.\end{aligned} \tag{48}$$

By assumptions (2)–(5) we have

$$\begin{aligned}
\alpha \int_\Omega \frac{|\nabla (u/t-g)|^2}{(1+|u/t-g|)^2}\,dx
&\leqslant 2\int_\Omega \frac{\langle \mathcal{A}(x,\nabla u/t),\nabla (u/t-g)\rangle}{(1+|u/t-g|)^2}\,dx\\
&\quad + 2\int_\Omega \frac{\langle \mathcal{A}(x,\nabla u/t),\nabla g\rangle}{(1+|u/t-g|)^2}\,dx\\
&\quad + 2\int_\Omega |\varphi(x)|\frac{|\nabla u/t|}{(1+|u/t-g|)^2}\,dx + 2\alpha\|\nabla g\|_2^2\\
&\leqslant 2\int_\Omega \left\langle \mathcal{A}(x,\nabla u/t),\frac{\nabla (u/t-g)}{(1+|u/t-g|)^2}\right\rangle dx\\
&\quad + \frac{\alpha}{2}\int_\Omega \frac{|\nabla (u/t-g)|^2}{(1+|u/t-g|)^2}\,dx + c\left(\|\nabla g\|_2^2+\|\varphi\|_2^2\right)
\end{aligned} \tag{49}$$

with $c=c(\alpha,\beta)$. Note that in the last inequality we used also Young's inequality.

Noting that

$$|\vartheta(x)\mathcal{B}(x,u)| \leqslant T_M b(x)|u| + |b_0(x)| \tag{50}$$

and that

$$\left|[1-\vartheta(x)]\mathcal{B}\left(x,\frac{u}{t}\right)\right| \leqslant (b(x)-T_M b(x))\frac{|u|}{t} + |b_0(x)| \tag{51}$$

and combining (48) and (49), by Hölder's inequality, we have

$$\begin{aligned}
\int_\Omega \frac{|\nabla (u/t-g)|^2}{(1+|u/t-g|)^2}dx
&\leqslant \frac{4}{\alpha}\int_\Omega |F|\frac{|\nabla (u/t-g)|}{(1+|u/t-g|)}\,dx\\
&\quad + \frac{4}{\alpha}\int_\Omega \left(T_M b(x)\frac{|u|}{t}+|b_0|\right)\frac{|\nabla (u/t-g)|}{(1+|u/t-g|)^2}\,dx\\
&\quad + \frac{4}{\alpha}\int_\Omega \left[(b(x)-T_M b(x))\frac{|u|}{t}+|b_0(x)|\right]\\
&\qquad \times \frac{|\nabla (u/t-g)|}{(1+|u/t-g|)^2}\,dx + c\left(\|\nabla g\|_2^2+\|\varphi\|_2^2\right)\\
&\leqslant \frac{4}{\alpha}\left\|\frac{\nabla (u/t-g)}{(1+|u/t-g|)}\right\|_2\\
&\qquad \times \left(\|F\|_2 + 2\|b_0\|_2 + \|b\|_2 + \|b\|_{N,\infty}\|g\|_{2^*,2}\right)\\
&\quad + c\|\nabla g\|_2^2 + \frac{2}{\alpha}\|\varphi\|_2^2.
\end{aligned} \tag{52}$$

Then, by the elementary relation $a\geqslant 0$, $b\geqslant 0$, $x^2\leqslant ax+b \Rightarrow x\leqslant a+\sqrt{b}$ we obtain

$$\begin{aligned}
\left\|\frac{\nabla (u/t-g)}{1+|u/t-g|}\right\|_2 &\leqslant \frac{4}{\alpha}\left(\|F\|_2+2\|b_0\|_2+\|b\|_2+\|b\|_{N,\infty}\|g\|_{2^*,2}\right)\\
&\quad + c\left(\|\nabla g\|_2+\|\varphi\|_2\right),
\end{aligned} \tag{53}$$

with $c=c(\alpha,\beta)$. This concludes our proof, observing that by Sobolev embedding theorem

$$\left\|\log\left(1+\left|\frac{u}{t}-g\right|\right)\right\|_{2^*} \leqslant C\left\|\frac{\nabla (u/t-g)}{1+|u/t-g|}\right\|_2 \tag{54}$$

with $C=C(N)$. □

Proof of Theorem 3. We will obtain the existence of a solution to problem (7) by applying the Leray-Schauder fixed point theorem stated in Section 2.3 to a suitable compact operator $\mathcal{F}$. Hence, we will now construct such operator.

