Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 946026, 8 pages
http://dx.doi.org/10.1155/2014/946026

Research Article

The Constants in A Posteriori Error Indicator for State-Constrained Optimal Control Problems with Spectral Methods

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Received 15 January 2014; Revised 7 April 2014; Accepted 26 April 2014; Published 19 May 2014

Academic Editor: Elena Berdysheva

We employ Legendre-Galerkin spectral methods to solve state-constrained optimal control problems. The constraint on the state variable is an integration form. We choose one-dimensional case to illustrate the techniques. Meanwhile, we investigate the explicit formulae of constants within a posteriori error indicator.

1. Introduction

Spectral methods provide higher accurate approximations with a relatively small number of unknowns and play increasingly important roles in design optimization, engineering design, and other scientific and engineering computations. Gottlieb and Orszag [1] summarized the theories and applications of spectral methods. There have been extensive researches on finite element methods for optimal control problems, most of which focus on control-constrained problems; see [2–5]. The authors studied the optimal control problems with the control constraint with spectral methods in [6]. In applications of engineering, one cares more about how to control the average value or L^2-norms of the state variable. The authors [7] discussed state-constrained optimal control problems with finite element methods. However, there are few work on the state-constrained optimal control problems with spectral methods.

In order to get a numerical solution with acceptable accuracy, spectral methods only increase the degree of basis when the error indicator is larger than the a posteriori error indicator, while the finite element methods refine meshes (see [8, 9]). There have been lots of papers on the a posteriori error estimates for h-version finite element methods but not for spectral methods. Guo [10] got a reliable and efficient error indicator for p-version finite element method in one dimension with a certain weight. The authors [11] deduced a simple error indicator for spectral Galerkin methods. In [12], the authors investigated Legendre-Galerkin spectral method for optimal control problems with integral constraint on state. It is difficult to obtain optimal a posteriori error indicators. Thus, if one gets the constants within upper bound a posteriori error estimates, it is easy to ensure the degree of polynomials to get an acceptable accuracy.

In this paper, we employ Legendre-Galerkin spectral methods to solve optimal control problems with state-constrained case and calculate constants in upper bound of the a posteriori error indicator, which can be used to decide the least unknowns for acceptable accuracy. With the help of auxiliary systems, we investigate explicit formulae of the constants in the a posteriori error indicator.

The outline of this paper is as follows. In Section 2, the model problem and its Legendre-Galerkin spectral approximations are listed. In Section 3, the constants within the a posteriori error indicator are investigated in detail and the explicit formulae are obtained. The conclusions are given in Section 4.

2. A Model Problem and Its Legendre-Galerkin Spectral Approximations

Throughout this paper we adopt the standard notations of Sobolev spaces [13]. Let $H^m(I)$ be a Sobolev space on $I = (-1,1)$, $L^2(I) = H^0(I)$ and $H_0^1(I) = \{v \in H^1(I) : v = 0 \text{ on } \partial I\}$, and the corresponding norms are denoted by $\|\cdot\|_m$, $\|\cdot\|_0$, and $\|\cdot\|_{0,1}$, respectively. This work focuses on the Legendre polynomials, which are orthogonal polynomials on [−1,1].

We concern the following distributed convex optimal control problems with integral constraint on state:

$$(\text{OCP}) \quad \begin{cases} \min\limits_{y\in K} & J(u,y) = \dfrac{1}{2}\displaystyle\int_I (y-y_d)^2 + \dfrac{\alpha}{2}\int_I u^2, \\ \text{s.t.} & -y'' = u \quad \text{in } I, \ y = 0 \ \text{on } \partial I, \ y \in K, \end{cases} \tag{1}$$

where $u \in U = L^2(I)$ is the control variable, $y \in K = \{w : \int_I w \geq \gamma\} \subset H_0^1(I) \triangleq V$ is the state, and $y_d \in L^2(I)$ is the observation.

In order to assure the existence and regularity of the solution, we assume that α is a given positive constant and y_d is an infinitely smooth function. It is well known that the problem (OCP) has a unique solution (see [3]).

