Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2015, Article ID 731068, 6 pages
http://dx.doi.org/10.1155/2015/731068

Research Article

On Sharp Hölder Estimates of the Cauchy-Riemann Equation on Pseudoconvex Domains in $\mathbb{C}^n$ with One Degenerate Eigenvalue

Sanghyun Cho[1] and Young Hwan You[2]

[1] *Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea*
[2] *Department of Mathematics, Indiana University East, Richmond, IN 47374, USA*

Correspondence should be addressed to Sanghyun Cho; shcho@sogang.ac.kr

Received 8 October 2015; Accepted 5 November 2015

Academic Editor: Chun-Gang Zhu

Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ with one degenerate eigenvalue and assume that there is a smooth holomorphic curve V whose order of contact with $b\Omega$ at $z_0 \in b\Omega$ is larger than or equal to η. We show that the maximal gain in Hölder regularity for solutions of the $\bar{\partial}$-equation is at most $1/\eta$.

1. Introduction

For any open set $U \subset \mathbb{C}^n$, we let $\Lambda_\delta(U)$ denote the space of functions in Hölder class $\delta \geq 0$ on U. Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ and $z_0 \in b\Omega$. Suppose that there exists a neighborhood U of z_0 such that, for all $\bar{\partial}$-closed forms α, with $\alpha \in \Lambda_\delta(\Omega)$, we can solve $\bar{\partial}u = \alpha$ in Ω with a gain of regularity of the solution u; that is,

$$\|u\|_{\Lambda_{\delta+\epsilon}(U\cap\Omega)} \leq C\|\alpha\|_{\Lambda_\delta(\Omega)}, \tag{1}$$

for some $\epsilon > 0$. In this event, we want to find a necessary condition and determine how large ϵ can be. When $z_0 \in \Omega$, it is well known that $\epsilon = 1$. However, when $z_0 \in b\Omega$, $\epsilon > 0$ depends on the boundary geometry of Ω near z_0.

Note that the Hölder estimates of $\bar{\partial}$-equation are well known when Ω is bounded strongly pseudoconvex domain in $\mathbb{C}^n$. However, for weakly pseudoconvex domains in $\mathbb{C}^n$, Hölder estimates are known only for special pseudoconvex domains, that is, pseudoconvex domains of finite type in $\mathbb{C}^2$, convex finite type domains in $\mathbb{C}^n$, and pseudoconvex domains of finite type with diagonal Levi-form in $\mathbb{C}^n$, and so forth. Proving Hölder estimates for general pseudoconvex domains in $\mathbb{C}^n$ is one of big questions in several complex variables. Meanwhile, it is of great interest to find a necessary condition or optimal possible gain of the Hölder estimates for $\bar{\partial}$.

Several authors have obtained necessary conditions for Hölder regularity of $\bar{\partial}$ on restricted classes of domains [1–4]. Let $T_{BG}(z_0)$, the "Bloom-Graham" type, be the maximum order of contact of $b\Omega$ with any $(n-1)$-dimensional complex analytic manifold at z_0. If $T_{BG}(z_0) = N$, then Krantz [2] showed that $\epsilon \leq 1/N$. Krantz's result is sharp for $\Omega \subset \mathbb{C}^2$ and when α is a $(0, n-1)$-form. Also McNeal [3] proved sharp Hölder estimates for $(0,1)$-form α under the condition that Ω has a holomorphic support function at $z_0 \in \Omega$. Note that the existence of holomorphic support function is satisfied for restricted domains and it is often the first step to prove the Hölder estimates for $\bar{\partial}$-equation [4].

Straube [5] proved necessary condition for Hölder regularity gain of Neumann operator N. More specifically, if Neumann operator N has Hölder regularity gain of 2ϵ, then $\epsilon \leq 1/\eta$, where η is larger than or equal to order of contact of an analytic variety (possibly singular) V at z_0. However, it should be emphasized that there is no natural machinery to pass between necessary conditions for Hölder regularity of $\bar{\partial}$-Neumann operator and that of $\bar{\partial}$, in contrast to the case of L^2-Sobolev topology.

Let $\Omega = \{z : r(z) < 0\}$, where r is a smooth defining function of Ω, and let V be a smooth 1-dimensional analytic variety passing through $z_0 \in b\Omega$. We say V has order of contact larger than or equal to η with $b\Omega$ at $z_0 \in b\Omega$ if there is a positive constant $C > 0$ such that

$$|r(z)| \le C|z - z_0|^{\eta}, \tag{2}$$

for all $z \in V$ sufficiently close to z_0. Here *smooth* means that $\gamma'(0) \ne 0$ if $\gamma(t)$ represents a parametrization of V. Recently, the second author, You [6], proved a necessary condition for Hölder estimates for bounded pseudoconvex domains of finite type in $\mathbb{C}^3$. That is, if there is a 1-dimensional smooth analytic variety V passing through $z_0 \in b\Omega$ and the order of contact of V with $b\Omega$ is larger than or equal to $\eta > 0$, then the gain of the regularity in Hölder norm should be less than or equal to $1/\eta$. To get a necessary condition for Hölder estimates, we first need a complete analysis of boundary geometry near $z_0 \in b\Omega$ of finite type.

