

Research Article

The Rate at Which the Energy of Solutions for a Class of p -Laplacian Wave Equation Decays

Soufiane Mokeddem¹ and Khaled Ben Walid Mansour²

¹Laboratory of Biomathematics, Djillali Liabes University, PB 89, 22000 Sidi Bel Abbes, Algeria

²Department of Mathematics, Djillali Liabes University, 22000 Sidi Bel Abbes, Algeria

Correspondence should be addressed to Soufiane Mokeddem; s_mokeddem@yahoo.com

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We will investigate the decay estimate of the energy of the global solutions to the p -Laplacian wave equation with dissipation of the form $u_{tt} - \operatorname{div}(|\nabla_x u|^{p-2} \nabla_x u) + \sigma(t)(u_t - \operatorname{div}(|\nabla_x u_t|^{m-2} \nabla_x u_t)) = 0$ under suitable assumptions on the positive function σ . For this end we use the multiplier method combined with nonlinear integral inequalities given by Martinez; the proof is based on the construction of a special weight function that depends on the behavior of σ .

1. Introduction

In this paper we are concerned with the energy decay rate of the p -Laplacian type wave equation of the form

$$\begin{aligned} u_{tt} - \operatorname{div}(|\nabla_x u|^{p-2} \nabla_x u) \\ + \sigma(t)(u_t - \operatorname{div}(|\nabla_x u_t|^{m-2} \nabla_x u_t)) &= 0 \\ &\text{in } \Omega \times [0, +\infty[, \\ u &= 0 \quad \text{on } \Gamma \times [0, +\infty[, \\ u(x, 0) &= u_0(x), \\ u_t(x, 0) &= u_1(x) \end{aligned} \quad (P)$$

where Ω is a bounded domain in \mathbb{R}^n with smooth boundary $\Gamma = \partial\Omega$, $p, m \geq 2$ are real numbers, and σ is a positive function satisfying some conditions to be specified later.

Problem (P) can be considered as a system describing the longitudinal motion of a viscoelastic configuration obeying a nonlinear Voigt model. The bibliography of works in this direction is so long. We mention, for instance, the works of

Andrews [1], Andrews and Ball [2], Ang and Dinh [3], and Kawashima and Shibata [4].

Existence of global solution and the decay property of the energy for the wave equation related to the problem (P) have been investigated by many authors through various approaches (see [5–8]).

In [5], Benaissa and Messaoudi considered the following problem:

$$\begin{aligned} u_{tt} - \Delta u + a(1 + |u_t|^{m-2})u_t &= -bu|u|^{p-2} \\ &\text{in } \Omega \times [0, +\infty[, \\ u &= 0 \quad \text{on } \Gamma \times [0, +\infty[, \\ u(x, 0) &= u_0(x), \\ u_t(x, 0) &= u_1(x) \end{aligned} \quad (1)$$

where $a, b > 0$. They showed that, for suitably chosen initial data, the problem has a global weak solution, which decays exponentially even if $m > 2$. Further they proved the global existence by using the potential well theory introduced by Sattinger [9].

Similar results have been established by Ye [7, 10]. In these works the author used the Faedo-Galerkin approximation together with compactness criteria and difference inequality introduced by Nakao [11].

More related studies of the nonlinear p -Laplacian wave equation type with damping term can be found in the papers [6, 8, 12, 13].

In [14], with considering $-\operatorname{div}(|\nabla_x u_t|^{m-2} \nabla_x u_t)$ instead of the damping term $\sigma(t)(u_t - \operatorname{div}(|\nabla_x u_t|^{m-2} \nabla_x u_t))$, we have obtained global existence result by using the argument in [15] combined with the concepts of so-called stable sets due to Sattinger [9]. We have also shown the asymptotic behavior of global solutions through the use of the integral inequality given by Komornik [16].

The purpose of this paper is to give an energy decay estimate of the solution of problem (P). Our proof is based on the multiplier method combined with nonlinear integral inequalities given by Martinez [17].

The plan of the paper is as follows. In the next section we present some assumptions, technical lemmas, and main result. Then in Section 3 we are devoted to the proof of decay estimate.

For simplicity of notation, we denote by $\|\cdot\|_p$ the Lebesgue space $L^p(\Omega)$ norm. In particular $\|\cdot\|_2$ denotes $L^2(\Omega)$ and (\cdot, \cdot) the inner product of $L^2(\Omega)$. We also write equivalent norm $\|\nabla\cdot\|_p$ instead of $W_0^{1,p}(\Omega)$ norm $\|\cdot\|_{W_0^{1,p}(\Omega)}$. As usual, we write, respectively, $u(t)$ and $u_t(t)$ instead of $u(x, t)$ and $u_t(x, t)$. Furthermore, throughout this paper the functions considered are all real valued.

