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*Research Article*

# On a System of Equations of a Non-Newtonian Micropolar Fluid

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We investigate a problem for a model of a non-Newtonian micropolar fluid coupled system. The problem has been considered in a bounded, smooth domain of $\mathbb{R}^3$ with Dirichlet boundary conditions. The operator stress tensor is given by $\tau(e(u)) = [(\nu + \nu_0 M(|e(u)|^2))e(u)]$. To prove the existence of weak solutions we use the method of Faedo-Galerkin and compactness arguments. Uniqueness and periodicity of solutions are also considered.

## 1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial\Omega$, and let $T > 0$. We denote by $Q_T$ the time space cylinder $I \times \Omega$, with lateral boundary $\Sigma = I \times \partial\Omega$, where $I = (0, T)$ is a time interval. The unsteady flows of incompressible fluids in a boundary domain $\Omega \subset \mathbb{R}^d$, $d > 1$, are described by the system of equations

$$\begin{gathered} \rho \frac{\partial u}{\partial t} - \nabla \cdot \tau(e(u)) + \rho (u \cdot \nabla) u = -\nabla p + \rho f \quad \text{in } Q_T, \\ \nabla \cdot u = 0 \quad \text{in } Q_T, \\ u = 0 \quad \text{on } \Sigma_T, \\ u(0) = u_0 \quad \text{in } \Omega, \end{gathered} \tag{1}$$

where $u = (u_1, u_2, \dots, u_d)$ is the velocity, $p$ represents the pressure, $\rho$ is a positive constant determining the density of a material, $f = (f_1, f_2, \dots, f_d)$ stands for the given external body forces, $\tau : \mathbb{R}^{d^2}_{\text{sym}} \to \mathbb{R}^{d^2}_{\text{sym}}$ denotes the extra stress tensor, $e : \mathbb{R}^d \to \mathbb{R}^{d^2}_{\text{sym}}$ denotes the symmetric part of the velocity gradient; that is,

$$e(u) = \frac{1}{2}\left[\nabla u + (\nabla u)^T\right], \tag{2}$$

whose components are defined as in [1] by

$$2e_{ij}(u) = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}, \quad i, j = 1, 2, \dots, d \tag{3}$$

and $\mathbb{R}^{d^2}_{\text{sym}}$ represents the set of all symmetric $d \times d$ matrices; that is,

$$\mathbb{R}^{d^2}_{\text{sym}} = \left\{D \in \mathbb{R}^{d^2}; D_{ij} = D_{ji},\ i, j = 1, 2, \dots, d\right\}. \tag{4}$$

Note, for example, that when $\tau(e(u))$ is of the form

$$\tau(e(u)) = \mu_0 \left(1 + |e(u)|^{p-2}\right) e(u), \tag{5}$$

with $p = 2$, problem (1) turns into the Navier-Stokes system, which is a model for Newtonian fluids. In the expression (5), $|e(u)|$ denotes the usual Euclidean matrix norm. We observe that (5) can be written in the form

$$\tau(e(u)) = \mu_0 M\left(|e(u)|^2\right) e(u), \tag{6}$$

where $M : \mathbb{R}^+_0 \to \mathbb{R}^+_0$, $M \in C^0(0, \infty)$ is the generalized viscosity function. Fluids constituted by (6) are sometimes named fluids with shear-dependent viscosity. Models belonging to this class of non-Newtonian fluid mechanics are frequently used in several fields of chemistry, glaciology, biology, and geology, as discussed by Malek et al. [2].

The first mathematical investigations of problem (1) was done by Ladyzhenskaya in 1963, where she proposed to study system (1) with (5) and $p = 4$. Combining monotone operator theory and compactness arguments, she proved the existence of weak solution to model (1), if $p \geq 1 + (2d/(d+2))$, and their uniqueness if $p \geq (d+2)/2$. See also Lions [3] for another proof of the same results. More results are known about problem (1) obtained in a series of papers, including those of Malek et al. [2], Malek et al. [4], Frehse and Málek [5], Malek et al. [1], and other mathematicians.

The equations below describe the motion of Newtonian micropolar fluids:

$$
\begin{gathered}
\frac{\partial u}{\partial t} - (\nu + \nu_r)\,\Delta u + (u\cdot\nabla)\,u + \nabla p = 2\nu_r \nabla\times w + f \quad \text{in } Q_T,\\
\frac{\partial w}{\partial t} - (c_a + c_d)\,\Delta w + (u\cdot\nabla)\,w - (c_o + c_d - c_a)\,\nabla(\nabla\cdot w)\\
+ \lambda w = 2\nu_r \nabla\times u + g \quad \text{in } Q_T,\\
\nabla\cdot u = 0 \quad \text{in } Q_T,\\
u = 0 \quad \text{on } \Sigma_T,\\
w = 0 \quad \text{on } \Sigma_T,\\
u(0) = u_0 \quad \text{in } \Omega,\\
w(0) = w_0 \quad \text{in } \Omega,
\end{gathered}
\tag{7}
$$

where $u(x,t), w(x,t) \in \mathbb{R}^3$ and $p(x,t) \in \mathbb{R}$, denoting for $(x,t) \in Q$, respectively, the unknown velocity, the microrotational velocity, and the hydrostatic pressure of the fluid and $\lambda$ is a positive constant. The positive constants $\nu$ and $\nu_r$ are, respectively, the *Newtonian* and *microrotational viscosity*. The positive constants $c_0$, $c_a$, and $c_d$ are called *coefficients of angular viscosities* and satisfy $c_0 + c_d > c_a$.

The main difference with respect to modeled fluids by the Navier-Stokes is that the rotation of the particles is taken into account. The above approach was introduced by Eringen [6]. The nonlinear coupled system (7) can be used to model the behavior of liquid crystals, polymeric fluids, and blood under some circumstances (see, e.g., [7]). These systems have been mainly analyzed in the book of Lukaszewicz [8].

The problem that we study in this work consists in supposing that in system (7) the fluid is of the type (5). More precisely, we investigate the mixed problem: let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, and let $T > 0$. We denote by $Q_T$ the time space cylinder $I\times\Omega$, with lateral boundary $\Sigma = I\times\partial\Omega$, where $I = (0,T)$ is a time interval. We find that $u, w : Q_T \to \mathbb{R}^3$ and $p : Q_T \to \mathbb{R}$ solving the following system of equations:

$$
\begin{gathered}
u' - \nabla\cdot\tau(e(u)) + (u\cdot\nabla)\,u + \nabla p = \nabla\times w + f \quad \text{in } Q_T,\\
w' - \nu_1 \nabla\cdot e(w) + (u\cdot\nabla)\,w + \lambda_1 w = \lambda_2 \nabla\times u + g \quad \text{in } Q_T,\\
\nabla\cdot u = 0 \quad \text{in } Q_T,\\
u = 0 \quad \text{on } \Sigma_T,\\
w = 0 \quad \text{on } \Sigma_T,\\
u(0) = u_0 \quad \text{in } \Omega,\\
w(0) = w_0 \quad \text{in } \Omega,
\end{gathered}
\tag{8}
$$

where the extra stress tensor is given by $\tau(e(u)) = (\nu + \nu_0 M(|e(u)|^2))e(u)$, $e(u)$ as in (2) and (3), $\nu_0$, $\nu_1$, $\lambda_1$, and $\lambda_2$ are positives constants, $u = (u_1, u_2, u_3)$, and $\nabla\times u$ is given by

$$
\nabla\times u = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right); \tag{9}
$$

the same holds for $\nabla\times w$. Let us consider $M : (0,\infty) \to (0,\infty)$ satisfying the hypothesis

$$
M \in C^1(0,\infty), \quad M > M_0 > 0, \quad M' > 0, \tag{10}
$$

$$
c_1 |e(u)|^2 \leq M\left(|e(u)|^2\right) \leq c_2 |e(u)|^2, \tag{11}
$$

where $M_0$, $c_1$, and $c_2$ are positive constants. We observe that if $M$ is a constant function, then problem (8) reduces to problem (7).

## 2. Notation and Main Results

In order to solve problem (8) we need some notations about Sobolev spaces. We use standard notation of $L^p(\Omega)$, $W^{m,p}(\Omega)$, and $C^p(\Omega)$ for functions that are defined on $\Omega$ and range in $\mathbb{R}$ and the notation of $\mathbf{L}^p(\Omega)$, $\mathbf{W}^{m,p}(\Omega)$, and $\mathbf{C}^p(\Omega)$ for functions that range in $\mathbb{R}^d$. We also work with the spaces $L^p(I; W^{m,p}(\Omega))$ or $L^p(Q_T)$.