In this paper we prove a necessary condition for the sharp Hölder estimates of $\bar\partial$-equation near $z_0 \in b\Omega$ when Ω is a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ and the Levi-form of $b\Omega$ at $z_0 \in b\Omega$ has $(n-2)$-positive eigenvalues. Our method used to prove the following main theorem will be useful for a study of necessary conditions of Hölder estimates of $\bar\partial$-equation for other kinds of finite type domains.

Theorem 1. *Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ and assume that the Levi-form of $b\Omega$ at $z_0 \in b\Omega$ has $(n-2)$-positive eigenvalues. Assume that there is a smooth holomorphic curve V whose order of contact with $b\Omega$ at $z_0 \in b\Omega$ is larger than or equal to η. If there exists a neighborhood U of z_0 and a constant $C > 0$ so that, for each $\alpha \in L^{0,1}_{\infty}(\Omega)$ with $\bar\partial\alpha = 0$, there is a $u \in \Lambda_{\epsilon}(U \cap \bar\Omega)$ such that $\bar\partial u = \alpha$ and*

$$\|u\|_{\Lambda_{\epsilon}(U\cap\bar\Omega)} \le C\|\alpha\|_{L_{\infty}(\Omega)}, \tag{3}$$

then $\epsilon \le 1/\eta$.

To prove Theorem 1 we use the analysis of the local geometry near $z_0 \in b\Omega$ in [7] and use the method developed in [6]. In particular Proposition 4 is a key coordinate change which shows that z_1 which represents the smooth variety V and the terms mixed with z_1 and strongly pseudoconvex directions vanishes up to order $m := [(\eta+1)/2]$, where $[x]$ denotes the largest integer less than or equal to x.

Remark 2. In general, we note that $N := T_{BG}(z_0) \le \eta$. Thus we have $\epsilon \le 1/\eta \le 1/N$ in (3). We also note that η is a positive integer.

2. Special Coordinates

Let (Ω, z_0, η) be as in the statement of Theorem 1 and let r be a smooth defining function of Ω near z_0. We may assume that there is a coordinate system $\tilde z = (\tilde z_1, \dots, \tilde z_n)$ about z_0 such that $z_0 = 0$ and $|\partial r/\partial \tilde z_n| \ge c > 0$, for some constant $c > 0$, in a small neighborhood U of z_0. In this section, we construct special coordinates $z = (z_1, \dots, z_n)$ near $z_0 \in b\Omega$ which change the given smooth holomorphic curve V into the z_1-axis. We will exclude the trivial case, $\eta = 2$, and hence we assume that $\eta \ge 3$ is a positive integer. Set $m := [(\eta+1)/2]$.

As in the proof of Proposition 2.2 in [7], after a linear change of coordinates followed by standard holomorphic changes of coordinates, we can remove inductively the pure terms such as $\tilde z_1^j$, $\bar{\tilde z}_1^k$ terms as well as $\tilde z_1^j \tilde z_\alpha$, $\bar{\tilde z}_1^j \bar{\tilde z}_\alpha$ terms, $2 \le \alpha \le n-1$, in the Taylor series expansion of $r(\tilde z)$ so that $r(\tilde z)$ can be written as

$$\begin{aligned} r(\tilde z) &= \operatorname{Re}\tilde z_n + \sum_{j+k\le\eta,\, j,k>0} \tilde a_{j,k}\tilde z_1^j \bar{\tilde z}_1^k + \sum_{\alpha=2}^{n-1}|\tilde z_\alpha|^2 \\ &\quad + \sum_{\alpha=2}^{n-1}\sum_{j+k\le m,\, k>0}\operatorname{Re}\left(\tilde a^{\alpha}_{j,k}\tilde z_1^j\bar{\tilde z}_1^k z_\alpha\right) \\ &\quad + \mathcal{O}\left(|\tilde z_n||\tilde z| + |\tilde z''|^2|\tilde z| + |\tilde z''||\tilde z_1|^{m+1} + |\tilde z_1|^{\eta+1}\right), \end{aligned} \tag{4}$$

where $\tilde z'' = (\tilde z_2, \dots, \tilde z_{n-1})$. Let V be the smooth 1-dimensional variety satisfying (2). Without loss of generality, we may assume that (2) is satisfied in $\tilde z$-coordinates defined in (4). Let $\gamma : \mathbb{C} \to V$, $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$, be a local parametrization of V. We may assume that $\gamma_1'(0) \ne 0$, and, hence, after reparametrization, we can write $\gamma(t) = (t, \gamma_2(t), \dots, \gamma_n(t))$ and it satisfies

$$|r(\gamma(t))| \le C|t|^{\eta}. \tag{5}$$

Lemma 3. *$\gamma_n(t)$ vanishes to order at least η.*

Proof. The proof is similar to the proof of Lemma 2.3 in [6]. Since $\gamma(0) = 0$, $\gamma_n(t)$ vanishes to order $s > 0$. Suppose that $s < \eta$; that is, $\gamma_n(t) = a_s t^s + \mathcal{O}(t^{s+1})$ for $s < \eta$. In terms of z coordinates in (4), we can write

$$\begin{aligned} r(\gamma(t)) &= \left(\frac{a_s}{2}t^s + \frac{\bar a_s}{2}\bar t^s\right) + \sum_{j+k\le\eta+1,\, j,k>0} c_{j,k}t^j\bar t^k \\ &\quad + \mathcal{O}\left(t^{s+1}\right). \end{aligned} \tag{6}$$