Let $\overline{u}\in W^{1,2}(\Omega)$. Using Theorem 8 we have that the problem

$$\begin{aligned}
&\int_\Omega \langle \mathcal{A}(x,\nabla u)+(1-\theta(x))\mathcal{B}(x,u),\nabla(v-u)\rangle\,dx\\
&\quad\geqslant \int_\Omega \langle F-\theta(x)\mathcal{B}(x,\overline{u}),\nabla(v-u)\rangle\,dx,\quad \forall v\in\mathcal{K}_{\psi,g},
\end{aligned} \tag{55}$$

admits a unique solution $u\in\mathcal{K}_{\psi,g}$, since the operator $\mathcal{A}: W^{1,2}(\Omega)\to (W^{1,2}(\Omega))^*$ defined by

$$\begin{aligned}
\langle \mathcal{A}u,v\rangle = \int_\Omega \langle \mathcal{A}(x,\nabla u)+(1-\theta(x))\mathcal{B}(x,u),\nabla v\rangle\,dx,&\\
u,v\in W^{1,2}&
\end{aligned} \tag{56}$$

is strictly monotone and coercive in $\mathcal{K}_{\psi,g}$ (see (45)).

Hence we can define an operator

$$\begin{gathered}
\mathcal{F}: W^{1,2}\longrightarrow \mathcal{K}_{\psi,g}\subseteq W^{1,2},\\
u=\mathcal{F}(\overline{u}).
\end{gathered} \tag{57}$$

We claim that such an operator $\mathcal{F}$ is compact.

Let us prove the compactness. (The proof that $\mathcal{F}$ is continuous is similar.) To this aim, suppose that $(\overline{u}_j)$ is a bounded sequence in $W^{1,2}(\Omega)$. Then, up to a subsequence, there exists $\overline{u}\in W^{1,2}(\Omega)$ such that $\overline{u}_j\to\overline{u}$ in $L^2(\Omega)$. Denoting $u_j=\mathcal{F}(\overline{u}_j)$ and $u=\mathcal{F}(\overline{u})$ we have that

$$\begin{aligned}
&\int_\Omega \left\langle \mathcal{A}\left(x,\nabla u_j\right)+(1-\theta(x))\mathcal{B}\left(x,u_j\right),\nabla\left(u-u_j\right)\right\rangle dx\\
&\quad\geqslant \int_\Omega \left\langle F-\theta(x)\mathcal{B}\left(x,\overline{u}_j\right),\nabla\left(u-u_j\right)\right\rangle dx,
\end{aligned} \tag{58}$$

$$\begin{aligned}
&\int_\Omega \left\langle \mathcal{A}(x,\nabla u)+(1-\theta(x))\mathcal{B}(x,u),\nabla\left(u_j-u\right)\right\rangle dx\\
&\quad\geqslant \int_\Omega \left\langle F-\theta(x)\mathcal{B}(x,\overline{u}),\nabla\left(u_j-u\right)\right\rangle dx.
\end{aligned} \tag{59}$$

Hence, adding (59) to (58) and using (3), (50), (51), and (4) we have

$$\begin{aligned}
\alpha \int_\Omega \left|\nabla\left(u-u_j\right)\right|^2 dx
&\leqslant \int_\Omega \left\langle \mathscr{A}(x,\nabla u)-\mathscr{A}\left(x,\nabla u_j\right), \nabla\left(u-u_j\right)\right\rangle dx \\
&\leqslant \int_\Omega \left|b(x)-T_M b(x)\right|\left|u-u_j\right|\left|\nabla\left(u-u_j\right)\right| dx \\
&\quad + \int_\Omega T_M b(x)\left|\bar{u}-\bar{u}_j\right|\left|\nabla\left(u-u_j\right)\right| dx \\
&\leqslant S_2\left\|b-T_M b\right\|_{N,\infty}\left\|\nabla\left(u-u_j\right)\right\|_2^2 \\
&\quad + M\left\|\bar{u}-\bar{u}_j\right\|_2\left\|\nabla(u-u_j)\right\|_2 .
\end{aligned} \tag{60}$$