By $\langle\cdot,\cdot\rangle$ we will represent the duality pairing between $X$ and $X'$, with $X'$ being the topological dual of the space $X$. We Also define the followings spaces:

$$
\mathcal{V} = \{\varphi \in \mathcal{D}(\Omega);\, \nabla\cdot\varphi = 0\}. \tag{12}
$$

$V_p = V_p(\Omega)$ is the closure of $\mathcal{V}$ in the space $\mathbf{W}^{1,p}(\Omega)$, $p \in (1,\infty)$. In particular, $V = V_2$. The norm of gradient in $V_p$ is given by

$$
\|\nabla u\|_p \equiv \left[\int_\Omega |\nabla u(x)|^p\, dx\right]^{1/p}. \tag{13}
$$

The inner product and norm in $V$ is given, respectively, by

$$
\begin{gathered}
((u,v)) = \sum_{i,j=1}^{3} \int_\Omega \frac{\partial u_i}{\partial x_j}(x)\,\frac{\partial v_i}{\partial x_j}(x)\,dx,\\
\|u\|^2 = \sum_{i,j=1}^{3} \int_\Omega \left(\frac{\partial u_i}{\partial x_j}(x)\right)^2 dx.
\end{gathered}
\tag{14}
$$

$H = H(\Omega)$ is the closure of $\mathcal{V}$ in the space $\mathbf{L}^2(\Omega)$, with inner product and norm defined, respectively, by

$$
\begin{gathered}
(u,v) = \sum_{i=1}^{3} \int_\Omega u_i(x)\,v_i(x)\,dx,\\
|u|^2 = \sum_{i=1}^{3} \int_\Omega |u_i(x)|^2\, dx.
\end{gathered}
\tag{15}
$$

*Remark 1.* $\mathbf{H}_0^1(\Omega)$ and $\mathbf{L}^2(\Omega)$ are Hilbert's spaces. We note that $\mathbf{H}_0^1(\Omega) \overset{c}{\hookrightarrow} \mathbf{L}^2(\Omega) \hookrightarrow \mathbf{H}^{-1}(\Omega)$, where the first embeddings are compact.

We introduce the following bilinear and the trilinear forms, as well as the convention of summation of indices, that is, $\alpha_i\beta_j$ instead of $\sum_{i,j=1}^{d} \alpha_i\beta_j$:

$$a(u,v) = \int_\Omega \frac{\partial u_i}{\partial x_j}(x)\frac{\partial v_i}{\partial x_j}(x)\,dx = ((u,v)), \quad \forall u,v \in V, \tag{16}$$

$$b(u,v,w) = \int_\Omega u_i(x)\frac{\partial v_j}{\partial x_i}(x)\,w_j(x)\,dx, \quad \forall u,v,w \in V. \tag{17}$$

We note that (see Lions [3])

$$b(u,v,w) = -b(u,w,v), \quad \forall u \in V,\ \forall v,w \in \mathbf{H}_0^1(\Omega). \tag{18}$$

We also introduce the notations

$$Au = -\Delta u, \quad B_u v = (u\cdot\nabla)v, \quad \forall u,v \in V, \tag{19}$$

$$\mathscr{K}u = -\nabla\cdot M\left(|e(u)|^2\right)e(u), \quad \forall u \in V. \tag{20}$$

According to this, we have

$$\langle Au, v\rangle = a(u,v), \quad \forall u,v \in V, \tag{21}$$

$$\langle B_u v, w\rangle = b(u,v,w), \quad \forall u \in V,\ \forall v,w \in \mathbf{H}_0^1(\Omega), \tag{22}$$

$$\langle \mathscr{K}u, v\rangle = \int_\Omega M\left(|e(u)|^2\right)e_{ij}(u)\,e_{ij}(v)\,dx, \quad \forall u,v \in V. \tag{23}$$

*Remark 2.* We observe that $M' > 0$ implies for all $u_1, u_2 \in V$ that

$$\begin{aligned}\langle \mathscr{K}u_1 - \mathscr{K}u_2, u_1 - u_2\rangle &= \int_\Omega \left[M\left(|e(u_1)|^2\right)e_{ij}(u_1) - M\left(|e(u_2)|^2\right)e_{ij}(u_2)\right] \\ &\quad\times\left[e_{ij}(u_1) - e_{ij}(u_2)\right]dx \ge 0.\end{aligned} \tag{24}$$

Therefore $\mathscr{K} : V \to V'$ is a monotonous operator.

*Definition 3.* Let $u_0 \in H, w_0 \in \mathbf{L}^2(\Omega)$, $f \in L^{4/3}(I, V')$, and $g \in L^2(I;\mathbf{H}^{-1}(\Omega))$. A weak solution to (8) is a pair of functions $\{u, w\}$, such that

$$\begin{gathered} u \in L^\infty(I;H)\cap L^4(I;V_4)\cap L^4(I;V), \\ w \in L^\infty\left(I;\mathbf{L}^2(\Omega)\right)\cap L^2\left(I;\mathbf{H}_0^1(\Omega)\right)\end{gathered} \tag{25}$$

satisfying the following identity:

$$\begin{aligned}
&\int_0^T \left\langle u'(t),\varphi\right\rangle dt + \nu\int_0^T a(u(t),\varphi)\,dt \\
&\quad + \int_0^T b(u(t),u(t),\varphi)\,dt \\
&\quad + \nu_0\int_0^T\int_\Omega M\left(|e(u(t))|^2\right)e_{ij}(u(t))\,e_{ij}(\varphi)\,dx\,dt \\
&= \int_0^T (\nabla\times w(t),\varphi)\,dt \\
&\quad + \int_0^T (f(t),\varphi)\,dt, \quad \forall\varphi\in\mathscr{D}(I;\mathscr{V}), \\
&\int_0^T \left\langle w'(t),\phi\right\rangle dt + \nu_1\int_0^T a(w(t),\phi)\,dt \\
&\quad + \nu_1\int_0^T (\nabla\cdot w(t),\nabla\cdot\phi)\,dt \\
&\quad + \int_0^T b(u(t),w(t),\phi)\,dt + \lambda_1\int_0^T (w(t),\phi)\,dt \\
&= \lambda_2\int_0^T (\nabla\times u(t),\phi)\,dt \\
&\quad + \int_0^T (g(t),\phi)\,dt, \quad \forall\phi\in\mathscr{D}(I;\mathscr{D}(\Omega)), \\
&u(0) = u_0, \qquad w(0) = w_0.
\end{aligned} \tag{26}$$

**Lemma 4** (Korn's inequality). *Let $1 < p < \infty$. Then, there exists a constant $K_p = K_p(\Omega)$, such that the inequality*

$$K_p\|v\|_{W^{1,p}(\Omega)} \le \|e(v)\|_{L^p(\Omega)} \tag{27}$$

*is fulfilled for all $v$ satisfying either $v \in W_0^{1,p}(\Omega)$, where $\Omega\subset\mathbb{R}^d$ is open and bounded with $\partial\Omega\subset C^1$.*

*Proof.* The proof of this lemma can be found in [1], page 169. □

**Lemma 5.** *Let $p \ge 2$, $\tau : \mathbb{R}^{d^2}_{sym} \to \mathbb{R}^{d^2}_{sym}$, and $\Phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and the assumptions below are satisfied for all $B, D \in \mathbb{R}^{d^2}_{sym}$ and $i,j,k,l = 1,\dots,d$:*

$$\begin{gathered}
\partial_{ij}\Phi\left(|D|^2\right) = \tau_{ij}(D), \\
\Phi(0) = \partial_{ij}\Phi(0) = 0, \\
\partial_{ij}\partial_{kl}\Phi\left(|D|^2\right)B_{ij}B_{kl} \ge C_1(1+|D|)^{p-2}|B|^2, \\
\left|\partial_{ij}\partial_{kl}\Phi\left(|D|^2\right)\right| \le C_2(1+|D|)^{p-2}.
\end{gathered} \tag{28}$$

*Then, there exist positive constants $C_i$, $i = 3,4,5$, such that*

$$C_3\left(1+|D|^{p-2}\right)|D|^2 \le \Phi\left(|D|^2\right) \le C_4(1+|D|)^p, \tag{29}$$

$$(\tau(B)-\tau(D))\cdot(B-D) \ge C_5|B-D|^2. \tag{30}$$

*Proof.* The proof of this lemma can be found in [4], page 263. □

**Lemma 6** (Vitali). *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and $f^m : \Omega \to \mathbb{R}$ integrable for every $m \in \mathbb{N}$. Assume that*

(1) $\lim_{m\to\infty} f^m(x)$ *exists and is finite for almost all* $x \in \Omega$;