Since $r(\gamma(t))$ vanishes to order at least η, there must be some cancelation between the parenthesis part and summation part. However, this is impossible because parenthesis part consists only of pure terms while summation part consists of mixed power terms. □

Proposition 4. *There is a holomorphic coordinate system z with $\Phi(z) = \tilde z$ such that, in terms of z coordinates, $\tilde r(z) := r \circ \Phi(z)$ can be written as*

$$r̃(z) = \operatorname{Re} z_n + \sum_{j+k=\eta,\, j,k>0} a_{j,k} z_1^j \bar z_1^k + \sum_{\alpha=2}^{n-1}|z_\alpha|^2$$

$$+\sum_{\alpha=2}^{n-1}\sum_{j+k=m,\,k>0}\operatorname{Re}\left(a^{\alpha}_{j,k}z_1^j\bar{z}_1^k z_\alpha\right)$$
$$+\mathcal{O}\left(|z_n||z|+|z''|^2|z|+|z''||z_1|^{m+1}+|z_1|^{\eta+1}\right), \tag{7}$$

and it satisfies

$$|\tilde{r}(t,0,\ldots,0,0)|\lesssim|t|^{\eta}. \tag{8}$$

Proof. With $\tilde{z}$-coordinates defined in (4), define $\Phi:\mathbb{C}^n\to\mathbb{C}^n$, $\tilde{z}=\Phi(z)$, by

$$\Phi(z)=\left(z_1,z_2+\gamma_2(z_1),\ldots,z_{n-1}+\gamma_{n-1}(z_1),z_n\right), \tag{9}$$

and set $\tilde{r}(z)=r\circ\Phi(z)$. In terms of z coordinates, $\tilde{r}(z)$ can be written as

$$\begin{aligned}\tilde{r}(z)&\\ &=\operatorname{Re}z_n+\sum_{j+k\le\eta,\,j,k>0}a_{j,k}z_1^j\bar{z}_1^k+\sum_{\alpha=2}^{n-1}|z_\alpha|^2\\ &\quad+\sum_{\alpha=2}^{n-1}\sum_{2\le j+k\le m,\,k>0}\operatorname{Re}\left(a^{\alpha}_{j,k}z_1^j\bar{z}_1^k z_\alpha\right)\\ &\quad+\mathcal{O}\left(|z_n||z|+|z''|^2|z|+|z''||z_1|^{m+1}+|z_1|^{\eta+1}\right).\end{aligned} \tag{10}$$

Since $\gamma_n(t)$ vanishes to order η, it follows from (5), (9), and (10) that

$$|\tilde{r}(t,0,\ldots,0)|=\left|r\left(t,\gamma_2(t),\ldots,\gamma_{n-1}(t),0\right)\right|\lesssim|t|^{\eta}, \tag{11}$$

and hence (8) is proved. Also we note that

$$\tilde{r}(t,0,\ldots,0)=\sum_{j+k\le\eta,\,j,k>0}a_{j,k}t^j\bar{t}^k+\mathcal{O}\left(|t|^{\eta+1}\right), \tag{12}$$

and hence $a_{j,k}=0$, for $j+k<\eta$, because of (8). This fact together with (10) proves that the first summation part in (7) is homogeneous polynomial of order η.

Now we want to show that $a^{\alpha}_{j,k}=0$, for $j+k<m$, in the third summation part in (7). On the contrary, let $0<s<m$ be the least integer such that $a^{\alpha}_{j,k}\ne0$ for some $j+k=s$ and α. In order to show that this is a contradiction, we use variants of the methods in Lemma 4.1 and Proposition 4.4 in [8]. For t with $0<t<1$, define a scaling map

$$z=H^s_t(z):=\left(t^{1/2s}w_1,t^{1/2}w_2,\ldots,t^{1/2}w_{n-1},tw_n\right), \tag{13}$$

and set $\rho^t_s=t^{-1}((H^s_t)*\tilde{r})$ and then set $\tilde{\rho}=\lim_{t\to0^+}\rho^t_s$. Note that $2s<\eta$, and hence the first summation part in (7) will be disappeared in this limiting process. Also note that $\tilde{\rho}$ is the limit in the C^∞-topology of ρ^t_s which, for each $t>0$, is a defining function of a pseudoconvex domain Ω_t, and hence $\tilde{\rho}$ is a defining function of a pseudoconvex domain $\tilde{\Omega}$ given by

$$\tilde{\rho}(w)=\operatorname{Re}w_n+\sum_{\alpha=2}^{n-1}|w_\alpha|^2+\operatorname{Re}\sum_{\alpha=2}^{n-1}P_\alpha(w_1,\bar{w}_1)w_\alpha, \tag{14}$$

where $P_\alpha(w_1,\bar{w}_1)$ is a plurisubharmonic, nonholomorphic, polynomial of order s provided it is nontrivial. Therefore the Hessian matrix $A:=(\partial^2\tilde{\rho}/\partial w_j\partial\bar{w}_k)_{1\le j,k\le n-1}$ is semidefinite Hermitian matrix and hence $\det A\ge0$. Note that

$$\det A=2\operatorname{Re}\sum_{\alpha=2}^{n-1}\frac{\partial^2P_\alpha}{\partial w_1\partial\bar{w}_1}w_\alpha-\sum_{\alpha=2}^{n-1}\left|\frac{\partial P_\alpha}{\partial\bar{w}_1}\right|^2\ge0. \tag{15}$$