We give some basic notations which will be used in the sequel. Let

$$\begin{aligned} (v,w) &= \int_I vw, \quad \forall v,w \in L^2(I), \\ a(v,w) &= \int_I v'w', \quad \forall v,w \in H_0^1(I). \end{aligned} \tag{2}$$

Hence, the state equation reduces to

$$a(y,w) = (u,w), \quad \forall w \in H_0^1(I). \tag{3}$$

Then (OCP) can be rewritten as finding (u,y) such that

$$(\mathscr{P}) \quad \begin{cases} \min\limits_{y\in K} & J(u,y) = \dfrac{1}{2}\displaystyle\int_I (y-y_d)^2 + \dfrac{\alpha}{2}\int_I u^2, \\ \text{s.t.} & a(y(u),w) = (u,w), \quad \forall w \in V. \end{cases} \tag{4}$$

We recall the following optimal conditions of $(\mathscr{P})$ (for details please refer to [7]).

Lemma 1. *The pair $(u,y) \in U \times V$ is the optimal solution of $(\mathscr{P})$ if and only if there exists a unique pair $(p,\lambda) \in V \times \mathbb{R}_-^1$ ($\mathbb{R}_-^1 \triangleq \{c \in \mathbb{R}^1;\ c \leq 0\}$) such that*

$$(OCP-OPT) \quad \begin{cases} a(y,w) = (u,w), \quad \forall w \in V, \\ a(q,p) \\ \quad = (y-y_d,q) + \lambda(1,q), \quad \forall q \in V, \\ \lambda(w-y) \leq 0, \quad \forall w \in K, \\ p + \alpha u = 0. \end{cases} \tag{5}$$

Let $\mathscr{P}_N(I) = \{\text{polynomials of degree} \leq N \text{ on } I\}$ and let $V_N = \mathscr{P}_N \cap H_0^1(I)$. One prefers to choose appropriate bases of V_N such that the resulting linear system is as simple as possible. We denote by $\{L_j\}_{j=0}^N$ the Legendre polynomial and employ the following basis functions (see [14]):

$$\begin{aligned} U_N &= \operatorname{span}\{L_0(x), L_1(x), \ldots, L_N(x)\}, \\ V_N &= \operatorname{span}\{\phi_0(x), \phi_1(x), \ldots, \phi_{N-2}(x)\}, \end{aligned} \tag{6}$$

where

$$\phi_i(x) = c_i(L_i(x) - L_{i+2}(x)), \quad c_i = \frac{1}{\sqrt{4i+6}}. \tag{7}$$

For $1 \leq j,\ k \leq N-2$, we denote $a_{jk} = a(\phi_k(x),\phi_j(x))$ and $b_{jk} = (\phi_k(x),\phi_j(x))$. By simple calculations, these coefficients satisfy

$$\begin{gathered} a_{jk} = \begin{cases} 1, & k = j, \\ 0, & k \neq j, \end{cases} \\ b_{jk} = b_{kj} = \begin{cases} c_k c_j\left(\dfrac{2}{2j+1} + \dfrac{2}{2j+5}\right), & k = j, \\ -c_k c_j \dfrac{2}{2k+1}, & k = j+2, \\ 0, & \text{otherwise}. \end{cases} \end{gathered} \tag{8}$$

Then Legendre-Galerkin spectral approximations of (OCP) can be read as finding (u_N, y_N) such that

$$(\mathscr{P}^N) \quad \begin{cases} \min\limits_{y_N\in K} & J(u_N,y_N) = \dfrac{1}{2}\displaystyle\int_I (y_N-y_d)^2 + \dfrac{\alpha}{2}\int_I u_N^2, \\ \text{s.t.} & a(y_N,w_N) = (u_N,w_N), \quad \forall w_N \in V_N. \end{cases} \tag{9}$$

The Legendre-Galerkin spectral approximations of (5) can be read as follows.