Dividing the last inequality by $\|\nabla(u-u_j)\|_2$, by (10) and (42),

$$\left\|\nabla\left(u-u_j\right)\right\|_2 \leqslant \frac{4M}{3\alpha}\left\|\bar{u}-\bar{u}_j\right\|_2 . \tag{61}$$

This implies that $\mathscr{F}(\bar{u}_j) \rightarrow \mathscr{F}(\bar{u})$ in $W^{1,2}(\Omega)$ so that we can conclude that $\mathscr{F}$ is compact.

A fixed point of $\mathscr{F}$ is a solution to problem (7). We will prove that $\mathscr{F}$ has a fixed point. In particular, we will find a constant $K > 1$ such that the a priori estimate $\|u\|_{W^{1,2}} < K$ holds for every $u \in W^{1,2}$ and $t \in [0,1]$ satisfying $u - t\mathscr{F}(u) = 0$.

To this aim let $t \in (0,1]$ and let $u \in W^{1,2}$ be a solution to the equation $u = t\mathscr{F}(u)$. Then, $u/t \in \mathscr{K}_{\psi,g}$ and

$$\begin{aligned}
&\int_\Omega \left\langle \mathscr{A}\left(x,\frac{\nabla u}{t}\right)+(1-\theta(x))\mathscr{B}\left(x,\frac{u}{t}\right), \nabla\left(v-\frac{u}{t}\right)\right\rangle dx \\
&\quad \geqslant \int_\Omega \left\langle F-\theta(x)\mathscr{B}(x,u), \nabla\left(v-\frac{u}{t}\right)\right\rangle dx, \\
&\qquad\qquad \forall v \in \mathscr{K}_{\psi,g}.
\end{aligned} \tag{62}$$

Now our aim is to estimate $\|\nabla T_k(u/t-g)\|_2$.

We use $v = u/t - T_k(u/t-g) \in \mathscr{K}_{\psi,g}$ as a test function in (62) obtaining

$$\begin{aligned}
&\int_\Omega \left\langle \mathscr{A}\left(x,\frac{\nabla u}{t}\right)+(1-\theta(x))\mathscr{B}\left(x,\frac{u}{t}\right), \nabla T_k\left(\frac{u}{t}-g\right)\right\rangle dx \\
&\quad \leqslant \int_\Omega \left\langle F-\theta(x)\mathscr{B}(x,u), \nabla T_k\left(\frac{u}{t}-g\right)\right\rangle dx .
\end{aligned} \tag{63}$$

Moreover, we observe that by (2) and (3)

$$\begin{aligned}
\alpha \int_{\{|u/t-g|\leqslant k\}} \left|\frac{\nabla u}{t}\right|^2 dx
&\leqslant \int_{\{|u/t-g|\leqslant k\}} \left\langle \mathscr{A}\left(x,\frac{\nabla u}{t}\right)-\mathscr{A}(x,0), \frac{\nabla u}{t}\right\rangle dx \\
&\leqslant \int_{\{|u/t-g|\leqslant k\}} \left\langle \mathscr{A}\left(x,\frac{\nabla u}{t}\right), \nabla\left(\frac{u}{t}-g\right)\right\rangle dx \\
&\quad + \beta\int_{\{|u/t-g|\leqslant k\}} \left|\frac{\nabla u}{t}\right|\left|\nabla g\right| dx \\
&\quad + \int_{\{|u/t-g|\leqslant k\}} \left|\varphi(x)\right|\left(\beta\left|\nabla g\right|+\left|\frac{\nabla u}{t}\right|\right) dx \\
&\leqslant \int_{\{|u/t-g|\leqslant k\}} \left\langle \mathscr{A}\left(x,\frac{\nabla u}{t}\right), \nabla\left(\frac{u}{t}-g\right)\right\rangle dx \\
&\quad + \frac{\alpha}{2}\int_{\{|u/t-g|\leqslant k\}} \left|\frac{\nabla u}{t}\right|^2 dx + c\left(\|\nabla g\|_2^2+\|\varphi\|_2^2\right)
\end{aligned} \tag{64}$$