(2) *for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\sup_{m\in\mathbb{N}} \int_H \left|f^m(x)\right| dx < \varepsilon, \quad \forall H \in \Omega,\ |H| < \delta; \tag{31}$$

*then*

$$\lim_{m\to\infty} \int_\Omega f^m(x)\,dx = \int_\Omega \lim_{m\to\infty} f^m(x)\,dx. \tag{32}$$

*Proof.* The proof of this lemma can be found in [9], page 63. □

**Lemma 7.** *Consider $d \geq 3$ and $s, r \in \mathbb{R}$, with $s > 2$, $r > d$, verifying $(2/s) + (d/r) = 1$. If $u \in \mathbf{L}^r(\Omega)$, then*

$$|b(u, v, w)| \leq c\,\|u\|_{L^r(\Omega)}\,\|v\|\,|w|^{2/s}\,\|w\|^{d/r} \tag{33}$$

*for all $v, w \in V$, where $c \geq 0$ is a constant independent of $u$, $v$, and $w$.*

*Proof.* The proof of this lemma can be found in [3], page 84. □

**Theorem 8.** *If $d \leq 3$, $u_0 \in H$, $w_0 \in \mathbf{L}^2(\Omega)$, $f \in L^{4/3}(I; V')$, and $g \in L^2(I; \mathbf{H}^{-1}(\Omega))$, then there exist a weak solution to problem (8).*

**Theorem 9.** *Under the assumptions of Theorem 8 with $d = 2$, problem (8) has a unique weak solution.*

**Theorem 10** (periodic solutions). *Under the assumptions of Theorem 8 there exist a pair of functions $(u, w)$ such that*

$$\begin{gathered}
u \in L^\infty(I; H) \cap L^4(I; V_4) \cap L^4(I; V), \\
w \in L^\infty\left(I; \mathbf{L}^2(\Omega)\right) \cap L^2\left(I; \mathbf{H}_0^1(\Omega)\right), \\
\left(u'(t), \varphi\right) + \nu a(u(t), \varphi) + \nu_0(\mathscr{K}u(t), \varphi) + \left(B_u u(t), \varphi\right) \\
= (\nabla \times w(t), \varphi) + (f(t), \varphi), \\
\left(w'(t), \phi\right) + \nu_1 a(w(t), \phi) + \nu_1(\nabla \cdot w(t), \nabla \cdot \phi) \\
+ \left(B_u w(t), \phi\right) + \lambda_1(w(t), \phi) \\
= \lambda_2(\nabla \times u(t), \phi) + (g(t), \phi), \\
u(0) = u(T), \qquad w(0) = w(T), \\
\forall \varphi \in V,\ \forall \phi \in \mathbf{H}_0^1(\Omega)\ \text{in the sense of}\ D'(0, T).
\end{gathered} \tag{34}$$

**Theorem 11.** *Assuming that $d \leq 3$, $u_0 \in V \cap V_4$, $w_0 \in \mathbf{H}_0^1(\Omega)$, $f \in L^{4/3}(I; V')$, and $g \in L^2(I; \mathbf{H}^{-1}(\Omega))$ there exist a unique weak solution to problem (8) such that*

$$\begin{gathered}
u' \in L^2(I; H), \\
u \in L^\infty(I; V_4), \\
u \in L^\infty(I; V), \\
w \in L^\infty\left(I; \mathbf{H}_0^1(\Omega)\right), \\
w' \in L^2\left(I; \mathbf{L}^2(\Omega)\right).
\end{gathered} \tag{35}$$

## 3. Proofs of the Results

*Proof of Theorem 8.* We will show the existence of a weak solution to system (8) employing the Galerkin approximations. For that purpose we consider $(\varphi_\nu)_{\nu\in\mathbb{N}} \subset V$, a basis of eigenvectors of the Stokes operator and $(\phi_\mu)_{\mu\in\mathbb{N}} \subset \mathbf{H}_0^1(\Omega)$ a basis of eigenvectors of Lamé. We represent by $V_m = [\varphi_1, \ldots, \varphi_m] \subset V$ the subspace generated by $\{\varphi_1, \ldots, \varphi_m\}$ and $W_m = [\phi_1, \ldots, \phi_m] \subset \mathbf{H}_0^1(\Omega)$ the subspace generated by $\{\phi_1, \ldots, \phi_m\}$. Let us also consider the pair $(u_m, w_m)$, such that

$$\begin{gathered}
u_m(x, t) = \sum_{r=1}^m g_{rm}(t)\,\varphi_r(x), \\
w_m(x, t) = \sum_{r=1}^m h_{rm}(t)\,\phi_r(x)
\end{gathered} \tag{36}$$

are the solution of the approximate problem

$$\begin{gathered}
\left(u_m'(t), \varphi_r\right) + \nu\left(Au_m(t), \varphi_r\right) + \nu_0\left(\mathscr{K}u_m(t), \varphi_r\right) \\
+ \left\langle B_{u_m} u_m(t), \varphi_r\right\rangle = \left(\nabla \times w_m(t), \varphi_r\right) + \left(f(t), \varphi_r\right), \\
r = 1, \ldots, m, \\
\left(w_m'(t), \phi_r\right) - \nu_1\left(\nabla \cdot e(w), \phi_r\right) + \left\langle B_{u_m} w_m(t), \phi_r\right\rangle \\
+ \lambda_1\left(w_m(t), \phi_r\right) = \lambda_2\left(\nabla \times u_m(t), \phi_r\right) + \left(g(t), \phi_r\right), \\
r = 1, \ldots, m, \\
u_m(0) = u_{0m} \longrightarrow u_0, \quad \text{strongly in } H, \\
w_m(0) = w_{0m} \longrightarrow w_0, \quad \text{strongly in } \mathbf{L}^2(\Omega).
\end{gathered} \tag{37}$$

The system of ordinary differential equations (37) has a local solution on an interval $[0, t_m[$, $0 < t_m < T$. The first estimate permits us to extend this solution to the whole interval $[0, T]$.

*First Estimate.* We sometimes omit the parameter $t$. Multiplying both sides of $(37)_1$ by $g_{r_m}$ and $(37)_2$ by $h_{r_m}$, next adding from $r = 1$ to $r = m$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left|u_m(t)\right|^2+\nu\left\|u_m(t)\right\|^2\\&\quad+\nu_0\int_\Omega M\left(\left|e\left(u_m(t)\right)\right|^2\right)\left|e_{ij}\left(u_m(t)\right)\right|^2dx\\&\le\left|w_m(t)\right|\left\|u_m(t)\right\|+\left\|f(t)\right\|_{V'}\left\|u_m(t)\right\|,\end{aligned}\tag{38}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left|w_m(t)\right|^2+\nu_1\left\|w_m(t)\right\|^2\\&\quad+\nu_1\left|\nabla\cdot w_m(t)\right|^2+\lambda_1\left|w_m(t)\right|^2\\&\le\lambda_2\left\|u_m(t)\right\|\left|w_m(t)\right|+\left\|g(t)\right\|_{H^{-1}(\Omega)}\left\|w_m(t)\right\|,\end{aligned}\tag{39}$$

because $b(u,u,u)=b(u,w,w)=0$, for all $u\in V$, for all $w\in\mathbf{H}_0^1(\Omega)$ (see Lions [3]), $|\nabla\times u_m|=|\nabla u_m|=\|u_m\|$ and $(\nabla\times w_m,u_m)=(w_m,\nabla\times u_m)$ (see Lukaszewicz [8]). Now using Young's inequality we obtain from (38) and (39), respectively:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left|u_m(t)\right|^2+\nu\left\|u_m(t)\right\|^2\\&\quad+\frac{\nu_0c_1K}{2}\left\|u_m(t)\right\|_{V_4}^4+\nu_2\left\|u_m(t)\right\|^4\\&\le\frac{\nu}{4}\left\|u_m(t)\right\|^2+c_\nu\left|w_m(t)\right|^2\\&\quad+\frac{\nu_2}{2}\left\|u_m(t)\right\|^4+c_{\nu_2}\left\|f(t)\right\|_{V'}^{4/3},\end{aligned}\tag{40}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left|w_m(t)\right|^2+\nu_1\left\|w_m(t)\right\|^2\\&\le\frac{\nu}{4}\left\|u_m(t)\right\|^2+c_\nu\left|w_m(t)\right|^2\\&\quad+\frac{\nu_1}{2}\left\|w_m(t)\right\|^2+c_{\nu_1}\left\|g(t)\right\|_{H^{-1}(\Omega)}^2.\end{aligned}\tag{41}$$

