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Research Article

Semi-Iterative Method for Computing the Generalized Inverse $A_{T,S}^{(2)}$

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The main aim of this paper is to compute the generalized inverse $A_{T,S}^{(2)}$ over Banach spaces by using semi-iterative method and to present the error bounds of the semi-iterative method for approximating $A_{T,S}^{(2)}$.

1. Introduction

The semi-iterative method (SIM) was originally inspired by the classical theory of summation and had been discussed by numerous authors [1, 2], which played an important role in many applications (see [3]). In 1962, Varga [3] defined the SIM:

$$y_k = \sum_{j=0}^{k} \pi_{k,j} x_j, \quad \left(\sum_{j=0}^{k} \pi_{k,j} = 1;\ k \geq 0 \right) \tag{1}$$

for the vectors $x_0, x_1, x_2, \ldots, x_k$. The weights $\pi_{k,j} \in C$ form an infinite lower triangular matrix

$$\psi = \left(\pi_{k,j} \right)_{k \geq j \geq 0}. \tag{2}$$

This semi-iterative formula is used to solve the Drazin inverse solution of linear equations in Banach spaces in [1] and the solution of singular linear systems of algebraic equations in [4]. In 1996, Chen [5] defined the iterative formula:

$$X_{k+1} = X_k + \beta Y \left(I - A X_k \right), \quad k = 0, 1, 2, \ldots, \ \beta \in C \setminus \{0\}. \tag{3}$$

In 2010, Liu et al. [2] extended the iterative method to compute the generalized inverse $A_{T,S}^{(2)}$ over Banach spaces.

As we know, the iterative method in Liu et al. [2] can be used to compute the generalized inverse $A_{T,S}^{(2)}$ when $\rho(I - \beta Y A) < 1$. However, we can not apply it to compute the generalized inverse $A_{T,S}^{(2)}$ when $\rho(I - \beta Y A) = 1$. Therefor, it is necessary to study the semi-iterative method for computing the generalized inverse $A_{T,S}^{(2)}$ when $\rho(I - \beta Y A) = 1$. In this paper, we use the semi-iterative method to compute the generalized inverse $A_{T,S}^{(2)}$ over Banach spaces and present the error bounds of the semi-iterative method for approximating $A_{T,S}^{(2)}$.

Now we list some notations used in this paper.

Let $\mathcal{X}$ and $\mathcal{Y}$ be arbitrary Banach spaces. The symbol $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ denotes the set of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$. Let $\mathcal{B}(\mathcal{X}) := \mathcal{B}(\mathcal{X}, \mathcal{X})$. For any $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, we denote its range, null space, and norm by $\mathcal{R}(A)$, $\mathcal{N}(A)$, and $\|A\|$, respectively. If $A \in \mathcal{B}(\mathcal{X})$, then we denote its spectrum and spectral radius by $\sigma(A)$ and $\rho(A)$. If $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ and $L \subset \mathcal{X}$, then the restriction $A|_L$ of A on L is defined by $x \mapsto Ax, x \in L$.

Let $L, M \subset \mathcal{X}$ with $L \oplus M = \mathcal{X}$. Denote $P_{L,M}$ by the projection from M onto L.

The paper is organized as follows. Some lemmas will be presented in the remainder of this section. In Section 2, we reconsider the method to compute the generalized inverse $A_{T,S}^{(2)}$ on a Banach space, and we also give some conditions for the existence of semi-iterative convergence to the generalized inverse $A_{T,S}^{(2)}$ and its existence and estimate the error bounds of the semi-iterative method for approximating $A_{T,S}^{(2)}$. In Section 3, we give an example for computing the generalized inverse $A_{T,S}^{(2)}$ when $\rho(I - \beta Y A) = 1$ in our semi-iterative method.

The following lemmas are needed in what follows.

Lemma 1 (see [6, Section 4]). *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, and T and S closed subspaces of $\mathcal{X}$ and $\mathcal{Y}$, respectively. Then the following statements are equivalent:*

(i) *A has a $\{2\}$-inverse $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ such that $\mathcal{R}(B) = T$ and $\mathcal{N}(B) = S$.*

(ii) *T is a complemented subspace of $\mathcal{X}$, $A(T)$ is closed, $A|_T : T \to A(T)$ is invertible, and $A(T) \oplus S = \mathcal{Y}$.*

In the case when (i) or (ii) holds, B is unique and one denotes it by $A^{(2)}_{T,S}$.

Lemma 2 (see [7, Section 3]). *Suppose that the conditions of Lemma 2 are satisfied. If one takes $T_1 = \mathcal{N}(A^{(2)}_{T,S}A)$, then $\mathcal{X} = T \oplus T_1$ holds and A has the following matrix form:*

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} : \begin{pmatrix} T \\ T_1 \end{pmatrix} \longrightarrow \begin{pmatrix} A(T) \\ S \end{pmatrix}, \tag{4}$$

where A_1 is invertible. Moreover, $A^{(2)}_{T,S}$ has the following matrix form:

$$A^{(2)}_{T,S} = \begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} A(T) \\ S \end{pmatrix} \longrightarrow \begin{pmatrix} T \\ T_1 \end{pmatrix}. \tag{5}$$

Consequently,

$$\begin{aligned} P_{A(T),S} &= AA^{(2)}_{T,S} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} A(T) \\ S \end{pmatrix} \longrightarrow \begin{pmatrix} A(T) \\ S \end{pmatrix}, \\ P_{T,T_1} &= A^{(2)}_{T,S}A = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} T \\ T_1 \end{pmatrix} \longrightarrow \begin{pmatrix} T \\ T_1 \end{pmatrix}. \end{aligned} \tag{6}$$

Lemma 3 (see [2, Section 2]). *Let $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ and $Y \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. Define the sequence $\{X_k\}$ in $\mathcal{B}(\mathcal{Y}, \mathcal{X})$ in the following way:*

$$X_k = X_{k-1} + \beta Y \left(I_{\mathcal{Y}} - AX_{k-1}\right), \quad k = 1, 2, 3, \ldots, \tag{7}$$

where $\beta \in \mathbb{C} \setminus \{0\}$ and $X_0 \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ with $Y \neq YAX_0$. Then the iteration (7) converges if and only if $\rho(I_{\mathcal{X}} - \beta YA) < 1$; equivalently, $\rho(I_{\mathcal{Y}} - \beta AY) < 1$.