with $c = c(\alpha,\beta)$. This gives, using (63),

$$\begin{aligned}
\frac{\alpha}{2}\int_{\{|u/t-g|\leqslant k\}} \left|\frac{\nabla u}{t}\right|^2 dx
&\leqslant \int_\Omega \left\langle F-\theta(x)\mathscr{B}(x,u), \nabla T_k\left(\frac{u}{t}-g\right)\right\rangle dx \\
&\quad + \int_\Omega \left|(1-\theta(x))\mathscr{B}\left(x,\frac{u}{t}\right)\right|\left|\nabla T_k\left(\frac{u}{t}-g\right)\right| dx \\
&\quad + c\left(\|\nabla g\|_2^2+\|\varphi\|_2^2\right) \\
&\leqslant \|F\|_2\left\|\nabla T_k\left(\frac{u}{t}-g\right)\right\|_2 \\
&\quad + \int_\Omega \left(T_M b(x)|u|+2\left|b_0(x)\right|\right)\left|\nabla T_k\left(\frac{u}{t}-g\right)\right| dx \\
&\quad + \int_\Omega \left(b(x)-T_M b(x)\right)\left|\frac{u}{t}\right|\left|\nabla T_k\left(\frac{u}{t}-g\right)\right| dx \\
&\quad + c\left(\|\nabla g\|_2^2+\|\varphi\|_2^2\right) \\
&\leqslant \|F\|_2\left\|\nabla T_k\left(\frac{u}{t}-g\right)\right\|_2 + 2\|b_0\|_2\left\|\nabla T_k\left(\frac{u}{t}-g\right)\right\|_2 \\
&\quad + \int_\Omega |b(x)|\left|\frac{u}{t}\right|\left|\nabla T_k\left(\frac{u}{t}-g\right)\right| dx + c\left(\|\nabla g\|_2^2+\|\varphi\|_2^2\right).
\end{aligned} \tag{65}$$

By (65), using Hölder's inequality and Theorem 5, we obtain

$$\begin{aligned}
\int_\Omega \left|\nabla T_k\left(\frac{u}{t}-g\right)\right|^2 dx
&= \int_{\{|u/t-g|\leqslant k\}} \left|\nabla\left(\frac{u}{t}-g\right)\right|^2 dx \\
&\leqslant 2\int_{\{|u/t-g|\leqslant k\}} \left(\left|\frac{\nabla u}{t}\right|^2+\left|\nabla g\right|^2\right) dx
\end{aligned}$$

$$\begin{aligned}&\leqslant \frac{4}{\alpha}\left\|\nabla T_k\left(\frac{u}{t}-g\right)\right\|_2\\&\quad\times\left(\|F\|_2+2\|b_0\|_2+k\|b\|_2+\|b\|_{N,\infty}\|g\|_{2^*,2}\right)\\&\quad+c\left(\|\nabla g\|_2^2+\|\varphi\|_2^2\right).\end{aligned}\tag{66}$$

Hence, we obtain, for every $k\in\mathbb{N}$,

$$\begin{aligned}&\left\|\nabla T_k\left(\frac{u}{t}-g\right)\right\|_2\\&\quad\leqslant\frac{4}{\alpha}\left(\|F\|_2+2\|b_0\|_2+k\|b\|_2+\|b\|_{N,\infty}\|g\|_{2^*,2}\right)\\&\quad+c\left(\|\nabla g\|_2+\|\varphi\|_2\right)\end{aligned}\tag{67}$$

with $c=c(\alpha,\beta)$.

Now, let us denote

$$G_k\left(\frac{u}{t}-g\right)=\left(\frac{u}{t}-g\right)-T_k\left(\frac{u}{t}-g\right).\tag{68}$$

And let us set

$$\Omega_k=\left\{x\in\Omega:\left|\frac{u}{t}-g\right|>k\right\}.\tag{69}$$

At this point our aim is to estimate $\|\nabla G_k(u/t-g)\|_2$.