From (27) (Korn's inequality) and (11) we can get

$$\begin{aligned}&\frac{\nu_0}{2}\int_\Omega M\left(\left|e\left(u_m\right)\right|^2\right)\left|e_{ij}\left(u_m\right)\right|^2dx\ge\frac{\nu_0c_1}{2}\left\|e(u_m)\right\|_{L^4(\Omega)}^4\\&\quad\ge\frac{\nu_0c_1K}{2}\left\|u_m\right\|_{V_4}^4,\\&\frac{\nu_0}{2}\int_\Omega M\left(\left|e\left(u_m\right)\right|^2\right)\left|e_{ij}\left(u_m\right)\right|^2dx\ge\frac{\nu_0c_1}{2}\left\|e(u_m)\right\|_{L^4(\Omega)}^4\\&\quad\ge\nu_2\left\|u_m\right\|^4.\end{aligned}\tag{42}$$

Adding inequalities (40) and (41) and integrating from 0 to $t$, with $0\le t\le T$, we conclude

$$\begin{aligned}&\left(\left|u_m(t)\right|^2+\left|w_m(t)\right|^2\right)+\nu_0c_1K\int_0^t\left\|u_m(s)\right\|_{V_4}^4ds\\&\quad+\nu_2\int_0^t\left\|u_m(s)\right\|^4ds+\nu_1\int_0^t\left\|w_m(s)\right\|^2ds\\&\le C+C\int_0^t\left(\left|u_m(s)\right|^2+\left|w_m(s)\right|^2\right)ds.\end{aligned}\tag{43}$$

By using Gronwall's inequality, we can write

$$\left|u_m(t)\right|^2+\left|w_m(t)\right|^2\le C.\tag{44}$$

Therefore, it follows from (43) that

$$\left(u_m\right)\text{ is bounded in }L^\infty(I;H),\tag{45}$$

$$\left(u_m\right)\text{ is bounded in }L^4\left(I;V_4\right),\tag{46}$$

$$\left(u_m\right)\text{ is bounded in }L^4(I;V),\tag{47}$$

$$\left(w_m\right)\text{ is bounded in }L^\infty\left(I;\mathbf{L}^2(\Omega)\right),\tag{48}$$

$$\left(w_m\right)\text{ is bounded in }L^2\left(I;\mathbf{H}_0^1(\Omega)\right).\tag{49}$$

*Second Estimate*. We consider $P_m:V\to V_m$ as the orthogonal projections from $V$ to $V_m$:

$$P_mu=\sum_{j=1}^m\left(u,\varphi_j\right)\varphi_j,\quad\forall u(t)\in V.\tag{50}$$

We also consider the adjoint operator to $P_m$ which is $P_m^*:V'\to V'$. We note that $P_m^*u_m'=u_m'$. By the choice of the special basis $(\varphi_\nu)$, we obtain

$$\left\|P_m\right\|_{\mathscr{L}(V,V)}\le1,\qquad\left\|P_m^*\right\|_{\mathscr{L}(V',V')}\le1.\tag{51}$$

It follows from $(37)_1$, (21), (22), and (23) that

$$u_m'=-\nu P_m^*Au_m-\nu_0P_m^*\mathscr{K}u_m-P_m^*Bu_m+P_m^*\nabla\times w_m+P_m^*f.\tag{52}$$

We have $|\langle Au_m,v\rangle|\le\|u_m\|\|v\|$, for all $u_m(t),v(t)\in V$. Therefore (47) implies

$$\left(Au_m\right)\text{ is bounded in }L^4\left(I;V'\right)\hookrightarrow L^{4/3}\left(I;V'\right).\tag{53}$$

Let $u_m(t),v(t)\in V$. From (20), Hölder's inequality, and (11) we take

$$\begin{aligned}\left|\left\langle\mathscr{K}u_m,v\right\rangle\right|&\le\left|\left\langle M\left(\left|e\left(u_m\right)\right|^2\right)e\left(u_m\right),\nabla v\right\rangle\right|\le c\left|e\left(u_m\right)\right|^3\|v\|\\&\le c\left|\nabla u_m\right|^3\|v\|\le\left\|u_m\right\|^3\|v\|.\end{aligned}\tag{54}$$

Therefore, from (47), we obtain

$$\left(\mathscr{K}u_m\right)\text{ is bounded in }L^{4/3}\left(I;V'\right).\tag{55}$$

From $d\le3$ we derive $H_0^1(\Omega)\hookrightarrow L^4(\Omega)$. Using (17) and Hölder's inequality we conclude

$$\left|\left\langle B_{u_m}u_m,v\right\rangle\right|\le\left\|u_m\right\|_{L^4(\Omega)}\left\|u_m\right\|\left\|v\right\|_{L^4(\Omega)}\le c\left\|u_m\right\|^2\|v\|\tag{56}$$

for all $u_m(t),v(t)\in V$. Therefore, from (47)

$$\left(B_{u_m}u_m\right)\text{ is bounded in }L^2\left(I;V'\right)\hookrightarrow L^{4/3}\left(I;V'\right).\tag{57}$$

On the other hand, let $w_m(t) \in \mathbf{H}_0^1(\Omega)$, and $v(t) \in V$

$$\left|\langle \nabla \times w_m, v\rangle\right| = \left|\langle w_m, \nabla \times v\rangle\right| \le \left|w_m\right| \|v\| \le c\left\|w_m\right\| \|v\| \tag{58}$$

(see Lukaszewicz [8], pp. 116). It follows from (49) that

$$\left(\nabla \times w_m\right) \text{ is bounded in } L^2\left(I; V'\right) \hookrightarrow L^{4/3}\left(I; V'\right). \tag{59}$$

It follows from (53)–(59), (51), and hypothesis about $f$ that

$$\left(u_m'\right) \text{ is bounded in } L^{4/3}\left(I; V'\right). \tag{60}$$

Analogously let $R_m : \mathbf{H}_0^1(\Omega) \to W_m$ be the orthogonal projections

$$R_m w = \sum_{j=1}^{m}\left(w, \phi_j\right)\phi_j, \quad \forall w \in \mathbf{H}_0^1(\Omega). \tag{61}$$

We also consider the adjoint operator to $R_m$, which is $R_m^* : \mathbf{H}^{-1}(\Omega) \to \mathbf{H}^{-1}(\Omega)$. We have $R_m^* w_m' = w_m'$ and by the choice of the special basis $(\phi_\nu)$, we can get

$$\left\|R_m\right\|_{\mathscr{L}(H_0^1(\Omega), H_0^1(\Omega))} \le 1, \qquad \left\|R_m^*\right\|_{\mathscr{L}(H^{-1}(\Omega), H^{-1}(\Omega))} \le 1. \tag{62}$$

From $(37)_2$, (21), (22), and (23)

$$\begin{aligned} w_m' = &-\nu_1 R_m^* \nabla \cdot e\left(w_m\right) - R_m^* B w_m - \lambda_1 R_m^* w_m \\ &+ \lambda_2 R_m^* \nabla \times u_m + R_m^* g. \end{aligned} \tag{63}$$

We note that $|\langle \nabla \cdot e(w_m), v\rangle| = |\langle \nabla w_m, \nabla v\rangle| \le \|w_m\|\|v\|$, for all $w_m(t), v(t) \in \mathbf{H}_0^1(\Omega)$. Thus, (49) implies

$$\left(\nabla \cdot e\left(w_m\right)\right) \text{ is bounded in } L^2\left(I; \mathbf{H}^{-1}(\Omega)\right). \tag{64}$$

Analogously and by using the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ we obtain

$$\left(w_m\right) \text{ is bounded in } L^2\left(I; \mathbf{H}^{-1}(\Omega)\right). \tag{65}$$

On the other hand $|\langle \nabla \times u_m, v\rangle| \le \|u_m\|\|v\|$, for all $u_m(t), v(t) \in \mathbf{H}_0^1(\Omega)$. Now, by using (47), we have

$$\left(\nabla \times u_m\right) \text{ is bounded in } L^2\left(I; \mathbf{H}^{-1}(\Omega)\right). \tag{66}$$

Finally assuming that $d = 2$, it follows from (18), (22), and Hölder's inequality that

$$\begin{aligned} \left|\left\langle B_{u_m} w_m, v\right\rangle\right| &= \left|b\left(u_m, w_m, v\right)\right| = \left|b\left(u_m, v, w_m\right)\right| \\ &\le c\left\|u_m\right\|_{L^4(\Omega)} \|v\| \left\|w_m\right\|_{L^4(\Omega)} \end{aligned} \tag{67}$$

for all $u_m(t), w_m(t), v(t) \in \mathbf{H}_0^1(\Omega)$. Therefore, $\|Bw_m\|_{H^{-1}(\Omega)}^2 \le c|u_m|\|u_m\||w_m|\|w_m\|$.