Assume P_α is nontrivial for some α; say, $\alpha=2$. For each $|w_1|<1$, take an appropriate argument of w_2 satisfying $\operatorname{Re}(\partial^2P_2/\partial w_1\partial\bar{w}_1)w_2\le0$. By (15), it follows that $\partial P_\alpha/\partial\bar{w}_1=0$ at $w=(w_1,w_2,0,\ldots,0)$, and hence P_α is holomorphic function of w_1 at w for each $2\le\alpha\le n-1$. This is a contradiction proving our proposition. □

3. A Construction of Special Functions

Let us take the coordinates $z=(z_1,\ldots,z_n)$ defined in Proposition 4 near $z_0\in b\Omega$. In this section, we construct a family of uniformly bounded holomorphic functions $\{f_\delta\}_{\delta>0}$ with large derivatives in z_n-direction along some curve $\Gamma\subset\Omega$ defined in (39).

In the sequel, we set $z'=(z_2,\ldots,z_n)$ and $z''=(z_2,\ldots,z_{n-1})$. We will consider slices of Ω in z_1-direction. From (7), $r_\delta(z'):=\tilde{r}(d\delta^{1/\eta},z_2,\ldots,z_n)$ can be written as

$$\begin{aligned}r_\delta(z')&=\operatorname{Re}z_n+b_\eta\delta+\sum_{k=2}^{n-1}|z_k|^2+\sum_{\alpha=2}^{n-1}\operatorname{Re}\left(b_\alpha\delta^{m/\eta}z_\alpha\right)\\ &\quad+\mathcal{O}\Big(|z_n|^{\delta^{1/\eta}}+|z_n||z'|+|z''|^3+|z''|\delta^{(m+1)/\eta}\\ &\quad+|z''|^2\delta^{1/\eta}+\delta^{1+1/\eta}\Big),\end{aligned} \tag{16}$$

where $b_\eta=d^\eta\sum_{j+k=\eta,\,j,k>0}a_{j,k}$ and where $a_{j,k}$'s are fixed constants in (7). Note that $b_\eta\in\mathbb{R}^1$. Define

$$\begin{aligned}w''&=z'',\\ w_n&=z_n+b_\eta\delta,\end{aligned} \tag{17}$$

and write $w'=z'$ for a convenience. Then $b_\eta\delta$ term is absorbed in the expression of (16).

Let π be the projection onto $b\Omega$ along z_n-direction. Set $z_\delta=(d\delta^{1/\eta},0,\ldots,0)$ and set $\tilde{z}_\delta=\pi(z_\delta):=(d\delta^{1/\eta},0,\ldots,0,\tilde{z}_n)$. Note that $|\tilde{z}_n|\le\delta$. Define a biholomorphism $\Phi_\delta:\mathbb{C}^{n-1}\to\mathbb{C}^{n-1}$, $\Phi_\delta(\zeta')=z'=(\zeta'',\Phi_n(\zeta'))$, by

$$\begin{aligned}\zeta''&=z'',\\ \Phi_n(\zeta')&=\zeta_n+\tilde{z}_n-\sum_{\alpha=2}^{n-1}b_\alpha\delta^{m/\eta}\zeta_\alpha,\end{aligned} \tag{18}$$

and set $\rho_\delta(\zeta'):=r_\delta\circ\Phi_\delta(\zeta')$. Then $\rho_\delta(0')=0$, and, in terms of ζ' coordinates, $\rho_\delta(\zeta')$ can be written as

$$\begin{aligned}\rho_\delta(\zeta')&=\operatorname{Re}\zeta_n+\sum_{k=2}^{n-1}|\zeta_k|^2+\mathcal{O}\left(|\zeta_n||\zeta'|+|\zeta''|^3\right)\\ &\quad+\mathcal{O}\left(|\zeta_n|\delta^{1/\eta}+|\zeta''|\delta^{(m+1)/\eta}+|\zeta''|^2\delta^{1/\eta}+\delta^{1+1/\eta}\right).\end{aligned} \tag{19}$$

Set $\widetilde{\Omega}_\delta := \Omega \cap \{(d\delta^{1/\eta}, z_2, \ldots, z_n)\}$, the z_1 slice of Ω, and set $\widetilde{U}_\delta = U \cap \{(d\delta^{1/\eta}, z_2, \ldots, z_n)\}$. Also set $\Omega_\delta = \Phi_\delta^{-1}(\widetilde{\Omega}_\delta)$, and set $U_\delta = \{(d\delta^{1/\eta}, \zeta'); \Phi(d\delta^{1/\eta}, \zeta') \in \widetilde{U}_\delta\}$. Then Ω_δ is pseudoconvex domain in $\mathbb{C}^{n-1}$ and $b\Omega_\delta \cap U_\delta$ is uniformly strongly pseudoconvex, independent of $\delta > 0$, provided U is sufficiently small. In the same manner as in Proposition 4.1 in [9] or Proposition 2.5 in [10] (our case is much simpler because $b\Omega_\delta \cap U_\delta$ is uniformly strongly pseudoconvex independent of δ), we can push out $b\Omega_\delta$ near $\widetilde{z}_\delta \in b\Omega_\delta \cap U_\delta$ uniformly independent of $\delta > 0$: For each small $\gamma > 0$, set $B_\gamma = \{\zeta' : |\zeta'| < \gamma\}$. Set

$$J_\delta(\zeta') = \left(\delta^2 + |\zeta_n|^2 + \sum_{k=2}^{n-1} |\zeta_k|^4\right)^{1/2}, \tag{20}$$

and for each small $\sigma > 0$ we set

$$W_{\delta,a,\sigma} = \{\zeta' : \rho_\delta(\zeta') < \sigma J_\delta(\zeta')\} \cap B_a, \tag{21}$$

where $a > 0$ is chosen so that $B_{2a} \subset U_\delta$. Then $W_{\delta,a,\sigma}$ is the maximally pushed out domain of Ω_δ near $\widetilde{z}_\delta$ reflecting strong pseudoconvexity.