In this case, assume that T and S are closed subspaces of $\mathcal{X}$ and $\mathcal{Y}$, respectively. If $\mathcal{R}(Y) = T$ and $\mathcal{N}(Y) = S$ and $\mathcal{R}(X_0) \subset T$, then $A^{(2)}_{T,S}$ exists and $\{X_k\}$ converges to $A^{(2)}_{T,S}$, and when $q = \min\{\|I_{\mathcal{X}} - \beta YA\|, \|I_{\mathcal{Y}} - \beta AY\|\} < 1$,

$$\left\|A^{(2)}_{T,S} - X_k\right\| \leq \frac{|\beta| q^k}{1 - q} \|Y\| \left\|I_{\mathcal{Y}} - AX_0\right\|. \tag{8}$$

2. Semi-Iterative Method for Computing Generalized Inverses $A^{(2)}_{T,S}$

In the section, we will discuss semi-iterative method for computing generalized inverses $A^{(2)}_{T,S}$. First, we deduce convergent conditions and error bounds of our semi-iterative method for computing generalized inverse $A^{(2)}_{T,S}$.

Theorem 4. *Let $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ and $Y \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. Define the sequence $\{X_k\}$ in $\mathcal{B}(\mathcal{Y}, \mathcal{X})$ in the following way:*

$$\begin{aligned} X_k &= X_{k-1} + \beta Y \left(I_{\mathcal{X}} - X_{k-1}A\right), \quad k = 1, 2, 3, \ldots, \\ y_m &= \sum_{j=0}^{m} \pi_{m,j} x_j, \quad m \geq 0, \end{aligned} \tag{9}$$

where $\beta \in \mathbb{C} \setminus \{0\}$ and $X_0 \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ with $Y \neq YAX_0$. Then the semi-iteration (9) converges if and only if

$$\begin{aligned} &\rho\left(I_{\mathcal{X}} - \beta YA\right) \leq 1, \ \text{equivalently}, \\ &\rho\left(I_{\mathcal{Y}} - \beta AY\right) \leq 1. \end{aligned} \tag{10}$$

In this case, assume that T and S are closed subspaces of $\mathcal{X}$ and $\mathcal{Y}$, respectively. If $\mathcal{R}(Y) = T$ and $\mathcal{N}(Y) = S$ and $\mathcal{R}(X_0) \subset T$, then $A^{(2)}_{T,S}$ exists and $\{y_m\}$ converges to $A^{(2)}_{T,S}$, and when $q = \min\{\|I_{\mathcal{X}} - \beta YA\|, \|I_{\mathcal{Y}} - \beta AY\|\} < 1$,

$$\begin{aligned} \left\|A^{(2)}_{T,S} - y_m\right\| \leq{}& \left\|\left(\frac{1}{2}\right)^m q^m \left(I_{\mathcal{X}} + \beta YA\right)^{-1} \beta Y\right\| \\ &+ \left\|\left(\frac{1}{2}\right)^{m+1} \left(I_{\mathcal{X}} - q^m\right) \left(\beta YA\right)^{-1} \beta Y\right\| \\ &+ \left\|\left(\frac{1}{2}\right)^{m+1} q^{m+1} \left(I_{\mathcal{X}} + \beta YA\right)^{-1} y_0\right\|. \end{aligned} \tag{11}$$

Proof. From (7),

$$\begin{aligned} X_k ={}& \left(I_{\mathcal{X}} - \beta YA\right)^k X_0 \\ &+ \left[\left(I_{\mathcal{X}} - \beta YA\right)^{k-1} + \left(I_{\mathcal{X}} - \beta YA\right)^{k-2} \right. \\ &\left. + \left(I_{\mathcal{X}} - \beta YA\right)^{k-3} + \cdots + \left(I_{\mathcal{X}} - \beta YA\right) + 1\right] \beta Y \\ ={}& \left(I_{\mathcal{X}} - \beta YA\right)^k X_0 + \sum_{j=0}^{k-1} \left(I_{\mathcal{X}} - \beta YA\right)^j \beta Y. \end{aligned} \tag{12}$$

Let $H = I_{\mathcal{X}} - \beta YA$, $C = \beta Y$, $X_0 = y_0$. Then we have

$$X_k = H^k X_0 + \sum_{j=0}^{k-1} H^k C; \tag{13}$$

hence,

$$X_j = H^j X_0 + \sum_{i=0}^{j-1} H^i C, \quad j = 1, 2, 3 \ldots,$$

$$
\begin{aligned}
y_m &= \sum_{j=0}^{m} \pi_{m,j} X_j \\
&= \pi_{m,0} X_0 + \sum_{j=1}^{m} \pi_{m,j} X_j \\
&= \pi_{m,0} X_0 + \sum_{j=1}^{m} \pi_{m,j} \left[H^j X_0 + \sum_{i=0}^{j-1} H^i C \right] \\
&= \pi_{m,0} X_0 + \sum_{j=1}^{m} \pi_{m,j} \sum_{i=0}^{j-1} H^i C + \sum_{j=1}^{m} \pi_{m,j} H^j X_0 \\
&= \sum_{j=1}^{m} \pi_{m,j} \left(\sum_{i=0}^{j-1} H^i C \right) + \left(\sum_{j=0}^{m} \pi_{m,j} H^j \right) X_0 \\
&= \left[\sum_{i=0}^{m-1} \left(\sum_{j=i+1}^{m} \pi_{m,j} \right) H^i \right] C + \sum_{j=0}^{m} \pi_{m,j} H^j y_0.
\end{aligned} \tag{14}
$$