Because if $d = 2$, then, $\|u\|_{L^4(\Omega)}^2 \le c|u|\|u\|$ (see Lions [3]). Now using Young's inequality we get

$$\left\|B_{u_m} w_m\right\|_{H^{-1}(\Omega)}^2 \le c\left|u_m\right|^2 \left\|u_m\right\|^2 + c\left|w_m\right|^2 \left\|w_m\right\|^2. \tag{68}$$

Therefore, (47)–(49) and (45) permit us to obtain

$$\left(B_{u_m} w_m\right) \text{ is bounded in } L^2\left(I; \mathbf{H}^{-1}(\Omega)\right), \quad d = 2. \tag{69}$$

Analogously and assuming that $d = 3$ we obtain from Lemma 7

$$\left\|B_{u_m} w_m\right\|_{H^{-1}(\Omega)} \le \left\|u_m\right\|_{L^6(\Omega)} \left|w_m\right|^{1/2} \left\|w_m\right\|^{1/2}. \tag{70}$$

$H_0^1(\Omega) \hookrightarrow L^6(\Omega)$, because $d = 3$. Thus,

$$\begin{aligned} \left\|B_{u_m} w_m\right\|_{H^{-1}(\Omega)}^2 &\le c\left\|u_m\right\|^2 \left|w_m\right| \left\|w_m\right\| \\ &\le c\left\|u_m\right\|^4 + c\left|w_m\right|^2 \left\|w_m\right\|^2. \end{aligned} \tag{71}$$

Therefore, (47)–(49) imply that

$$\left(B_{u_m} w_m\right) \text{ is bounded in } L^2\left(I; \mathbf{H}^{-1}(\Omega)\right), \quad d = 3. \tag{72}$$

It follows from (64)–(72) and hypothesis about $g$ that

$$\left(w_m'\right) \text{ is bounded in } L^2\left(I; \mathbf{H}^{-1}(\Omega)\right). \tag{73}$$

We note that (45)–(49), (60), (73), and the Aubin-Lions lemma imply that there exist subsequences of $(u_m)$ and $(w_m)$, still denoted by $(u_m)$ and $(w_m)$, such that

$$u_m \longrightarrow u \text{ strongly in } L^2(I; H), \quad \text{a.e. in } Q_T, \tag{74}$$

$$u_m \longrightarrow u \text{ weak star in } L^\infty(I; H), \tag{75}$$

$$u_m \longrightarrow u \text{ weakly in } L^4(I; V), \tag{76}$$

$$u_m' \longrightarrow u' \text{ weakly in } L^{4/3}\left(I; V'\right), \tag{77}$$

$$w_m \longrightarrow w \text{ strongly in } L^2\left(I; \mathbf{L}^2(\Omega)\right), \quad \text{a.e. in } Q_T, \tag{78}$$

$$w_m \longrightarrow w \text{ weak star in } L^\infty\left(I; \mathbf{L}^2(\Omega)\right), \tag{79}$$

$$w_m \longrightarrow w \text{ weakly in } L^2\left(I; \mathbf{H}_0^1(\Omega)\right), \tag{80}$$

$$w_m' \longrightarrow w' \text{ weakly in } L^2\left(I; H^{-1}(\Omega)\right), \tag{81}$$

$$\mathscr{K} u_m \longrightarrow \chi \text{ weakly in } L^{4/3}\left(I; V'\right). \tag{82}$$

We note that (46) and (60) imply that $u \in C^0(I; \mathbf{H})$. Similarly, (49) and (73) imply that $w \in C^0(I; \mathbf{L}^2(\Omega))$. Thus, it does make sense to consider $u(0) = u_0$ and $w(0) = w_0$.

In order to prove that

$$\begin{gathered} \int_0^T b\left(u_m, u_m, \varphi\right) \longrightarrow \int_0^T b(u, u, \varphi), \quad \forall \varphi \in \mathscr{D}(I; \mathscr{V}), \\ \int_0^T b\left(u_m, w_m, \phi\right) \longrightarrow \int_0^T b(u, w, \phi), \quad \forall \phi \in \mathscr{D}(I; \mathscr{D}(\Omega)), \end{gathered} \tag{83}$$

we use (74) and (78) (see [1], pp. 210). Now, we note that

$$\int_{Q_T} e_{ij}\left(w_m\right) e_{ij}(\phi)\, dx\, dt \longrightarrow \int_{Q_T} e_{ij}(w)\, e_{ij}(\phi)\, dx\, dt, \tag{84}$$

or equivalently

$$\int_0^T \langle \nabla \cdot e(w_m), \phi \rangle \, dt \longrightarrow \int_0^T \langle \nabla \cdot e(w), \phi \rangle \, dt \tag{85}$$

results from (80). The other terms of $(37)_2$ are obtained in the usual manner. In order to prove that

$$\begin{aligned} &\int_{Q_T} M\left(|e(u_m)|^2\right) e_{ij}(u_m) e_{ij}(\varphi) \, dx \, dt \\ &\quad \longrightarrow \int_{Q_T} M\left(|e(u)|^2\right) e_{ij}(u) e_{ij}(\varphi) \, dx \, dt, \end{aligned} \tag{86}$$

we use the fact $\nabla u_m \to \nabla u$ a.e. in $Q_T$, (see [5] pp. 565-566). Therefore,

$$|\nabla u_m|^2 \longrightarrow |\nabla u|^2 \quad \text{a.e. in } Q_T; \tag{87}$$

that is,

$$|e(u_m)|^2 \longrightarrow |e(u)|^2 \quad \text{a.e. in } Q_T. \tag{88}$$

Since $M \in C^1(0, \infty)$ we obtain from (88)

$$M\left(|e(u_m)|^2\right) \longrightarrow M\left(|e(u)|^2\right) \quad \text{a.e. in } Q_T. \tag{89}$$

Thus,

$$M\left(|e(u_m)|^2\right) e_{ij}(u_m) e_{ij}(\varphi) \longrightarrow M\left(|e(u)|^2\right) e_{ij}(u) e_{ij}(\varphi) \tag{90}$$

a.e. in $Q_T$ and for all $\varphi \in \mathscr{D}(I; \mathscr{D}(\Omega))$. Using (46) and (11) we obtain

$$\begin{aligned} &\int_{Q_T} M\left(|e(u_m)|^2\right) e_{ij}(u_m) e_{ij}(\varphi) \, dx \, dt \\ &\quad \leq C \int_{Q_T} |e(u_m)|^3 \left|e_{ij}(\varphi)\right| dx \, dt \\ &\quad \leq C \int_{Q_T} |e(u_m)|^3 \, dx \, dt \leq C \int_{Q_T} |\nabla u_m|^3 \, dx \, dt \leq C. \end{aligned} \tag{91}$$

It follows that

$$M\left(|e(u_m)|^2\right) e_{ij}(u_m) e_{ij}(\varphi) \in L^1(Q_T). \tag{92}$$

Moreover, if $H \subset Q_T$ is a measurable set, we have from (11), (46), and Hölder's inequality that

$$\begin{aligned} &\int_H M\left(|e(u_m)|^2\right) e_{ij}(u_m) e_{ij}(\varphi) \, dx \, dt \\ &\quad \leq c \int_H |e(u_m)|^3 |e(\varphi)| \, dx \, dt \\ &\quad \leq c \left( \int_{Q_T} |e(u_m)|^4 \, dx \, dt \right)^{3/4} \left( \int_H |e(\varphi)|^4 \, dx \, dt \right)^{1/4} \\ &\quad \leq c \left( \int_{Q_T} |\nabla u_m|^4 \, dx \, dt \right)^{3/4} |H|^{1/4} \leq c \, |H|^{1/4}. \end{aligned} \tag{93}$$

Therefore,

$$\sup_{m \in \mathbb{N}} \int_H M\left(|e(u_m)|^2\right) e_{ij}(u_m) e_{ij}(\varphi) \, dx \, dt \leq c \, |H|^{1/4}. \tag{94}$$

Assuming that $|H|$ is sufficiently small, we obtain

$$\sup_{m \in \mathbb{N}} \int_H M\left(|e(u_m)|^2\right) e_{ij}(u_m) e_{ij}(\varphi) \, dx \, dt \leq \varepsilon, \tag{95}$$

for all $\varepsilon \in \mathbb{R}$. Now using (90), (92), (95), and Vitali's lemma we can derive (86). Therefore, we can write $\chi = \mathscr{K} u$ in $L^{4/3}(I; \mathbf{H}^{-1}(\Omega))$. The convergences (74)–(82) and (85) and (86) allow us to pass the limit on system (37), with $\varphi_r$ and $q_r$ being fixed to obtain

$$\begin{aligned} &u' + \nu A u + \nu_0 \mathscr{K} u + B_u u = \nabla \times w + f \quad \text{in } L^{4/3}\left(I; V'\right), \\ &\quad w' + \nu_1 \nabla \cdot e(w) + B_u w + \lambda_1 w \\ &\quad\quad = \lambda_2 \nabla \times u + g \quad \text{in } L^2\left(I; \mathbf{H}^{-1}(\Omega)\right). \end{aligned} \tag{96}$$