2. Preliminaries and Main Result

First assume that $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a nonincreasing positive function of class C^1 on \mathbb{R}_+ , satisfying

$$\int_0^{+\infty} \sigma(\tau) d\tau = +\infty. \quad (2)$$

Let us define the energy equality associated with the solution of the problem (P) by the following formula:

$$E(t) = \frac{1}{2} \|u_t\|_2^2 + \frac{1}{p} \|\nabla u\|_p^p, \quad (3)$$

for $u \in W_0^{1,p}(\Omega)$ and $t \geq 0$.

We state, without proof, a global existence result for the problem (P). For more details we refer the reader to [6].

Theorem 1. *Let $2 \leq m \leq p$ and assume that $(u_0, u_1) \in W_0^{1,p}(\Omega) \times L^2(\Omega)$. Then, for any $T > 0$, the problem (P) has a unique strong solution $u(t)$ on $\Omega \times [0, T]$ in the class*

$$u(t) \in L^\infty([0, T], W_0^{1,p}(\Omega)) \cap W^{1,\infty}([0, T], L^2(\Omega)) \cap W^{1,m}([0, T], W_0^{1,m}(\Omega)). \quad (4)$$

We now present the following well-known lemmas which will be needed later.

Lemma 2 (energy identity). *Let $u(x, t)$ be a global solution to the problem (P) on $[0, \infty)$. Then one has*

$$E(t) + \int_\Omega \int_0^t \sigma(s) u_t \left(u_t - \operatorname{div}(|\nabla_x u_t|^{m-2} \nabla_x u_t) \right) ds dx = E(0), \quad (5)$$

for all $t \in [0, \infty)$.

Lemma 3 (Sobolev-Poincaré inequality). *Let $p \in \mathbb{N}^*$ and $r \in \mathbb{R}$ with $2 \leq r < +\infty$ ($n = 1, 2, \dots, p$) or $2 \leq r \leq np/(n-p)$ ($n \geq p+1$). Then there is a constant $c_* = c_*(\Omega, p, r)$ such that*

$$\|u\|_r \leq c_* \|\nabla u\|_p \quad \text{for } u \in W_0^{1,p}(\Omega). \quad (6)$$

The case $p = r = 2$ gives the known Poincaré's inequality.

Before stating our main result, we introduce the following lemma which plays an important part in studying the decay estimate of energy associated with the solution of the problem (P).

Lemma 4 (see [17]). *Let $E : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a nonincreasing function and $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ an increasing C^2 function such that*

$$\begin{aligned} \phi(0) &= 0, \\ \phi(t) &\longrightarrow +\infty \quad \text{as } t \longrightarrow +\infty. \end{aligned} \quad (7)$$

Assume that there exist $q \geq 0$ and $\gamma > 0$ such that

$$\begin{aligned} \int_S^{+\infty} E(t)^{q+1} \phi'(t) dt &\leq \gamma^{-1} E(0)^q E(S), \\ 0 &\leq S < +\infty. \end{aligned} \quad (8)$$

Then one has

$$\begin{aligned} E(t) &\leq E(0) \left(\frac{1+q}{1+q\gamma\phi(t)} \right)^{1/q} \quad \forall t \geq 0 \text{ if } q > 0, \\ E(t) &\leq E(0) \exp(1-\gamma\phi(t)) \quad \forall t \geq 0 \text{ if } q = 0. \end{aligned} \quad (9)$$

Now we are in position to state and prove our main result.

Theorem 5. *Let $(u_0, u_1) \in W_0^{1,p}(\Omega) \times L^2(\Omega)$ and $p \geq m \geq 2$. Suppose that (2) holds. Then the solution $u(x, t)$ of the problem (P) satisfies the following energy decay estimates.*

(1) *If $p = m$, then there exists a positive constant ω such that*

$$E(t) \leq c(E(0)) \exp\left(1 - \omega \int_0^t \sigma(\tau) d\tau\right) \quad \forall t > 0. \quad (10)$$

(2) *If $p > m$, then there exists a positive constant $c(E(0))$ depending continuously on $E(0)$ such that*

$$E(t) \leq \left(\frac{c(E(0))}{\int_0^t \sigma(\tau) d\tau} \right)^{p(m-1)/(p-m)} \quad \forall t > 0. \quad (11)$$

3. Proof of Main Result

From now on, we denote by c various positive constants depending on the known constants and they may be different at each appearance.