Let $\pi_{m,n} = (1/2)^{n+1}$ $(n = 1, 2, \ldots)$. We have

$$
\begin{aligned}
&\sum_{i=0}^{m-1} \left(\sum_{j=i+1}^{m} \pi_{m,j} \right) H^i \\
&\quad= \sum_{j=1}^{m} \pi_{m,1} H^0 + \sum_{j=2}^{m} \pi_{m,j} H^1 \\
&\qquad + \sum_{j=3}^{m} \pi_{m,j} H^2 + \cdots + \sum_{j=m}^{m} \pi_{m,j} H^{m-1} \\
&\quad= \left(\pi_{m,1} + \pi_{m,2} + \pi_{m,3} + \cdots + \pi_{m,m} \right) H^0 \\
&\qquad + \left(\pi_{m,2} + \pi_{m,3} + \pi_{m,4} + \cdots + \pi_{m,m} \right) H^1 \\
&\qquad + \left(\pi_{m,3} + \pi_{m,4} + \pi_{m,5} + \cdots + \pi_{m,m} \right) H^2 \\
&\qquad + \cdots + \pi_{m,m} H^{m-1} \\
&\quad= \left[\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^{m+1} \right] H^0 \\
&\qquad + \left[\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \cdots + \left(\frac{1}{2}\right)^{m+1} \right] H^1 \\
&\qquad + \left[\left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 + \cdots + \left(\frac{1}{2}\right)^{m+1} \right] H^2 \\
&\qquad + \cdots + \left(\frac{1}{2}\right)^{m+1} H^{m-1} \\
&\quad= \frac{1}{2} \left[1 - \left(\frac{1}{2}\right)^m \right] H^0 + \left(\frac{1}{2}\right)^2 \left[1 - \left(\frac{1}{2}\right)^{m-1} \right] H^1 \\
&\qquad + \left(\frac{1}{2}\right)^3 \left[1 - \left(\frac{1}{2}\right)^{m-2} \right] H^2 + \cdots + \left(\frac{1}{2}\right)^{m+1} H^{m-1} \\
&\quad= \frac{1}{2} T^0 - \left(\frac{1}{2}\right)^{m+1} H^0 + \left(\frac{1}{2}\right)^2 H^1 \\
&\qquad - \left(\frac{1}{2}\right)^{m+1} H^1 + \left(\frac{1}{2}\right)^3 H^2 - \left(\frac{1}{2}\right)^{m+1} H^2 \\
&\qquad + \cdots + \left(\frac{1}{2}\right)^m \left[1 - \left(\frac{1}{2}\right)^{m-(m-1)} \right] H^{m-1} \\
&\quad= \frac{1}{2} H^0 + \left(\frac{1}{2}\right)^2 H^1 + \left(\frac{1}{2}\right)^3 H^2 \\
&\qquad + \cdots + \left(\frac{1}{2}\right)^m H^{m-1} - \left(\frac{1}{2}\right)^{m+1} H^0 - \left(\frac{1}{2}\right)^{m+1} H^1 \\
&\qquad - \cdots - \left(\frac{1}{2}\right)^{m+1} H^{m-1} \\
&\quad= \frac{1}{2} H^0 + \left(\frac{1}{2}\right)^2 H^1 + \left(\frac{1}{2}\right)^3 H^2 + \cdots + \left(\frac{1}{2}\right)^m H^{m-1} \\
&\qquad - \left(\frac{1}{2}\right)^{m+1} \left[H^0 + H^1 + T^2 + \cdots + H^{m-1} \right], \\
&H^0 + H^1 + H^2 + \cdots + H^{m-1} \\
&\quad= \left(H^0 + H^1 + H^2 + \cdots + H^{m-1} \right) (I - H) (I - H)^{-1} \\
&\quad= \left[H^0 + H^1 + H^2 + \cdots + H^{m-1} - H^1 \right. \\
&\qquad \left. - H^2 - \cdots - H^{m-1} - H^m \right] (I - H)^{-1} \\
&\quad= \left(H^0 - H^m \right) (I - H)^{-1}, \\
&\sum_{j=0}^{m} \pi_{m,j} H^j \\
&\quad= \pi_{m,0} H^0 + \pi_{m,1} H^1 + \pi_{m,2} H^2 + \cdots + \pi_{m,m} H^m \\
&\quad= \frac{1}{2} H^0 + \left(\frac{1}{2}\right)^2 H^1 + \left(\frac{1}{2}\right)^3 H^2 + \cdots + \left(\frac{1}{2}\right)^{m+1} H^m \\
&\quad= \left[\frac{1}{2} H^0 + \left(\frac{1}{2}\right)^2 H^1 + \left(\frac{1}{2}\right)^3 H^2 + \cdots + \left(\frac{1}{2}\right)^{m+1} H^m \right] \\
&\qquad \cdot \left(I - \frac{1}{2} H \right) \left(I - \frac{1}{2} H \right)^{-1} \\
&\quad= \left[\frac{1}{2} H^0 + \left(\frac{1}{2}\right)^2 H^1 + \left(\frac{1}{2}\right)^3 H^2 + \cdots + \left(\frac{1}{2}\right)^{m+1} H^m \right. \\
&\qquad \left. - \left(\frac{1}{2}\right)^2 H^1 - \left(\frac{1}{2}\right)^3 H^2 - \cdots - \left(\frac{1}{2}\right)^{m+2} H^{m+1} \right] \\
&\qquad \cdot \left(I - \frac{1}{2} H \right)^{-1} \\
&\quad= \left[\frac{1}{2} H^0 - \left(\frac{1}{2}\right)^{m+2} H^{m+1} \right] \left(I - \frac{1}{2} H \right)^{-1}.
\end{aligned} \tag{15}
$$