To connect the pushed out part $W_{\delta,a,\sigma}$ and Ω_δ, we use a bumping family $\{\Omega_\delta^t\}_{0\le t\le\tau} \subset \mathbb{C}^{n-1}$ with front B_a as in Theorem 2.3 in [11] or Theorem 2.6 in [10] (again the construction of a bumping family is much simpler because Ω_δ is uniformly strongly pseudoconvex). Set

$$D_{\delta,\sigma}^t = (\Omega_\delta^t \setminus B_a) \cup (W_{\delta,a,\sigma} \cap \Omega_\delta^t). \tag{22}$$

Then $D_{\delta,\sigma}^t$ becomes a pseudoconvex domain in $\mathbb{C}^{n-1}$ which is pushed out near the origin provided $t > 0$ and $\sigma > 0$ are sufficiently small. In the sequel, we fix these t_0 and σ_0 and we note that these choices of t_0 and $\sigma_0 > 0$ are independent of $\delta > 0$. Set $D_\delta := D_{\delta,\sigma_0}^{t_0} \subset \mathbb{C}^{n-1}$.

According to Section 3 of [10], or by a method similar to dimension two case of [9], there exists $L^2(D_\delta)$ holomorphic function f_δ satisfying

$$\left|\frac{\partial f_\delta}{\partial \zeta_n}\left(0, \ldots, 0, -\frac{b\delta}{2}\right)\right| \ge \frac{1}{\delta}, \tag{23}$$

for some $b \in \mathbb{R}$ independent of δ where b is taken so that $(0, \ldots, 0, -b\delta/2) \in \Omega_\delta \subset \mathbb{C}^{n-1}$. Note that f_δ is independent of z_1.

Recall that the domains Ω_δ or D_δ are the domains in $\mathbb{C}^{n-1}$ obtained by fixing $\zeta_1 = d\delta^{1/\eta}$. Define a biholomorphism $\Psi : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\Psi(\zeta_1, \zeta') = (\zeta_1, \Phi_\delta(\zeta')), \tag{24}$$

and set $\rho(\zeta) = \widetilde{r} \circ \Psi(\zeta)$. For a small constant $0 < c < d$ to be determined, set

$$P_{\delta,c} := \{\zeta : |\zeta_1 - d\delta^{1/\eta}| < c\delta^{1/\eta},\ |\zeta_k| < a_1,\ k = 2, \ldots, n\}, \tag{25}$$

where $a_1 = a/2n$. In terms of ζ coordinates, for each $0 < \sigma \le \sigma_0$, and for each $0 < c < d$, set

$$\Omega_{\delta,c}^\sigma = P_{\delta,c} \cap \{(\zeta_1, \zeta') : \rho(d\delta^{1/\eta}, \zeta') < \sigma J_\delta(\zeta')\}, \tag{26}$$

which is obtained by moving $W_{\delta,a,\sigma}$ along ζ_1 direction, and set

$$\Omega_{\delta,c} = P_{\delta,c} \cap \{\zeta : \rho(\zeta) < 0\}. \tag{27}$$

Note that $\Omega_{\delta,c}^\sigma$ and $\Omega_{\delta,c}$ are small neighborhoods of z_δ including ζ_1 direction.

Lemma 5. *For sufficiently small $c > 0$, we have $\Omega_{\delta,c} \Subset \Omega_{\delta,c}^{\sigma/2}$, or, equivalently,*

$$\rho(d\delta^{1/\eta}, \zeta') - \rho(\zeta) < \frac{\sigma}{2} J_\delta(\zeta'), \quad \text{for } \zeta = (\zeta_1, \zeta') \in P_{\delta,c}. \tag{28}$$

Proof. Assume $\zeta \in \Omega_{\delta,c}$. Then

$$|\rho(\zeta) - \rho(d\delta^{1/\eta}, \zeta')| \le c\delta^{1/\eta} \max_{|\widetilde{\zeta}_1 - d\delta^{1/\eta}| < c\delta^{1/\eta}} |D_1\rho(\widetilde{\zeta}_1, \zeta')|. \tag{29}$$

Note that Φ_δ is independent of ζ_1. Since $\rho(\zeta) = \widetilde{r} \circ \Psi(\zeta)$, it follows from (7) and (24) that

$$|D_1\rho(\widetilde{\zeta}_1, \zeta')| \lesssim \delta^{1-1/\eta} + |\zeta_n| + |\zeta''|^2 + \delta^{(m-1)/\eta}|\zeta''| \lesssim \delta^{-1/\eta} J_\delta(\zeta'), \tag{30}$$

because $\delta^{(m-1)/\eta}|\zeta''| \le \delta^{-1/\eta}(\delta^{2m/\eta} + |\zeta''|^2)$ and $2m \ge \eta$. Combining (29) and (30), we obtain (28) provided $c > 0$ is sufficiently small. □