Multiplying by $E^q \phi'(t) u$ on both sides of the first equation of (P) and integrating over $\Omega \times [T, S]$, where ϕ is a function satisfying all the hypotheses of Lemma 4 and $0 \leq S \leq T \leq +\infty$, we obtain that

$$0 = \int_S^T E^q \phi' \int_{\Omega} u \left[u_{tt} - \operatorname{div}(|\nabla_x u|^{p-2} \nabla_x u) + \sigma(t) (u_t - \operatorname{div}(|\nabla_x u_t|^{m-2} \nabla_x u_t)) \right] dx dt. \quad (12)$$

By an integration by parts we see that

$$\begin{aligned} 0 = & \left[E^q \phi' \int_{\Omega} uu_t \right]_S^T - \int_S^T (qE' E^{q-1} \phi' + E^q \phi'') \int_{\Omega} uu_t dx dt \\ & - \int_{\Omega} uu_t dx dt - \int_S^T E^q \phi' \int_{\Omega} |u_t|^2 dx dt \\ & + \int_S^T E^q \phi' \int_{\Omega} |\nabla u|^p dx dt + \int_S^T E^q \phi' \sigma(t) \\ & \cdot \int_{\Omega} u (u_t - \operatorname{div}(|\nabla_x u_t|^{m-2} \nabla_x u_t)) dx dt. \end{aligned} \quad (13)$$

Hence from the definition of energy and a simple argument we can obtain

$$\begin{aligned} p \int_S^T E^{q+1} \phi' dt = & - \left[E^q \phi' \int_{\Omega} uu_t \right]_S^T \\ & + \int_S^T (qE' E^{q-1} \phi' + E^q \phi'') \int_{\Omega} uu_t dx dt + \left(\frac{p}{2} + 1 \right) \\ & \cdot \int_S^T E^q \phi' \int_{\Omega} |u_t|^2 dx dt - \int_S^T E^q \phi' \sigma \\ & \cdot \int_{\Omega} u (u_t - \operatorname{div}(|\nabla_x u_t|^{m-2} \nabla_x u_t)) dx dt. \end{aligned} \quad (14)$$

Now we must estimate both sides of (14) to arrive at a similar inequality as (8).

Define

$$\phi(t) = \int_0^t \sigma(\tau) d\tau. \quad (15)$$

It is clear that ϕ is a nondecreasing function of class C^2 on \mathbb{R}_+ and hypothesis (2) ensures that

$$\phi(t) \longrightarrow +\infty \quad \text{as } t \longrightarrow +\infty. \quad (16)$$

Since E is nonincreasing, ϕ' is a bounded nonnegative function on \mathbb{R}_+ (we denote by μ its maximum) and, using the definition of energy, Cauchy-Schwartz inequality, and Sobolev-Poincaré inequality, we have

$$- \left[E^q \phi' \int_{\Omega} uu_t dx \right]_S^T \leq c\mu E(S)^{q+1/2+1/p}, \quad (17)$$

where the above estimate follows from the fact that

$$\begin{aligned} \int_{\Omega} uu_t dx & \leq \|u\|_2 \|u_t\|_2 \leq c \|\nabla u\|_p \|u_t\|_2 \\ & \leq cE(t)^{1/p} E(t)^{1/2}. \end{aligned} \quad (18)$$

Again, exploiting Cauchy-Schwartz inequality, Sobolev-Poincaré inequality, the definition of energy, and (18), we obtain

$$\begin{aligned} & \left| \int_S^T (qE' E^{q-1} \phi' + E^q \phi'') \int_{\Omega} uu_t dx dt \right| \\ & \leq c\mu \int_S^T -E' E^{q-1/2+1/p} dt \\ & \quad + \int_S^T cE^{q+1/2+1/p} (-\phi'') dt \leq c\mu E(S)^{q+1/2+1/p}, \end{aligned} \quad (19)$$

where the fact that $E(t)$ is nonincreasing is used.

Furthermore, by using Lemma 2, we have

$$\begin{aligned} & \left(1 + \frac{p}{2} \right) \int_S^T E^q \phi' \int_{\Omega} |u_t|^2 dx dt \\ & \leq \left(1 + \frac{p}{2} \right) \int_S^T E^q \phi' \int_{\Omega} (|u_t|^2 + |\nabla u_t|^m) dx dt \\ & \leq \left(1 + \frac{p}{2} \right) \int_S^T E^q \phi' \left(-\frac{E'(t)}{\sigma(t)} \right) dt \leq cE^{q+1}(S). \end{aligned} \quad (20)$$