From (15), we have

$$\begin{aligned} y_m &= \left[\sum_{i=0}^{m-1}\left(\sum_{j=i+1}^{m}\pi_{m,j}\right)H^i\right]C+\sum_{j=0}^{m}\pi_{m,j}H^j y_0 \\ &= \left\{\frac{1}{2}H^0+\left(\frac{1}{2}\right)^2 H^1+\left(\frac{1}{2}\right)^3 H^2+\cdots+\left(\frac{1}{2}\right)^m H^{m-1}\right. \\ &\qquad \left.-\left(\frac{1}{2}\right)^{m+1}\left[H^0+H^1+H^2+\cdots+H^{m-1}\right]\right\}C \\ &\quad +\left[\frac{1}{2}H^0+\left(\frac{1}{2}\right)^2 H^1+\left(\frac{1}{2}\right)^3 H^2\right. \\ &\qquad \left.+\cdots+\left(\frac{1}{2}\right)^{m+1} H^m\right]y_0 \\ &= \left\{\left[\frac{1}{2}H^0-\left(\frac{1}{2}\right)^{m+1}H^m\right]\left(I_{\mathcal{X}}-\frac{1}{2}H\right)^{-1}\right. \\ &\qquad \left.-\left(\frac{1}{2}\right)^{m+1}\left(H^0-H^m\right)\left(I_{\mathcal{X}}-H\right)^{-1}\right\}C \\ &\quad +\left\{\left[\frac{1}{2}H^0-\left(\frac{1}{2}\right)^{m+2}H^{m+1}\right]\left(I_{\mathcal{X}}-\frac{1}{2}H\right)^{-1}\right\}y_0; \end{aligned} \tag{16}$$

thus,

$$\begin{aligned} &y_m-y_{m-1} \\ &\quad= \left[\left(\frac{1}{2}\right)^m H^{m-1}-\left(\frac{1}{2}\right)^{m+1}H^m\right]\left(I_{\mathcal{X}}-\frac{1}{2}H\right)^{-1}C \\ &\qquad +\left(\frac{1}{2}\right)^m\left(\frac{1}{2}H^0+\frac{1}{2}H^m-H^{m-1}\right)\left(I_{\mathcal{X}}-H\right)^{-1}C \\ &\qquad +\left[\left(\frac{1}{2}\right)^{m+1}H^m-\left(\frac{1}{2}\right)^{m+2}H^{m+1}\right]\left(I_{\mathcal{X}}-\frac{1}{2}H\right)^{-1}y_0 \\ &\quad= \left(\frac{1}{2}\right)^m\left[H^{m-1}-\frac{1}{2}H^m\right]\left(I_{\mathcal{X}}-\frac{1}{2}H\right)^{-1}C \\ &\qquad +\left(\frac{1}{2}\right)^m\left(\frac{1}{2}H^0+\frac{1}{2}H^m-H^{m-1}\right)\left(I_{\mathcal{X}}-H\right)^{-1}C \\ &\qquad +\left(\frac{1}{2}\right)^{m+1}\left[H^m-\left(\frac{1}{2}\right)^m H^{m+1}\right]\left(I_{\mathcal{X}}-\frac{1}{2}H\right)^{-1}y_0 \\ &\quad= \left(\frac{1}{2}\right)^m H^{m-1}\left[I_{\mathcal{X}}-\frac{1}{2}H\right]\left(I_{\mathcal{X}}-\frac{1}{2}H\right)^{-1}C \\ &\qquad +\left(\frac{1}{2}\right)^{m+1}\left(I_{\mathcal{X}}-H\right)^{-1}C \\ &\qquad +\left(\frac{1}{2}\right)^m\left(\frac{1}{2}H^m-H^{m-1}\right)\left(I_{\mathcal{X}}-H\right)^{-1}C \\ &\qquad +\left(\frac{1}{2}\right)^{m+1}H^m\left[I_{\mathcal{X}}-\frac{1}{2}H\right]\left(I_{\mathcal{X}}-\frac{1}{2}H\right)^{-1}y_0 \\ &\quad= \left(\frac{1}{2}\right)^m H^{m-1}C+\left(\frac{1}{2}\right)^{m+1}\left(I_{\mathcal{X}}-H\right)^{-1}C \\ &\qquad +\left(\frac{1}{2}\right)^m H^{m-1}\left(\frac{1}{2}H-I_{\mathcal{X}}\right)\left(I_{\mathcal{X}}-H\right)^{-1}C \\ &\qquad +\left(\frac{1}{2}\right)^{m+1}H^m y_0 \\ &\quad= \left(\frac{1}{2}\right)^m\left(I_{\mathcal{X}}-\beta YA\right)^{m-1}\beta Y \\ &\qquad +\left(\frac{1}{2}\right)^{m+1}\left(I_{\mathcal{X}}-\left(I_{\mathcal{X}}-\beta YA\right)\right)^{-1}\beta Y \\ &\qquad +\left(\frac{1}{2}\right)^m\left(I_{\mathcal{X}}-\beta YA\right)^{m-1}\left(\frac{1}{2}\left(I_{\mathcal{X}}-\beta YA\right)-I_{\mathcal{X}}\right) \\ &\qquad \cdot\left(I_{\mathcal{X}}-\left(I_{\mathcal{X}}-\beta YA\right)\right)^{-1}\beta Y \\ &\qquad +\left(\frac{1}{2}\right)^{m+1}\left(I_{\mathcal{X}}-\beta YA\right)^m y_0 \\ &\quad= \left(\frac{1}{2}\right)^m\left(I_{\mathcal{X}}-\beta YA\right)^{m-1}\beta Y+\left(\frac{1}{2}\right)^{m+1}A^{-1} \\ &\qquad +\left(\frac{1}{2}\right)^m\left(I_{\mathcal{X}}-\beta YA\right)^{m-1}\left(-\frac{1}{2}\left(I_{\mathcal{X}}+\beta YA\right)\right)A^{-1} \\ &\qquad +\left(\frac{1}{2}\right)^{m+1}\left(I_{\mathcal{X}}-\beta YA\right)^{m-1}y_0 \\ &\quad= \left(\frac{1}{2}\right)^m\left(I_{\mathcal{X}}-\beta YA\right)^{m-1}\beta Y+\left(\frac{1}{2}\right)^{m+1}A^{-1} \\ &\qquad +\left(\frac{1}{2}\right)^m\left(I_{\mathcal{X}}-\beta YA\right)^{m-1}\left(-\frac{1}{2}\right)A^{-1} \\ &\qquad +\left(\frac{1}{2}\right)^m\left(I_{\mathcal{X}}-\beta YA\right)^{m-1}\left(-\frac{1}{2}\beta YA\right)A^{-1} \\ &\qquad +\left(\frac{1}{2}\right)^{m+1}\left(I_{\mathcal{X}}-\beta YA\right)^m y_0 \\ &\quad= \left(\frac{1}{2}\right)^{m+1}\left(I_{\mathcal{X}}-\beta YA\right)^{m-1}\beta Y \\ &\qquad +\left(\frac{1}{2}\right)^{m+1}\left[I_{\mathcal{X}}-\left(I_{\mathcal{X}}-\beta YA\right)^{m-1}\right]A^{-1} \\ &\qquad +\left(\frac{1}{2}\right)^{m+1}\left(I_{\mathcal{X}}-\beta YA\right)^m y_0; \end{aligned} \tag{17}$$