For each $\sigma > 0$ and $a_2 > 0$, set $U_{\delta,a_2}^\sigma := \overline{\Omega}_{\delta,a_2}^\sigma$. Since f_δ is independent of ζ_1, we see that f_δ is holomorphic on $\Omega_{\delta,c}^\sigma$. We will show that f_δ is bounded uniformly on $U_{\delta,a_2}^{\sigma/8}$ for some $0 < a_2 < c \le a_1$ to be determined. For each $q = (q_1, q') \in U_{\delta,a_2}^{\sigma/8}$, set $\tau_1 = a_2\delta^{1/\eta}$, $\tau_k = (a_2 J_\delta(q'))^{1/2}$, $2 \le k \le n-1$, and $\tau_n = a_2 J_\delta(q')$, and define a nonisotropic polydisc $Q_{a_2}(q)$ by

$$Q_{a_2}(q) := \{\zeta : |\zeta_k - q_k| < \tau_k,\ 1 \le k \le n\}. \tag{31}$$

In order to proceed as in Section 7 of [9], we first show the following lemma which is similar to Lemma 4.3 in [9].

Lemma 6. *There is an independent constant $0 < a_2 < c$ such that*

$$Q_{a_2}(q) \subset U_{\delta,a_2}^{3\sigma/4} \subset \Omega_{\delta,c}^\sigma, \quad \text{for } q = (q_1, q') \in U_{\delta,a_2}^{\sigma/8}. \tag{32}$$

Proof. Assume $\zeta = (\zeta_1, \zeta') \in Q_{a_2}(q)$. Then we have

$$\begin{aligned} J_\delta(q')^2 &\le \delta^2 + 2|\zeta_n|^2 + 2|\zeta_n - q_n|^2 \\ &\quad + 8\sum_{k=2}^{n-1}\left(|\zeta_k|^4 + |\zeta_k - q_k|^4\right) \\ &\le 8J_\delta(\zeta')^2 + 2\left(a_2 J_\delta(q')\right)^2 \\ &\quad + 8\sum_{k=2}^{n-1}\left(a_2 J_\delta(q')\right)^2 \\ &= 8J_\delta(\zeta')^2 + (8n-14)\,a_2^2 J_\delta(q')^2. \end{aligned} \tag{33}$$

If we take $a_2 > 0$ so that $(8n-14)a_2^2 \le 1/2$, we obtain that $(1/4)J_\delta(q') \le J_\delta(\zeta')$. This shows that $q \in Q_{\tilde{a}_2}(\zeta)$, where $\tilde{a}_2 = 4a_2$. By the same argument, we have $J_\delta(\zeta') \le 4J_\delta(q')$ provided $(8n-14)\tilde{a}_2^2 \le 1/2$. Therefore, if $0 < a_2 \le 1/8 \cdot 1/\sqrt{4n-7}$, we obtain that

$$\frac{1}{4}J_\delta(q') \le J_\delta(\zeta') \le 4J_\delta(q'), \quad \text{for } \zeta \in Q_{a_2}(q). \tag{34}$$

Since $\delta^{m/\eta} \le \delta^{1/2} \le J_\delta(\zeta')^{1/2}$, it follows from (7) that

$$\begin{aligned} |D_k\rho(\zeta)| &\lesssim J_\delta(\zeta')^{1/2}, \\ &\qquad \zeta = (\zeta_1, \zeta') \in Q_{a_2}(q),\ 2 \le k \le n-1. \end{aligned} \tag{35}$$

Combining (34) and (35), one obtains

$$\left|\nabla'\rho\left(d\delta^{1/\eta}, \tilde{\zeta}'\right)\cdot\left(\zeta' - q'\right)\right| \le C_2 {a_2}^{1/2} J_\delta(q'), \tag{36}$$

for each $\zeta, \tilde{\zeta} \in Q_{a_2}(q)$, for some $C_2 > 0$, where ∇' denotes the gradient of $\zeta' = (\zeta_2, \dots, \zeta_n)$ variables.

Now we prove (32). Assume $q \in U^{\sigma/8}_{\delta,a_2}$ and $\zeta \in Q_{a_2}(q)$. Since $\rho(d\delta^{1/\eta}, q') \le (\sigma/8)J_\delta(q')$, we can write

$$\rho\left(d\delta^{1/\eta}, \zeta'\right) \le \frac{\sigma}{8}J_\delta(q') + \left|\nabla'\rho\left(d\delta^{1/\eta}, \tilde{\zeta}'\right)\left(\zeta' - q'\right)\right|, \tag{37}$$

for some $\tilde{\zeta} = (\tilde{\zeta}_1, \tilde{\zeta}') \in Q_{a_2}(q)$. Combining (34), (36), and (37), we obtain that

$$\begin{aligned} \rho\left(d\delta^{1/\eta}, \zeta'\right) &\le \frac{\sigma}{2}J_\delta(\zeta') + C_2 a_2^{1/2} J_\delta(q') \\ &< \frac{3\sigma}{4}J_\delta(\zeta'), \end{aligned} \tag{38}$$

provided $16C_2 a_2^{1/2} < \sigma$. This proves (32). □

Let $b_\eta = d^\eta \sum_{j+k=\eta,\, j,k>0} a_{j,k}$ be the number in (16), and define

$$\Gamma = \left\{z : z = \left(d\delta^{1/\eta}, 0, \dots, 0, -\frac{b\delta}{2} - b_\eta\delta\right),\ \delta > 0\right\}. \tag{39}$$

Then $\Gamma = \Psi(\tilde{\Gamma})$, where $\tilde{\Gamma} = \{\zeta : \zeta = (d\delta^{1/\eta}, 0, \dots, 0, -b\delta/2)\}$ and where Ψ is defined in (24), and $b > 0$ is the number in (23). Note that $\Gamma \subset \Omega$ for all sufficiently small $\delta > 0$ provided $d > 0$ is sufficiently small.