We then estimate the last term in (14) as follows:

$$\begin{aligned} & \left| \int_S^T E^q \phi' \sigma(t) \int_{\Omega} u (u_t - \operatorname{div}(|\nabla_x u_t|^{m-2} \nabla_x u_t)) dx dt \right| \\ & \leq \left| \int_S^T E^q \phi' \sigma(t) \int_{\Omega} uu_t dx dt \right| \\ & \quad + \left| \int_S^T E^q \phi' \sigma(t) \int_{\Omega} u \operatorname{div}(|\nabla_x u_t|^{m-2} \nabla_x u_t) dx dt \right|, \end{aligned} \quad (21)$$

which implies that

$$\begin{aligned} & \left| \int_S^T E^q \phi' \sigma(t) \int_{\Omega} u \operatorname{div}(|\nabla_x u_t|^{m-2} \nabla_x u_t) dx dt \right| \\ & \leq \int_S^T E^q \phi' \sigma(t) \int_{\Omega} |\nabla u| |\nabla u_t|^{m-1} dx dt. \end{aligned} \quad (22)$$

Using Hölder's and Sobolev-Poincaré's inequalities, we see that

$$\begin{aligned} & \left| \int_S^T E^q \phi' \sigma(t) \int_{\Omega} u \operatorname{div}(|\nabla_x u_t|^{m-2} \nabla_x u_t) dx dt \right| \\ & \leq \int_S^T E^q \phi' \sigma(t) \|\nabla u\|_p \|\nabla u_t\|_{p(m-1)/(p-1)}^{m-1} dt. \end{aligned} \quad (23)$$

We also have

$$\begin{aligned} \|\nabla u_t\|_{p(m-1)/(p-1)}^{m-1} & \leq c |\Omega|^{(p-m)/pm} \|\nabla u_t\|_m^{m-1}, \\ \|\nabla u_t\|_m^{m-1} & \leq \left(\frac{-E'(t)}{\sigma(t)} \right)^{(m-1)/m}. \end{aligned} \quad (24)$$

This gives

$$\begin{aligned} & \left| \int_S^T E^q \phi' (t) \sigma (t) \int_{\Omega} u \operatorname{div} \left(|\nabla_x u_t|^{m-2} \nabla_x u_t \right) \right| \\ & \leq c \int_S^T E^{q+1/p} \phi' \sigma (t) \left(\frac{-E' (t)}{\sigma (t)} \right)^{(m-1)/m} dt \quad (25) \\ & = c \int_S^T E^{q+1/p} \phi' \sigma (t)^{1/m} (-E' (t))^{(m-1)/m} dt. \end{aligned}$$

Further, by Young inequality, we have for $\varepsilon > 0$

$$\begin{aligned} & \left| \int_S^T E^q \phi' (t) \sigma (t) \int_{\Omega} u \operatorname{div} \left(|\nabla_x u_t|^{m-2} \nabla_x u_t \right) \right| \\ & \leq \frac{1}{m} \varepsilon^m \int_S^T E^{(q+1/p)m} (\phi' (t))^m \sigma (t) dt \\ & \quad + \frac{m-1}{m} \frac{1}{\varepsilon (m/(m-1))} E (S) \quad (26) \\ & = \frac{1}{m} \varepsilon^m \int_S^T E^{(q+1/p)m} \phi' (t) (\sigma (t))^m dt \\ & \quad + \frac{m-1}{m} \frac{1}{\varepsilon (m/(m-1))} E (S). \end{aligned}$$

We choose q such that

$$m \left(q + \frac{1}{p} \right) = q + 1. \quad (27)$$

Thus $q = (p-m)/p(m-1)$.

Combining estimates (17)–(26), (14) becomes

$$\begin{aligned} & \int_S^T E^{q+1} \phi' (t) dt \\ & \leq c E (S)^{q+1/2+1/p} + c' E (S)^{q+1} + c'' E (S) \quad (28) \\ & \leq \left(\frac{c E (0)^{q-1/2+1/p} + c' E (0)^q + c''}{E (0)^q} \right) E (0)^q E (S), \end{aligned}$$

where c , c' , and c'' are different positive constants independent of $E(0)$.

Letting $T \rightarrow +\infty$, this yields the following estimate:

$$\begin{aligned} & \int_S^{+\infty} E^{q+1} \phi' (t) dt \\ & \leq \left(\frac{c E (0)^{q-1/2+1/p} + c' E (0)^q + c''}{E (0)^q} \right) E (0)^q E (S), \quad (29) \end{aligned}$$

$$\forall S \geq 0,$$

and we conclude from Lemma 4 that

$$\begin{aligned} E (t) & \leq \left(c E (0)^{q-1/2+1/p} + c' E (0)^q + c'' \right)^{1/q} \\ & \quad \cdot \left(\frac{1+q}{q} \right)^{1/q} \int_0^t \sigma (t) dt. \quad (30) \end{aligned}$$

It is clear that, for $p = m$, we have $q = 0$ and the energy $E(t)$ associated with the solution of the problem (P) satisfies the decay property in (10).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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