hence,

$$\begin{aligned} &\left\|y_m-y_{m-1}\right\| \\ &\qquad \leq \left\|\left(\frac{1}{2}\right)^{m+1}\left(I_{\mathcal{X}}-\beta YA\right)^{m-1}\beta Y\right\| \end{aligned}$$

$$+\left\|\left(\frac{1}{2}\right)^{m+1}\left[I_{\mathcal{X}}-\left(I_{\mathcal{X}}-\beta YA\right)^{m-1}\right]A^{-1}\right\|$$
$$+\left\|\left(\frac{1}{2}\right)^{m+1}\left(I_{\mathcal{X}}-\beta YA\right)^{m}y_0\right\|. \tag{18}$$

Therefore, we have

$$\begin{aligned}
&\lim_{m\to\infty} y_m\\
&=\lim_{m\to\infty}\left\{\left\{\left[\frac{1}{2}H^0-\left(\frac{1}{2}\right)^{m+1}H^m\right]\left(I_{\mathcal{X}}-\frac{1}{2}H\right)^{-1}\right.\right.\\
&\qquad\left.-\left(\frac{1}{2}\right)^{m+1}\left(H^0-T^m\right)\left(I_{\mathcal{X}}-H\right)^{-1}\right\}C\\
&\qquad\left.+\left\{\left[\frac{1}{2}H^0-\left(\frac{1}{2}\right)^{m+2}H^{m+1}\right]\left(I_{\mathcal{X}}-\frac{1}{2}H\right)^{-1}\right\}y_0\right\}\\
&=\frac{1}{2}H^0\left(I_{\mathcal{X}}-\frac{1}{2}H\right)^{-1}C+\frac{1}{2}H^0\left(I_{\mathcal{X}}-\frac{1}{2}H\right)^{-1}y_0\\
&=\frac{1}{2}H^0\left(I_{\mathcal{X}}-\frac{1}{2}H\right)^{-1}\left(C+y_0\right).
\end{aligned} \tag{19}$$

Denote $\lim_{m\to\infty} y_m = y_\infty$, and $H = I_{\mathcal{X}} - \beta YA$, $C = \beta Y$, $X_0 = y_0$. Then from (19), we have

$$\begin{aligned}
y_\infty &= \frac{1}{2}\left(I_{\mathcal{X}}-\frac{1}{2}H\right)^{-1}\left(C+y_0\right)\\
&=\frac{1}{2}\left[I_{\mathcal{X}}-\frac{1}{2}\left(I_{\mathcal{X}}-\beta YA\right)\right]^{-1}\left(\beta Y+y_0\right)\\
&=\frac{1}{2}\left(\frac{1}{2}I_{\mathcal{X}}+\frac{1}{2}\beta YA\right)^{-1}\left(\beta Y+y_0\right)\\
&=\left(I_{\mathcal{X}}+\beta YA\right)^{-1}\left(\beta Y+y_0\right).
\end{aligned} \tag{20}$$

So

$$\begin{gathered}
\left(I_{\mathcal{X}}+\beta YA\right)y_\infty=\beta Y+y_0,\\
y_\infty+\beta YAy_\infty=\beta Y+y_0,\\
y_\infty=\beta Y+y_0-\beta YAy_\infty.
\end{gathered} \tag{21}$$