Remark 7. In the above discussion, $\sigma > 0$ is any number such that $0 < \sigma \le \sigma_0$. Thus, in particular, we can fix $\sigma = \sigma_0$.

Theorem 8. *f_δ is bounded holomorphic function in $\Omega^{\sigma/8}_{\delta,a_2}$ and, along Γ, f_δ satisfies*

$$\left|\frac{\partial f_\delta}{\partial \zeta_n}\left(d\delta^{1/\eta}, 0, \dots, 0, -\frac{b\delta}{2}\right)\right| \ge \frac{1}{\delta}, \tag{40}$$

for some $b \in \mathbb{R}$ independent of δ.

Proof. By (23) and (24), we already know that there is a L^2 holomorphic function f_δ on $\Omega^\sigma_{\delta,c}$ satisfying estimate (40). We only need to show that f_δ is bounded in $\Omega^{\sigma/8}_{\delta,a_2}$. Assume $q \in U^{\sigma/8}_{\delta,a_2} = \overline{\Omega}^{\sigma/8}_{\delta,a_2} \Subset \Omega^\sigma_{\delta,c}$. Then $Q_{a_2}(q) \subset \Omega^\sigma_{\delta,c}$ by Lemma 6. Now if we use the mean value theorem on polydisc $Q_{a_2}(q) \subset \Omega^\sigma_{\delta,c}$ and the fact that $f_\delta \in L^2(\Omega^\sigma_{\delta,c})$ is holomorphic we will get the boundedness of f_δ on $\overline{\Omega}^{\sigma/8}_{\delta,a_2}$. □

4. Proof of Theorem 1

Without loss of generality, we may assume that $\Omega = \{\zeta \in \mathbb{C}^n; \rho(\zeta) < 0\}$, where $\rho(\zeta) = \rho \circ \Psi(\zeta)$ and where Ψ is given in (24). Let $f = f_\delta$ be the bounded holomorphic function in $\Omega^{\sigma/8}_{\delta,a_2}$ defined in Theorem 8, and set $\alpha = \bar{\partial} g_\delta$, where

$$\begin{aligned} g_\delta = {} & \phi\left(\frac{|\zeta_1 - d\delta^{1/\eta}|}{c\delta^{1/\eta}}\right)\phi\left(\frac{|\zeta_2|}{a_2}\right)\phi\left(\frac{|\zeta_3|}{a_2}\right) \\ & \cdots\phi\left(\frac{|\zeta_n|}{a_2}\right) f(0, \zeta_2, \dots, \zeta_n) \end{aligned} \tag{41}$$

and where

$$\phi(t) = \begin{cases} 1, & |t| \le \dfrac{1}{2}, \\ 0, & |t| \ge \dfrac{3}{4}. \end{cases} \tag{42}$$

Note that

$$\|\alpha\|_{L^\infty} \lesssim \delta^{-1/\eta}. \tag{43}$$

Now set

$$h(\zeta_1, \dots, \zeta_n) = u(\zeta_1, \zeta_2, \dots, \zeta_n) - g_\delta, \tag{44}$$

where $u \in \Lambda_\epsilon(U \cap \Omega)$ solves $\bar{\partial} u = \alpha$ as in the statement of Theorem 1, and hence h is holomorphic. Set $q_1^\delta(\theta) = (d\delta^{1/\eta} + (4/5)c\delta^{1/\eta}e^{i\theta}, 0, \dots, 0, -b\delta/2)$ and $q_2^\delta(\theta) = (d\delta^{1/\eta} + (4/5)c\delta^{1/\eta}e^{i\theta}, 0, \dots, 0, -b\delta)$, where $\theta \in \mathbb{R}$. Let us estimate the lower and upper bounds of the integral

$$H_\delta = \left|\frac{1}{2\pi}\int_0^{2\pi}\left[h\left(q_1^\delta(\theta)\right) - h\left(q_2^\delta(\theta)\right)\right]d\theta\right|. \tag{45}$$

From the definition of ϕ we have $g_\delta(q_1^\delta(\theta)) = g_\delta(q_2^\delta(\theta)) = 0$, and it follows from (3) and (43) that

$$\begin{aligned} H_\delta &= \left| \frac{1}{2\pi} \int_0^{2\pi} \left[u\left(q_1^\delta(\theta)\right) - u\left(q_2^\delta(\theta)\right) \right] d\theta \right| \\ &\lesssim \delta^\epsilon \|\alpha\|_{L^\infty} \lesssim \delta^{\epsilon - 1/\eta}. \end{aligned} \tag{46}$$