Let us preliminarily observe that using $v=u/t-G_k(u/t-g)\in\mathscr{K}_{\psi,g}$ as a test function in (62) we obtain

$$\begin{aligned}&\int_\Omega\left\langle\mathscr{A}\left(x,\frac{\nabla u}{t}\right)+(1-\theta(x))\mathscr{B}\left(x,\frac{u}{t}\right),\nabla G_k\left(\frac{u}{t}-g\right)\right\rangle dx\\&\quad\leqslant\int_\Omega\left\langle F-\theta(x)\mathscr{B}(x,u),\nabla G_k\left(\frac{u}{t}-g\right)\right\rangle dx.\end{aligned}\tag{70}$$

Using (2) and (3) and arguing as above by (70) we obtain

$$\begin{aligned}&\frac{\alpha}{4}\int_\Omega\left|\nabla G_k\left(\frac{u}{t}-g\right)\right|^2dx\\&\leqslant\left(\|F\|_2+2\|b_0\|_2+\|b\|_{N,\infty}\|g\|_{2^*,2}\right)\left\|\nabla G_k\left(\frac{u}{t}-g\right)\right\|_2\\&\quad+\int_{\Omega_k}T_Mb(x)\left(\left|G_k\left(\frac{u}{t}-g\right)\right|+k\right)\left|\nabla G_k\left(\frac{u}{t}-g\right)\right|dx\\&\quad+\int_{\Omega_k}(b(x)-T_Mb(x))\left(\left|G_k\left(\frac{u}{t}-g\right)\right|+k\right)\left|\nabla G_k\left(\frac{u}{t}-g\right)\right|dx\\&\quad+c\left(\|\nabla g\|_2^2+\|\varphi\|_2^2\right)\\&\leqslant\left\|\nabla G_k\left(\frac{u}{t}-g\right)\right\|_2\Big\{\|F\|_2+2\|b_0\|_2+\|b\|_{N,\infty}\|g\|_{2^*,2}+k\|b\|_2\\&\qquad+S_2\|T_Mb\|_{L^{N,\infty}(\Omega_k)}\left\|\nabla G_k\left(\frac{u}{t}-g\right)\right\|_2\\&\qquad+S_2\|b-T_Mb\|_{L^{N,\infty}}\left\|\nabla G_k\left(\frac{u}{t}-g\right)\right\|_2\Big\}\\&\quad+c\left(\|\nabla g\|_2^2+\|\varphi\|_2^2\right)\end{aligned}\tag{71}$$

with $c=c(\alpha,\beta)$. Using (42), this leads to the estimate

$$\begin{aligned}&\left\|\nabla G_k\left(\frac{u}{t}-g\right)\right\|_2^2\\&\quad\leqslant c\left\|\nabla G_k\left(\frac{u}{t}-g\right)\right\|_2\\&\qquad\times\left(\|F\|_2+\|b_0\|_2+k\|b\|_2+\|b\|_{N,\infty}\|g\|_{2^*,2}\right)\\&\qquad+c\|T_Mb\|_{L^{N,\infty}(\Omega_k)}\left\|\nabla G_k\left(\frac{u}{t}-g\right)\right\|_2^2+c\left(\|\nabla g\|_2^2+\|\varphi\|_2^2\right)\end{aligned}\tag{72}$$

with $c=c(\alpha,\beta,b,N,\Omega)$.

On the other hand, since $u\in W^{1,2}$ is a solution to $u-t\mathscr{F}(u)=0$, we can apply Lemma 10 to obtain

$$|\Omega_k|\leqslant\frac{C}{[\log(1+k)]},\tag{73}$$

where $C=C(\alpha,\beta,N,\Omega,b,g,\|b_0\|_2,\|F\|_2,\|\varphi\|_2)$. Moreover, by (20), we have

$$\|T_Mb\|_{L^{N,\infty}(\Omega_k)}\leqslant M|\Omega_k|^{1/N}.\tag{74}$$

Then, combining (73) and (74), we can now fix $k=k_0$, independent of t and such that

$$\|T_Mb\|_{L^{N,\infty}(\Omega_{k_0})}\leqslant\frac{1}{2c}.\tag{75}$$

By (75) and (72), we obtain

$$\begin{aligned}&\left\|\nabla G_{k_0}\left(\frac{u}{t}-g\right)\right\|_2\\&\leqslant c\left(\|F\|_2+\|b_0\|_2+k_0\|b\|_2+\|b\|_{N,\infty}\|g\|_{2^*,2}+\|\nabla g\|_2+\|\varphi\|_2\right)\end{aligned}\tag{76}$$

with $c=c(\alpha,\beta,b,N,\Omega)$.