This concludes the proof of Theorem 8. □

*Proof of Theorem 9.* Let $(u_1, w_1)$ and $(u_2, w_2)$ be weak solutions to Problem (8). Then,

$$\begin{aligned} &u_1, u_2 \in L^\infty(I; H) \cap L^4(I; V) \cap L^4(I, V_4), \\ &w_1, w_2 \in L^\infty\left(I; \mathbf{L}^2(\Omega)\right) \cap L^\infty\left(I; \mathbf{H}_0^1(\Omega)\right). \end{aligned} \tag{97}$$

Consider $\widetilde{u} = u_1 - u_2$ and $\widetilde{w} = w_1 - w_2$. Then, $(\widetilde{u}, \widetilde{w})$ satisfies

$$\begin{aligned} &\widetilde{u}' + \nu A \widetilde{u} + \nu_0 (\mathscr{K} u_1 - \mathscr{K} u_2) + \left(B_{u_1} u_1 - B_{u_2} u_2\right) = \nabla \times \widetilde{w}, \\ &\quad \widetilde{w}' + \nu_1 A \widetilde{w} + \nu_1 \nabla (\nabla \cdot \widetilde{w}) + \left(B_{u_1} w_1 - B_{u_2} w_2\right) \\ &\quad\quad + \lambda_1 \widetilde{w} = \lambda_2 \nabla \times \widetilde{u}, \\ &\quad\quad\quad \widetilde{u}(0) = \widetilde{w}(0) = 0, \end{aligned} \tag{98}$$

where the first equality has been considered in $L^{4/3}(I; V')$ and the second in $L^2(\mathbf{H}^{-1}(\Omega))$. We take the duality in $(98)_1$ and $(98)_2$ with $\widetilde{u}$ and $\widetilde{w}$, respectively, to obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} |\widetilde{u}|^2 + \nu \|\widetilde{u}\|^2 + \nu_0 \langle \mathscr{K} u_1 - \mathscr{K} u_2, \widetilde{u} \rangle \\ &\quad + \left\langle B_{u_1} u_1 - B_{u_2} u_2, \widetilde{u} \right\rangle = \langle \nabla \times \widetilde{w}, \widetilde{u} \rangle, \\ &\frac{1}{2} \frac{d}{dt} |\widetilde{w}|^2 + \nu_1 \|\widetilde{w}\|^2 + \nu_1 |\nabla \cdot \widetilde{w}|^2 \\ &\quad + \left\langle B_{u_1} w_1 - B_{u_2} w_2, \widetilde{w} \right\rangle + \lambda_1 |\widetilde{w}|^2 = \lambda_2 \langle \nabla \times \widetilde{u}, \widetilde{w} \rangle, \\ &\quad\quad \widetilde{u}(0) = \widetilde{w}(0) = 0. \end{aligned} \tag{99}$$

We note that

$$\left\langle B_{u_1}u_1(t) - B_{u_2}u_2(t), \tilde{u}(t)\right\rangle = b(\tilde{u}(t), u_1(t), \tilde{u}(t)),$$
$$\left\langle B_{u_1}w_1(t) - B_{u_2}w_2(t), \widetilde{w}(t)\right\rangle = b(\tilde{u}(t), w_2(t), \widetilde{w}(t)). \tag{100}$$

From the monotonicity of $\mathscr{K}$ we have $\langle \mathscr{K}u_1 - \mathscr{K}u_2, \tilde{u}\rangle \geq 0$. Thus,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\tilde{u}|^2 + \nu\|\tilde{u}\|^2 + b(\tilde{u}(t), u_1(t), \tilde{u}(t)) \leq \langle \nabla\times\widetilde{w}, \tilde{u}\rangle, \\
&\frac{1}{2}\frac{d}{dt}|\widetilde{w}|^2 + \nu_1\|\widetilde{w}\|^2 + \lambda_1|\widetilde{w}|^2 + b(\tilde{u}(t), w_2(t), \widetilde{w}(t)) \\
&\quad \leq \lambda_2\langle \nabla\times\tilde{u}, \widetilde{w}\rangle.
\end{aligned} \tag{101}$$

Adding the inequalities above, we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(|\tilde{u}|^2 + |\widetilde{w}|^2\right) + \nu\|\tilde{u}\|^2 + \nu_1\|\widetilde{w}\|^2 \\
&\quad \leq |\widetilde{w}|\,\|\tilde{u}\| + \lambda_2\|\tilde{u}\|\,|\widetilde{w}| \\
&\qquad + |b(\tilde{u}, u_1, \tilde{u})| + |b(\tilde{u}, w_2, \widetilde{w})|.
\end{aligned} \tag{102}$$

In other words,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(|\tilde{u}|^2 + |\widetilde{w}|^2\right) + \nu\|\tilde{u}\|^2 + \nu_1\|\widetilde{w}\|^2 \\
&\quad \leq \frac{\nu}{4}\|\tilde{u}\|^2 + \frac{\nu}{4}\|\tilde{u}\|^2 + c_\nu|\widetilde{w}|^2 \\
&\qquad + |b(\tilde{u}, u_1, \tilde{u})| + |b(\tilde{u}, w_2, \widetilde{w})|.
\end{aligned} \tag{103}$$

Considering $d = 2$, we get $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$. Moreover (see Lions [3])

$$|u|_{L^4(\Omega)} \leq c|u|^{1/2}\|u\|^{1/2}. \tag{104}$$

Thus, using (17), Hölder's inequality, (104), and Young's inequality we take

$$\begin{aligned}
&|b(\tilde{u}, u_1, \tilde{u})| + |b(\tilde{u}, w_2, \widetilde{w})| \\
&\quad \leq \|\tilde{u}\|^2_{L^4(\Omega)}\|u_1\| + \|\tilde{u}\|_{L^4(\Omega)}\|w_2\|\,\|\widetilde{w}\|_{L^4(\Omega)} \\
&\quad \leq c\|u_1\|\,|\tilde{u}|\,\|\tilde{u}\| + \|w_2\|\,|\tilde{u}|^{1/2}\|\tilde{u}\|^{1/2}|\widetilde{w}|^{1/2}\|\widetilde{w}\|^{1/2} \\
&\quad \leq \frac{\nu}{4}\|\tilde{u}\|^2 + c_\nu\|u_1\|^2|\tilde{u}|^2 \\
&\qquad + \sqrt{\frac{\nu}{2}}\|\tilde{u}\|\sqrt{2\nu_1}\|\widetilde{w}\| + c|\tilde{u}|\,|\widetilde{w}|\,\|w_2\|^2 \\
&\quad \leq \frac{\nu}{4}\|\tilde{u}\|^2 + c_\nu\|u_1\|^2|\tilde{u}|^2 + \frac{\nu}{4}\|\tilde{u}\|^2 + \nu_1\|\widetilde{w}\|^2 \\
&\qquad + c\|w_2\|^2|\tilde{u}|^2 + c\|w_2\|^2|\widetilde{w}|^2.
\end{aligned} \tag{105}$$

It follows from (103) that we can write

$$\frac{1}{2}\frac{d}{dt}\left(|\tilde{u}|^2 + |\widetilde{w}|^2\right) \leq c\left(1 + \|u_1\|^2 + \|w_2\|^2\right)\left(|\tilde{u}|^2 + |\widetilde{w}|^2\right). \tag{106}$$

Integrating from 0 to $t$ we obtain

$$\begin{aligned}
&|\tilde{u}(t)|^2 + |\widetilde{w}(t)|^2 \\
&\quad \leq c\int_0^t\left(1 + \|u_1(s)\|^2 + \|w_2(s)\|^2\right)\left(|\tilde{u}(s)|^2 + |\widetilde{w}(s)|^2\right)ds.
\end{aligned} \tag{107}$$

Applying Gronwall's inequality in (107), we deduce by using (47) and (49) that

$$u_1(t) = u_2(t), \qquad w_1(t) = w_2(t), \quad \forall t \in [0, T]. \tag{108}$$