Since $y_0 \in \mathcal{R}(Y)$ and $\mathcal{R}(YAy_\infty) \subset \mathcal{R}(Y)$, then $\mathcal{R}(y_\infty) \subset \mathcal{R}(Y)$. On the other hand,

$$\begin{gathered}
\beta Y-\beta YAy_\infty+y_0=y_\infty,\\
\beta Y\left(I_{\mathcal{X}}-Ay_\infty+\frac{1}{\beta}\right)=y_\infty,\\
\mathcal{R}(Y)\subset\mathcal{R}\left(y_\infty\right),
\end{gathered} \tag{22}$$

so $\mathcal{R}(y_\infty) = \mathcal{R}(Y) = T$. Since

$$\begin{gathered}
y_\infty+\beta YAy_\infty=\beta Y+y_0,\\
\mathcal{N}\left(y_\infty+\beta YAy_\infty\right)=\mathcal{N}\left(\beta Y+y_0\right),\\
\mathcal{N}(Y)=\mathcal{N}\left(y_\infty+\beta YAy_\infty\right),
\end{gathered} \tag{23}$$

then $\mathcal{N}(Y) \subset \mathcal{N}(y_\infty)$. As

$$\begin{aligned}
y_\infty&=\left(I_{\mathcal{X}}+\beta YA\right)^{-1}\left(C+y_0\right)\\
&=\left(I_{\mathcal{X}}+\beta YA\right)^{-1}\left(\beta Y+y_0\right),
\end{aligned} \tag{24}$$

and $\mathcal{N}(y_\infty) \subset \mathcal{N}(Y)$, then $\mathcal{N}(y_\infty) = \mathcal{N}(Y) = S$; hence $y_\infty = A^{(2)}_{T,S}$. Furthermore,

$$\begin{aligned}
&\left\|\lim_{m\to\infty} y_\infty-y_m\right\|\\
&=\left\|\frac{1}{2}H_0\left(I_{\mathcal{X}}-\frac{1}{2}H\right)^{-1}\left(C+y_0\right)\right.\\
&\qquad-\left[\left(\frac{1}{2}H_0-\left(\frac{1}{2}\right)^{m+1}H^m\right)\left(I_{\mathcal{X}}-\frac{1}{2}H\right)^{-1}\right.\\
&\qquad\left.-\left(\frac{1}{2}\right)^{m+1}\left(H^0-H^m\right)\left(I_{\mathcal{Y}}-H^m\right)^{-1}\right]C\\
&\qquad\left.-\left(\frac{1}{2}H_0-\left(\frac{1}{2}\right)^{m+2}H^{m+1}\right)\right\|\\
&=\left\|\frac{1}{2}\left(I_{\mathcal{X}}-\frac{1}{2}H\right)^{-1}C+\frac{1}{2}\left(I_{\mathcal{X}}-\frac{1}{2}H\right)^{-1}y_0\right.\\
&\qquad-\frac{1}{2}\left(I_{\mathcal{X}}-\frac{1}{2}H\right)^{-1}C\\
&\qquad+\left(\frac{1}{2}\right)^{m+1}H^m\left(I_{\mathcal{X}}-\frac{1}{2}H\right)^{-1}C\\
&\qquad+\left(\frac{1}{2}\right)^{m+1}\left(I_{\mathcal{X}}-H\right)^{-1}C\\
&\qquad-\left(\frac{1}{2}\right)^{m+1}H^m\left(I_{\mathcal{X}}-H\right)^{-1}C\\
&\qquad-\frac{1}{2}\left(I_{\mathcal{X}}-\frac{1}{2}H\right)^{-1}y_0\\
&\qquad\left.+\left(\frac{1}{2}\right)^{m+2}H^{m+1}\left(I_{\mathcal{X}}-\frac{1}{2}H\right)^{-1}y_0\right\|\\
&=\left\|\left(\frac{1}{2}\right)^{m+1}H^m\left(I_{\mathcal{X}}-\frac{1}{2}H\right)^{-1}C\right.\\
&\qquad+\left(\frac{1}{2}\right)^{m+1}\left(I_{\mathcal{X}}-H^m\right)\left(I_{\mathcal{X}}-H\right)^{-1}C\\
&\qquad\left.+\left(\frac{1}{2}\right)^{m+2}H^{m+1}\left(I_{\mathcal{X}}-\frac{1}{2}H\right)^{-1}y_0\right\|\\
&=\left\|\left(\frac{1}{2}\right)^{m+1}\left(I_{\mathcal{X}}-\beta YA\right)^m\left[\frac{1}{2}\left(I_{\mathcal{X}}+\beta YA\right)\right]^{-1}\beta Y\right.\\
&\qquad+\left(\frac{1}{2}\right)^{m+1}\left[I_{\mathcal{X}}-\left(I_{\mathcal{X}}-\beta YA\right)^m\right]\\
&\qquad\left[I_{\mathcal{X}}-\left(I_{\mathcal{X}}-\beta YA\right)\right]^{-1}\beta Y
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{1}{2}\right)^{m+2}\left(I_{\mathscr{X}}-\beta YA\right)^{m+1}\left[\frac{1}{2}\left(I_{\mathscr{X}}+\beta YA\right)\right]^{-1}y_0\Bigg\| \\
&=\Bigg\|\left(\frac{1}{2}\right)^{m}\left(I_{\mathscr{X}}-\beta YA\right)^{m}\left(I_{\mathscr{X}}+\beta YA\right)^{-1}\beta Y \\
&\quad+\left(\frac{1}{2}\right)^{m+1}\left[I_{\mathscr{X}}-\left(I_{\mathscr{X}}-\beta YA\right)^{m}\right]\left(\beta YA\right)^{-1}\beta Y \\
&\quad+\left(\frac{1}{2}\right)^{m+1}\left(I_{\mathscr{X}}-\beta YA\right)^{m+1}\left(I_{\mathscr{X}}+\beta YA\right)^{-1}y_0\Bigg\| \\
&=\Bigg\|\left(\frac{1}{2}\right)^{m}q^{m}\left(I_{\mathscr{X}}+\beta YA\right)^{-1}\beta Y \\
&\quad+\left(\frac{1}{2}\right)^{m+1}\left[I_{\mathscr{X}}-q^{m}\right]\left(\beta YA\right)^{-1}\beta Y \\
&\quad+\left(\frac{1}{2}\right)^{m+1}q^{m+1}\left(I_{\mathscr{X}}+\beta YA\right)^{-1}y_0\Bigg\| \\
&=\Bigg\|\left(\frac{1}{2}\right)^{m}q^{m}\left(I_{\mathscr{X}}+\beta YA\right)^{-1}\beta Y \\
&\quad+\left(\frac{1}{2}\right)^{m+1}\left(\beta YA\right)^{-1}\beta Y \\
&\quad-\left(\frac{1}{2}\right)^{m+1}q^{m}\left(\beta YA\right)^{-1}\beta Y \\
&\quad+\left(\frac{1}{2}\right)^{m}q^{m+1}\left(I_{\mathscr{X}}+\beta YA\right)^{-1}y_0\Bigg\| \\
&=\left(\frac{1}{2}\right)^{m}\Bigg\|q^{m}\left(I_{\mathscr{X}}+\beta YA\right)^{-1}\beta Y+\frac{1}{2}\left(\beta YA\right)^{-1}\beta Y \\
&\quad-\frac{1}{2}q^{m}\left(\beta YA\right)^{-1}\beta Y \\
&\quad+\frac{1}{2}q^{m+1}\left(I_{\mathscr{X}}+\beta YA\right)^{-1}y_0\Bigg\| \\
&=\Bigg\|\left(\frac{1}{2}\right)^{m}q^{m}\left(I_{\mathscr{X}}+\beta YA\right)^{-1}\beta Y \\
&\quad+\left(\frac{1}{2}\right)^{m+1}\left(I_{\mathscr{X}}-q^{m}\right)\left(\beta YA\right)^{-1}\beta Y \\
&\quad+\left(\frac{1}{2}\right)^{m+1}q^{m+1}\left(I_{\mathscr{X}}+\beta YA\right)^{-1}y_0\Bigg\| \\
&\leq\left\|\left(\frac{1}{2}\right)^{m}q^{m}\left(I_{\mathscr{X}}+\beta YA\right)^{-1}\beta Y\right\| \\
&\quad+\left\|\left(\frac{1}{2}\right)^{m+1}\left(I_{\mathscr{X}}-q^{m}\right)\left(\beta YA\right)^{-1}\beta Y\right\| \\
&\quad+\left\|\left(\frac{1}{2}\right)^{m+1}q^{m+1}\left(I_{\mathscr{X}}+\beta YA\right)^{-1}y_0\right\|.
\end{aligned}
\tag{25}
$$