For the lower bound estimate, we start with an estimate of the holomorphic function $f = f_\delta$ with a large nontangential derivative constructed in Theorem 8. For each sufficiently small $\delta > 0$, set $\zeta'_\delta = (0, \ldots, 0, -b\delta/2)$ and $\widetilde{\zeta}'_\delta = (0, \ldots, 0, -b\delta)$, and set $\zeta_\delta = (d\delta^{1/\eta}, \zeta'_\delta)$ and $\widetilde{\zeta}_\delta = (d\delta^{1/\eta}, \widetilde{\zeta}'_\delta)$. Then Taylor's theorem of f in ζ_n variable shows that

$$\begin{aligned} f(0, \ldots, 0, \zeta_n) &= f\left(\zeta'_\delta\right) + \frac{\partial f}{\partial \zeta_n}\left(\zeta'_\delta\right)\left(\zeta_n + \frac{b\delta}{2}\right) \\ &\quad + \mathcal{O}\left(\left|\zeta_n + \frac{b\delta}{2}\right|^2\right). \end{aligned} \tag{47}$$

Now we take $\zeta_n = -b\delta$. Since $|(\partial f/\partial \zeta_n)(\zeta'_\delta)| \geq 1/\delta$, it follows that

$$\left| f\left(\widetilde{\zeta}'_\delta\right) - f\left(\zeta'_\delta\right) \right| = \left| \frac{\partial f}{\partial \zeta_n}\left(\zeta'_\delta\right)\left(-\frac{b\delta}{2}\right) + \mathcal{O}\left(\delta^2\right) \right| \gtrsim 1, \tag{48}$$

for all sufficiently small $\delta > 0$. Returning to the lower bound estimate of H_δ, the mean value property, (3), (43), and (48) give us

$$\begin{aligned} H_\delta &= \left| \frac{1}{2\pi} \int_0^{2\pi} \left[h\left(q_1^\delta(\theta)\right) - h\left(q_2^\delta(\theta)\right) \right] d\theta \right| \\ &= \left| h\left(\zeta_\delta\right) - h\left(\widetilde{\zeta}_\delta\right) \right| \\ &\geq \left| f\left(\widetilde{\zeta}'_\delta\right) - f\left(\zeta'_\delta\right) \right| - \left| u\left(\widetilde{\zeta}_\delta\right) - u\left(\zeta_\delta\right) \right| \\ &\gtrsim 1 - \delta^{\epsilon - 1/\eta}, \end{aligned} \tag{49}$$

because $g_\delta(\zeta_\delta) = f(\zeta'_\delta)$ and $g_\delta(\widetilde{\zeta}_\delta) = f(\widetilde{\zeta}'_\delta)$. If we combine (46) and (49), we obtain that

$$1 \lesssim \delta^{\epsilon - 1/\eta}. \tag{50}$$

If we assume $\epsilon > 1/\eta$ and $\delta \to 0$, (50) will be a contradiction. Therefore, $\epsilon \leq 1/\eta$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] N. Kerzman, "Hölder and L_p estimates on the solutions of $\overline{\partial} = f$ in a strongly pseudoconvex domains," *Communications on Pure and Applied Mathematic*, vol. 24, pp. 301–379, 1971.

[2] S. G. Krantz, "Characterizations of various domains of holomorphy via $\overline{\partial}$ estimates and applications to a problem of Kohn," *Illinois Journal of Mathematics*, vol. 23, no. 2, pp. 267–285, 1979.

[3] J. McNeal, "On sharp Hölder estimates for the solutions of the $\overline{\partial}$ equations, Several complex variables and complex geometry," in *Proceedings of Symposia in Pure Mathematics*, vol. 52, part 3, pp. 277–285, American Mathematical Society, Providence, RI, USA, 1989.

[4] R. M. Range, "On Hölder estimates for $\overline{\partial} u = f$ on weakly pseudoconvex domains," in *Proceedings of International Conference on Onseveral Complex Variables*, pp. 247–267, Cortona, Italy, 1976-1977.

[5] E. J. Straube, "A Remark on hölder smoothing and subellipticity of the ∂-neumann operator," *Communications in Partial Differential Equations*, vol. 20, no. 1-2, pp. 267–275, 1995.

[6] Y. H. You, "Necessary conditions for Hölder regularity gain of $\overline{\partial}$ equation in C^n," http://arxiv.org/abs/1504.05432.

[7] S. Cho, "Boundary behavior of the Bergman kernel function on some pseudo convex domains in $\mathbb{C}^n$," *Transactions of the American Mathematical Society*, vol. 345, no. 2, pp. 803–817, 1994.

[8] D. W. Catlin, "Boundary invariants of pseudoconvex domains," *Annals of Mathematics*, vol. 120, no. 3, pp. 529–586, 1984.

[9] D. W. Catlin, "Estimates of invariant metrics on pseudoconvex domains of dimension two," *Mathematische Zeitschrift*, vol. 200, no. 3, pp. 429–466, 1989.

[10] S. Cho, "Estimates of invariant metrics on some pseudoconvex domains in $\mathbb{C}^n$," *Journal of the Korean Mathematical Society*, vol. 32, pp. 661–678, 1995.

[11] S. Cho, "Extension of complex structures on weakly pseudoconvex compact complex manifolds with boundary," *Mathematische Zeitschrift*, vol. 211, no. 1, pp. 105–120, 1992.