Theorem 9 has been proved. □

*Proof of Theorem 10.* Under the assumptions and notations defined in the proof of the Theorem 8 we know that the system (37) has a solution whatever initial value $(u_m(0), w_m(0)) \in V_m \times W_m$. To prove Theorem 10, we first show that there exist an approximate solution for (37), such that

$$(u_m(0), w_m(0)) = (u_m(T), w_m(T)). \tag{109}$$

For this purpose, let us take $\varphi = u_m$ and $\phi = w_m$ in (37) to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u_m|^2 + \nu\|u_m\|^2 + \nu_0\int_\Omega M\left(|e(u_m)|^2\right)|e_{ij}(u_m)|^2 dx \\
&\quad \leq |w_m|\,\|u_m\| + \|f\|_{V'}\|u_m\|,
\end{aligned} \tag{110}$$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|w_m|^2 + \nu_1\int_\Omega |e_{ij}(w_m)|^2 dx + \lambda_1|w_m|^2 \\
&\quad \leq \lambda_2\|u_m\|\,|w_m| + |g|\,|w_m|,
\end{aligned} \tag{111}$$

because $b(u, u, u) = 0$ (see Lions [3]), $|\nabla\times u_m| = |\nabla u_m| = \|u_m\|$, and $(\nabla\times w_m, u_m) = (w_m, \nabla\times u_m)$ (see Lukaszewicz [8]). Using (11) and (10) and (27) (Korn's inequality), we obtain from (110) and (111), respectively:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u_m|^2 + \nu\|u_m\|^2 + \nu_0 M_0|e(u_m)|^2 \\
&\quad \leq |w_m|\,\|u_m\| + \|f\|_{V'}\|u_m\|,
\end{aligned} \tag{112}$$

$$\frac{1}{2}\frac{d}{dt}|w_m|^2 + \nu_1 K\|w_m\|^2 \leq \lambda_2\|u_m\|\,|w_m| + |g|\,|w_m|. \tag{113}$$

After usual computations, we can derive

$$\begin{aligned}
&\frac{d}{dt}\left(|u_m|^2 + |w_m|^2\right) + \nu\|u_m\|^2 + \nu_1 K\|w_m\|^2 \\
&\quad \leq C\left(|u_m|^2 + |w_m|^2 + \|f\|^2_{V'} + |g|^2\right).
\end{aligned} \tag{114}$$

Considering the embeddings $V \hookrightarrow H$ and $\mathbf{H}_0^1(\Omega) \hookrightarrow \mathbf{L}^2(\Omega)$, there exists a constant $c_2$ such that

$$\begin{aligned}
&\frac{d}{dt}\left(|u_m|^2 + |w_m|^2\right) + c_2|u_m|^2 + c_2|w_m|^2 \\
&\quad \leq C\left(|u_m|^2 + |w_m|^2 + \|f\|^2_{V'} + |g|^2\right).
\end{aligned} \tag{115}$$

Multiplying by $e^{c_2 t}$ and integrating on $[0,t)$, we obtain

$$\begin{aligned}|u_m(t)|^2 + |w_m(t)|^2 \le{}& e^{-c_2 t}\left(|u_m(0)|^2 + |w_m(0)|^2\right)\\ &+ C + C\int_0^t \left(|u_m(s)|^2 + |w_m(s)|^2\right) ds.\end{aligned} \tag{116}$$

By Gronwall's inequality, we can write

$$|u_m(t)|^2 + |w_m(t)|^2 \le e^{-c_2 t}\left(|u_m(0)|^2 + |w_m(0)|^2\right) + C, \tag{117}$$

for all $t \in [0,T]$. Let $\theta(t) = e^{-c_2 t}$; we have $0 < \theta(t) < 1$. Thus,

$$|u_m(T)|^2 + |w_m(T)|^2 \le \theta\left(|u_m(0)|^2 + |w_m(0)|^2\right) + C, \tag{118}$$

where $\theta = \theta(t)$ is a positive constant, such that $0 < 1-\theta < 1$. Therefore, $C < C/(1-\theta)$. Taking $R > 0$, such that $C/(1-\theta) < R^2$, we obtain $C < (1-\theta)R^2$. Choosing the initial data $(u_m(0), w_m(0)) \in V_m \times W_m$, such that

$$|u_m(0)|^2 < \frac{R^2}{2}, \qquad |w_m(0)|^2 < \frac{R^2}{2}. \tag{119}$$

We obtain from (118)

$$\begin{aligned}|u_m(T)|^2 + |w_m(T)|^2 \le{}& \theta\left(|u_m(0)|^2 + |w_m(0)|^2\right)\\ &+ C < \theta R^2 + (1-\theta)R^2 = R^2.\end{aligned} \tag{120}$$

Therefore, $|u_m(0)|^2 + |w_m(0)|^2 < R^2$ implies that $|u_m(T)|^2 + |w_m(T)|^2 < R^2$.

Now we define $\sigma : \mathscr{B}_R(0) \cap (V_m \times W_m) \to \mathscr{B}_R(0) \cap (V_m \times W_m)$, such that

$$\sigma(u_m(0), w_m(0)) = (u_m(T), w_m(T)), \tag{121}$$

where $\mathscr{B}_R(0) = \{(u,w) \in H \times \mathbf{L}^2(\Omega);\ |u|^2 + |w|^2 < R\}$. We note that $\sigma$ is a continuous function because the solution of the (37) depends continuously of the initial data. We also note that (118) implies $\sigma(\mathscr{B}_R(0)) \subset \mathscr{B}_R(0)$. Therefore, it follows from Brower fixed-point theorem that $\sigma$ has a fixed point:

$$(u_{0m}, w_{0m}) \in B_R(0) \subset V_m \times W_m. \tag{122}$$

In other words, $\sigma(u_{0m}, w_{0m}) = (u_{0m}, w_{0m})$. Taking the initial data $(u_{0m}, w_{0m})$ in (37), that is, $(u_m(0), w_m(0)) = (u_{0m}, w_{0m})$, we obtain

$$(u_m(0), w_m(0)) = (u_m(T), w_m(T)). \tag{123}$$

Therefore, (37) has a periodic solution. Next, we obtain estimates to (37) with the initial data $(u_{0m}, w_{0m})$ as in the proof of Theorem 8. We obtain

$$u_m \longrightarrow u \text{ weakly in } L^4(I; V_4), \tag{124}$$

$$u_m' \longrightarrow u' \text{ weakly in } L^{4/3}(I; V'), \tag{125}$$

$$w_m \longrightarrow w \text{ weakly in } L^2\left(I; \mathbf{H}_0^1(\Omega)\right), \tag{126}$$

$$w_m' \to w' \text{ weakly in } L^2\left(I; \mathbf{H}^{-1}(\Omega)\right), \tag{127}$$

where $u$ is the solution to problem (8) in the sense of Definition 3. The convergences (124) and (125) allow us to derive

$$\int_0^T \frac{d}{dt}(u_m(s), v)\theta(s)\, ds \longrightarrow \int_0^T \frac{d}{dt}(u(s), v)\theta(s)\, ds, \tag{128}$$

for all $v \in V$ and $\theta \in \mathscr{D}(0,T)$, with $\theta(T) = 0$. In other words,

$$(u_m(0), v) \longrightarrow (u(0), v), \quad \forall v \in V. \tag{129}$$

The same argument with $\theta \in \mathscr{D}(0,T)$ and $\theta(0) = 0$ allows us to derive

$$(u_m(T), v) \longrightarrow (u(T), v), \quad \forall v \in V. \tag{130}$$

It follows from (129) and (130) that $u(0) = u(T)$. Analogously, from (126) and (127), we obtain

$$\begin{aligned}(w_m(0), \overline{v}) &\longrightarrow (w(0), \overline{v}), \quad \forall \overline{v} \in \mathbf{L}^2(\Omega),\\ (w_m(T), \overline{v}) &\longrightarrow (w(T), \overline{v}), \quad \forall \overline{v} \in \mathbf{L}^2(\Omega).\end{aligned} \tag{131}$$

Therefore, $w(0) = w(T)$. Theorem 10 has been proved. □

*Proof of Theorem 11.* Let us consider $\tau : \mathbb{R}^{d^2}_{\text{sym}} \to \mathbb{R}^{d^2}_{\text{sym}}$ and the corresponding potential $\Phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that

$$\begin{aligned}\tau(D) &= M\left(|D|^2\right) D, \quad \forall D \in \mathbb{R}^{d^2}_{\text{sym}},\\ \Phi\left(|D|^2\right) &= \int_0^{|D|^2} M(s)\, ds, \quad \forall D \in \mathbb{R}^{d^2}_{\text{sym}}.\end{aligned} \tag{132}$$

It is possible to verify (see Malek et al. [1]) that (132) satisfy the assumptions (28) of Lemma 5.