Now we are in a position to estimate $\|\nabla(u/t-g)\|_2$. We obtain, combining (67) and (76),

$$\begin{aligned}\left\|\nabla\left(\frac{u}{t}-g\right)\right\|_2&\leqslant\left\|\nabla T_{k_0}\left(\frac{u}{t}-g\right)\right\|_2+\left\|\nabla G_{k_0}\left(\frac{u}{t}-g\right)\right\|_2\\&\leqslant c\left(\|F\|_2+\|b_0\|_2+k_0\|b\|_2+\|b\|_{N,\infty}\|g\|_{2^*,2}\right.\\&\qquad\left.+\|\nabla g\|_2+\|\varphi\|_2\right)\\&=\overline{K}.\end{aligned}\tag{77}$$

Hence, for all $t\in[0,1]$ and all $u\in W^{1,2}(\Omega)$ solution to $u-t\mathscr{F}(u)=0$, we have $\|u\|_{W^{1,2}(\Omega)}<\overline{K}+\|g\|_{W^{1,2}}=K$, with $K=K(\alpha,\beta,N,\Omega,b,g,\|b_0\|_2,\|F\|_2,\|\varphi\|_2)$.

Since $\mathscr{F}$ is a compact operator, Theorem 9 implies that $\mathscr{F}$ has a fixed point, which is a solution u of (7). □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, vol. 88 of *Pure and Applied Mathematics*, Academic Press, Harcourt Brace Jovanovich, New York, NY, USA, 1980.

[2] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires*, Dunod, Paris, France, 1969.

[3] M. Chipot, *Variational Inequalities and Flow in Porous Media*, Springer, Berlin, Germany, 1984.

[4] J.-F. Rodrigues, *Obstacle Problems in Mathematical Phisics*, vol. 134 of *North-Holland Mathematics Studies*, Elsevier, Amsterdam, The Netherlands, 1987.

[5] J. Droniou, "Non-coercive linear elliptic problems," *Potential Analysis*, vol. 17, no. 2, pp. 181–203, 2002.

[6] G. Moscariello, "Existence and uniqueness for elliptic equations with lower-order terms," *Advances in Calculus of Variations*, vol. 4, no. 4, pp. 421–444, 2011.

[7] L. Boccardo, "Some developments on Dirichlet problems with discontinuous coefficients," *Bollettino della Unione Matematica Italiana*, vol. 9, no. 2, pp. 285–297, 2009.

[8] G. Zecca, "Existence and uniqueness for nonlinear elliptic equations with lower-order terms," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 75, no. 2, pp. 899–912, 2012.

[9] F. Giannetti, L. Greco, and G. Moscariello, "Linear elliptic equations with lower-order terms," *Differential and Integral Equations*, vol. 26, no. 5-6, pp. 623–638, 2013.

[10] L. Greco, G. Moscariello, and G. Zecca, "Regularity for solutions to nonlinear elliptic equations," *Differential and Integral Equations*, vol. 26, no. 9-10, pp. 1105–1113, 2013.

[11] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, Boston, Mass, USA, 1988.

[12] R. O'Neil, "Integral transforms and tensor products on Orlicz spaces and *L(p, q)* spaces," *Journal d'Analyse Mathematique*, vol. 21, pp. 1–276, 1968.

[13] M. Carozza and C. Sbordone, "The distance to L^∞ in some function spaces and applications," *Differential and Integral Equations*, vol. 10, no. 4, pp. 599–607, 1997.

[14] A. Alvino, "Sulla disuguaglianza di Sobolev in spazi di Lorentz," *Bollettino della Unione Matematica Italiana A*, vol. 5, no. 14, pp. 148–156, 1977.

[15] L. Greco and G. Moscariello, "An embedding theorem in Lorentz-Zygmund spaces," *Potential Analysis*, vol. 5, no. 6, pp. 581–590, 1996.

[16] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, Germany, 1983.