The proof is complete. □

Similarly, we can obtain the following theorem.

Theorem 5. *Let $A\in\mathscr{B}(\mathscr{X},\mathscr{Y})$ and $Y\in\mathscr{B}(\mathscr{Y},\mathscr{X})$. Define the sequence $\{X_k\}$ in $\mathscr{B}(\mathscr{Y},\mathscr{X})$ in the following way:*

$$
\begin{aligned}
X_k&=X_{k-1}+\beta Y\left(I_{\mathscr{Y}}-AX_{k-1}\right),\quad k=1,2,3,\ldots, \\
y_m&=\sum_{i=0}^{m}\pi_{m,j}x_j,\quad m\geq 0,
\end{aligned}
\tag{26}
$$

where $\beta\in\mathbb{C}\setminus\{0\}$ and $X_0\in\mathscr{B}(\mathscr{Y},\mathscr{X})$ with $Y\neq X_0AY$. Then the semi-iteration (26) converges if and only if

$$
\begin{aligned}
&\rho\left(I_{\mathscr{Y}}-\beta AY\right)\leq 1,\ \text{equivalently}, \\
&\rho\left(I_{\mathscr{X}}-\beta YA\right)\leq 1.
\end{aligned}
\tag{27}
$$

In this case, assume that T and S are closed subspaces of $\mathscr{X}$ and $\mathscr{Y}$, respectively. If $\mathscr{R}(Y)=T$ and $\mathscr{N}(Y)=S$ and $\mathscr{N}(X_0)\supset S$, then $A^{(2)}_{T,S}$ exists and $\{y_m\}$ converges to $A^{(2)}_{T,S}$, and when $q=\min\{\|I_{\mathscr{X}}-\beta YA\|,\|I_{\mathscr{Y}}-\beta AY\|\}<1$,

$$
\begin{aligned}
\left\|A^{(2)}_{T,S}-y_m\right\|\leq&\left\|\left(\frac{1}{2}\right)^{m}q^{m}\left(I_{\mathscr{Y}}+\beta AY\right)^{-1}\beta Y\right\| \\
&+\left\|\left(\frac{1}{2}\right)^{m+1}\left(I_{\mathscr{Y}}-q^{m}\right)\left(\beta AY\right)^{-1}\beta Y\right\| \\
&+\left\|\left(\frac{1}{2}\right)^{m+1}q^{m+1}\left(I_{\mathscr{Y}}+\beta AY\right)^{-1}y_0\right\|.
\end{aligned}
\tag{28}
$$

3. Examples

We give an example for computing $A^{(2)}_{T,S}$ by the semi-iterative (9). Let the symbol $\|\cdot\|$ denote the Frobenius norm.