To obtain some estimate for $u_m'$, we make $\varphi_r = u_m'$ in $(37)_1$, to obtain applying Schwarz's inequality

$$\begin{aligned}&\left|u_m'\right|^2 + \nu \int_\Omega e_{ij}(u_m)\, e_{ij}\left(u_m'\right) dx\\ &\quad + \nu_0 \int_\Omega \tau_{ij}(e(u_m))\, e_{ij}\left(u_m'\right) dx\\ &\le \int_\Omega \left|u_{m_i}\right| \left|\frac{\partial u_{m_j}}{\partial x_i}\right| \left|u_{m_j}'\right| dx\\ &\quad + \left|u_m'\right| \left|\nabla \times w_m\right| + |f| \left|u_m'\right|.\end{aligned} \tag{133}$$

*Remark 12.* We note that (see [4])

$$\int_\Omega \tau_{ij}(e(v))\, e_{ij}\left(v'\right) dx = \frac{d}{dt}\int_\Omega \Phi\left(|e(v)|^2\right) dx. \tag{134}$$

Applying Young's inequality, (49) and (134), we obtain from (141) that

$$\begin{aligned}&\frac{1}{2}\left|u_m'\right|^2 + \frac{\nu}{2}\frac{d}{dt}\left|e(u_m)\right|^2 + \nu_0 \frac{d}{dt}\int_\Omega \Phi\left(\left|e(u_m)\right|^2\right) dx\\ &\quad \le c + c\int_\Omega \left|u_m\right|^2 \left|\nabla u_m\right|^2 dx.\end{aligned} \tag{135}$$

We observe that (46) implies $u_m(t) \in V_4 \hookrightarrow \mathbf{L}^\infty(\Omega)$ because $d \le 3$. On the other hand, (47) implies $\nabla u_m(t) \in \mathbf{L}^2(\Omega)$. Thus, applying the Hölder's inequality in (135), integrating on $(0,t)$, with $t \le T$, and after applying Korn's inequality, we obtain

$$\begin{aligned}\int_0^t \left|u_m'(s)\right|^2 ds + \nu_1 K \|u_m\|^2 + 2\nu_0 \int_\Omega \Phi\left(\left|e(u_m)\right|^2\right) dx \\ \le c + 2\nu_0 \int_\Omega \Phi\left(\left|e(u_m(0))\right|^2\right) dx + \nu_1 K \|u_m(0)\|^2 \\ + c\int_0^t \left|u_m(s)\right|^2_{L^\infty(\Omega)} \|u_m(s)\|^2 ds.\end{aligned} \quad (136)$$

Observing that (46) implies $\|u_m(t)\|^2_{L^\infty(\Omega)} \in L^2(0,T)$ and also $\|u_m(t)\|^2 \in L^2(0,T)$, because of (47), so we can obtain by using Hölder's inequality in (136)

$$\begin{aligned}\int_0^t \left|u_m'(s)\right|^2 ds + \nu_1 K \|u_m\|^2 \\ + 2\nu_0 \int_\Omega \Phi\left(\left|e(u_m)\right|^2\right) dx \le c\end{aligned} \quad (137)$$

because (29) and $u_0 \in V \cap V_4$. Now we note that (29) and Korn's inequality imply

$$\int_\Omega \Phi\left(\left|e(u_m)\right|^2\right) dx \ge C_3 \|e(u_m)\|^4_{L^4(\Omega)} \ge C_3 K \|u_m\|^4_{V_4}. \quad (138)$$

It follows from (137) that

$$\int_0^t \left|u_m'(s)\right|^2 ds + \nu_1 K \|u_m\|^2 + C_3 K \|u_m\|^4_{V_4} \le c. \quad (139)$$

Inequality (139) permits us to obtain the following estimates:

$$\begin{aligned}&(u_m') \text{ is bounded in } L^2(I;H), \\ &(u_m) \text{ is bounded in } L^\infty(I;V), \\ &(u_m) \text{ is bounded in } L^\infty(I;V_4).\end{aligned} \quad (140)$$

Now we will obtain some estimate for $w_m'$. For that purpose, we make $\phi_r = w_m'$ in $(37)_2$ to obtain

$$\begin{aligned}\left|w_m'\right|^2 + \nu_1 \int_\Omega e_{ij}(w_m) e_{ij}(w_m') dx + \frac{\lambda_1}{2}\frac{d}{dt}\left|w_m\right|^2 \\ \le \int_\Omega \left|u_{m_i}\right| \left|\frac{\partial w_{m_j}}{\partial x_i}\right| \left|w_{m_j}'\right| dx + \lambda_2 \|u_m\| \left|w_m'\right| + |g| \left|w_m'\right|.\end{aligned} \quad (141)$$

Applying Young's inequality and using (47) we derive

$$\begin{aligned}\frac{1}{2}\left|w_m'\right|^2 + \frac{\nu_1}{2}\frac{d}{dt}\left|e(w_m)\right|^2 + \frac{\lambda_1}{2}\frac{d}{dt}\left|w_m\right|^2 \\ \le c + c\int_\Omega \left|u_m\right|^2 \left|\nabla w_m\right|^2 dx.\end{aligned} \quad (142)$$

We observe that (46) implies $u_m(t) \in V_4 \hookrightarrow \mathbf{L}^\infty(\Omega)$ because $d \le 3$. On the other hand, (49) implies $\nabla w_m(t) \in \mathbf{L}^2(\Omega)$. Thus, applying Hölder's inequality in (142), integrating on $(0,t)$, with $t \le T$, and after applying Korn's inequality, we obtain

$$\begin{aligned}\int_0^t \left|w_m'(s)\right|^2 ds + \nu_1 K \|w_m\|^2 \\ \le c + c\int_0^t \left|u_m(s)\right|^2_{L^\infty(\Omega)} \|w_m(s)\|^2 ds\end{aligned} \quad (143)$$

because $w_0 \in \mathbf{H}_0^1(\Omega)$. Furthermore (46) implies $\|u_m(t)\|^2_{L^\infty(\Omega)} \in L^2(0,T)$ and (49) implies $\|u_m(t)\| \in L^1(0,T)$. Thus, by using Gronwall's inequality in (143), we conclude

$$(w_m') \text{ is bounded in } L^2\left(I;\mathbf{L}^2(\Omega)\right), \quad (144)$$

$$(w_m) \text{ is bounded in } L^\infty\left(I;\mathbf{H}_0^1(\Omega)\right). \quad (145)$$

In order to prove uniqueness of solution to the case $d = 3$ we observe that by assuming $d = 3$ we have $H_0^1 \hookrightarrow L^6(\Omega)$. From Lemma 7 with $s = 4$, $r = 6$ and Young's inequality we derive

$$\begin{aligned}&\left|b(\tilde u, u_1, \tilde u)\right| + \left|b(\tilde u, w_2, \tilde w)\right| \\ &\le c\|\tilde u\|_{L^6(\Omega)} \|u_1\| |\tilde u|^{1/2} \|\tilde u\|^{1/2} \\ &\quad + c\|\tilde u\|_{L^6(\Omega)} \|w_2\| |\tilde w|^{1/2} \|\tilde w\|^{1/2} \\ &\le c\|u_1\| |\tilde u|^{1/2} \|\tilde u\|^{3/2} \\ &\quad + c\|w_2\| |\tilde w|^{1/2} \|\tilde u\| \|\tilde w\|^{1/2} \\ &\le \frac{\nu}{4}\|\tilde u\|^2 + c_\nu \|u_1\|^4 |\tilde u|^2 \\ &\quad + \frac{\nu}{4}\|\tilde u\|^2 + c_\nu \|w_2\|^2 |\tilde w| \|\tilde w\| \\ &\le \frac{\nu}{4}\|\tilde u\|^2 + c_\nu \|u_1\|^4 |\tilde u|^2 \\ &\quad + \frac{\nu}{4}\|\tilde u\|^2 + \nu_1 \|\tilde w\|^2 + c_{\nu_1} \|w_2\|^4 |\tilde w|^2.\end{aligned} \quad (146)$$

It follows from (103) that

$$\frac{1}{2}\frac{d}{dt}\left(|\tilde u|^2 + |\tilde w|^2\right) \le c\left(1 + \|u_1\|^4 + \|w_2\|^4\right)\left(|\tilde u|^2 + |\tilde w|^2\right). \quad (147)$$

Integrating from 0 to $t$ we obtain

$$\begin{aligned}&|\tilde u(t)|^2 + |\tilde w(t)|^2 \\ &\le c\int_0^t \left(1 + \|u_1(s)\|^4 + \|w_2(s)\|^4\right)\left(|\tilde u(s)|^2 + |\tilde w(s)|^2\right) ds.\end{aligned} \quad (148)$$

Applying Gronwall's inequality in (148), we deduce by using (47) and (145) that

$$u_1(t) = u_2(t), \quad w_1(t) = w_2(t), \quad \forall t \in [0,T]. \quad (149)$$

□

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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