Example 1. Consider the matrix

$$
\begin{aligned}
A&=\begin{bmatrix}0.5&0&0&0\\0&0.5&0&0\\0&0&0.25&0\\0&0&0&1\end{bmatrix}, \\
Y=X_0=y_0&=\begin{bmatrix}0.5&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&0\end{bmatrix},
\end{aligned}
\tag{29}
$$

$T=\mathbb{C}^4$, $e=(0,0,0,0,1)^T\in\mathbb{C}^4$, and $S=\operatorname{span}\{e\}$. Obviously, $\mathscr{R}(Y)=T$ and $\mathscr{N}(Y)=S$. Moreover, we choose

$$
I=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}.
\tag{30}
$$

Therefore, for any $\beta\in\mathbb{C}\setminus\{0\}$, we have $\rho(I-\beta YA)=1$. If the scalar $\beta=0.9$, it is easily verified that

$$
A^{(2)}_{T,S}=\begin{bmatrix}0.7755&0&0&0\\0&2.0000&0&0\\0&0&2.6207&0\\0&0&0&0\end{bmatrix}.
\tag{31}
$$

Table 1: Results for computing $A_{T,S}^{(2)}$ by using iteration (20).

β	Step	$E(m)=\|A_{T,S}^{(2)}-y_m\|$	$\|y_m-y_{m-1}\|$
$\beta=0.9$	$m=6$	$3.820778657434200E-02$	$3.753942246016400E-02$
	$m=10$	$2.398173793546000E-03$	$2.387899534974000E-03$
	$m=11$	$1.198734582542000E-03$	$1.195013235056000E-03$
	$m=17$	$1.869961136884640E-05$	$1.868898369984000E-05$
	$m=21$	$1.168271567800560E-06$	$1.168036362390750E-06$
	$m=26$	$3.650255486528950E-08$	$3.650050835579940E-08$
	$m=31$	$1.140652761558560E-09$	$1.140634888089180E-09$
	$m=32$	$5.703241064454260E-10$	$5.703171804724460E-10$
	$m=64$	$1.327871055533770E-19$	$1.327871050907670E-19$
$\beta=1.0$	$m=3$	$3.012230808118550E-01$	$2.881862247962140E-01$
	$m=5$	$7.658271287825600E-02$	$7.500354183760500E-02$
	$m=7$	$1.921616264226400E-02$	$1.904402615464200E-02$
	$m=10$	$2.398878340030000E-03$	$2.392152983091000E-03$
	$m=13$	$2.994086021157650E-04$	$2.990957245386160E-04$
	$m=14$	$1.496563849083790E-04$	$1.495411757049050E-04$
	$m=30$	$2.281288673173540E-09$	$2.281271586655230E-09$
	$m=32$	$5.703195890929840E-10$	$5.703171861702270E-10$
	$m=64$	$1.327871051078640E-19$	$1.327871050516590E-19$
$\beta=1.2$	$m=3$	$3.074437258505530E-01$	$2.943566993012280E-01$
	$m=5$	$7.720793593447200E-02$	$7.622976300737400E-02$
	$m=8$	$9.608716640119000E-03$	$9.584156541330000E-03$
	$m=10$	$2.397229438700000E-03$	$2.394566834436000E-03$
	$m=14$	$1.495829839649130E-04$	$1.495447708030870E-04$
	$m=25$	$7.300123408684290E-08$	$7.300086429541690E-08$
	$m=31$	$1.140633927430880E-09$	$1.140633247516680E-09$
	$m=32$	$5.703167566295210E-10$	$5.703165186574720E-10$
	$m=64$	$1.327871050315660E-19$	$1.327871050309540E-19$
$\beta=1.25$	$m=4$	$1.545274774285740E-01$	$1.521939388599310E-01$
	$m=6$	$3.856094399263400E-02$	$3.834943769150000E-02$
	$m=8$	$9.606257104476000E-03$	$9.586990214228000E-03$
	$m=10$	$2.396666491289000E-03$	$2.394595536570000E-03$
	$m=13$	$2.991967359449660E-04$	$2.991146333829520E-04$
	$m=14$	$1.495691457980860E-04$	$1.495409285906900E-04$
	$m=25$	$7.300098966535540E-08$	$7.300076512906710E-08$
	$m=32$	$5.703165607729410E-10$	$5.703164334061720E-10$
	$m=64$	$1.327871050307390E-19$	$1.327871050305550E-19$
$\beta=1.3$	$m=3$	$3.110631215879030E-01$	$2.986869244201630E-01$
	$m=4$	$1.547020913728830E-01$	$1.533981948482430E-01$
	$m=6$	$3.855659973720000E-02$	$3.840382345353100E-02$
	$m=12$	$5.984798515077750E-04$	$5.983002588957640E-04$
	$m=14$	$1.495572846785640E-04$	$1.495367682194500E-04$
	$m=15$	$7.477013266862850E-05$	$7.476319565060200E-05$
	$m=30$	$2.281266572051920E-09$	$2.281265986830480E-09$
	$m=32$	$5.703164418238200E-10$	$5.703163751627730E-10$
	$m=64$	$1.327871050304590E-19$	$1.327871050304050E-19$

The error estimate $\|A_{T,S}^{(2)} - y_m\|$ for diverse scalar β and parameter m of (25) and $\|y_m - y_{m-1}\|$ of (18) is given in Table 1.

Table 1 shows that within limits the larger the parameters m and β, the smaller the error bounds. But m can not be infinitely large, because when m is large enough, the error bounds are large as well. So we must choose the best β and m.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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