Theorem 2. *The pair $(u_N,y_N) \in U_N \times V_N$ is the optimal solution of $(\mathscr{P}^N)$ if and only if there exists a unique pair $(p_N,\lambda_N) \in V_N \times \mathbb{R}_-^1$ such that*

$$(OCP-OPT)^N \quad \begin{cases} a(y_N,v_N) \\ \quad = (u_N,v_N), \quad \forall v_N \in V_N, \\ a(q_N,p_N) \\ \quad = (y_N-y_d,q_N) \\ \qquad +\lambda_N(1,q_N), \quad \forall q_N \in V_N, \\ \lambda_N(1,w_N-y_N) \leq 0, \quad \forall w_N \in K_N, \\ \alpha u_N + p_N = 0. \end{cases} \tag{10}$$

3. Constants within the A Posteriori Error Estimates

In this section, we calculate all constants within the a posteriori error estimates. Here, we analyze the constant in the Poincaré inequality.

For all $v \in W_0^{1,p}(I)$, $1 \leq p < \infty$, we recall the Poincaré inequality with L^2-norm as (see [15])

$$\|v\|_0 \leq \frac{|I|}{2}\|v'\|_0. \tag{11}$$

Now, we are at the point to investigate all constants in detail. We introduce an auxiliary state $y(u_N) \in H_0^1(I)$, which satisfies

$$a(y(u_N), w) = (u_N, w), \quad \forall w \in H_0^1(I). \tag{12}$$

Subtracting (12) from (3), we get

$$a(y - y(u_N), w) = (u - u_N, w), \quad \forall w \in H_0^1(I). \tag{13}$$

Let $w = y(u_N) - y \in H_0^1(I)$. It is clear that

$$a(y(u_N) - y,\ y(u_N) - y) = (u_N - u,\ y(u_N) - y). \tag{14}$$

Then

$$\begin{aligned} \left\|(y(u_N) - y)'\right\|_0^2 &\le \|u_N - u\|_0 \|y(u_N) - y\|_0 \\ &\le \frac{|I|}{2} \|u_N - u\|_0 \left\|(y(u_N) - y)'\right\|_0, \end{aligned} \tag{15}$$

which means

$$\left\|(y(u_N) - y)'\right\|_0 \le \frac{|I|}{2} \|u_N - u\|_0. \tag{16}$$

Hence

$$\begin{aligned} &\|y(u_N) - y\|_1 \\ &\quad \le \left(\left\|(y(u_N) - y)'\right\|_0^2 + \left(\frac{|I|}{2}\right)^2 \left\|(y(u_N) - y)'\right\|_0^2 \right)^{1/2} \\ &\quad = \left(1 + \left(\frac{|I|}{2}\right)^2\right)^{1/2} \left\|(y(u_N) - y)'\right\|_0. \end{aligned} \tag{17}$$

Then

$$\|y(u_N) - y\|_1 \le \left(1 + \left(\frac{|I|}{2}\right)^2\right)^{1/2} \frac{|I|}{2} \|u_N - u\|_0. \tag{18}$$

We denote by c_1 the constant in (18); that is,

$$c_1 = \left(1 + \left(\frac{|I|}{2}\right)^2\right)^{1/2} \frac{|I|}{2}. \tag{19}$$

Similarly, we introduce an auxiliary state $p(u_N) \in H^1(I)$, which satisfies

$$a(q, p(u_N)) = (y(u_N) - y_d, q) + \lambda(1, q), \quad \forall q \in H_0^1(I). \tag{20}$$

Subtracting (20) from the continuous systems (5), we get

$$\begin{aligned} &a(p - p(u_N), w) \\ &\quad = (y - y(u_N), w) + (\lambda - \lambda_N)(1, w), \\ &\qquad\qquad \forall w \in H_0^1(I). \end{aligned} \tag{21}$$

We select $\varphi \in C_0^\infty(I)$ which satisfies $\overline{\varphi} = 1$, where $\overline{\varphi} \triangleq \int_I \varphi/|I| = 1$ denotes the integral average on I of the function φ and $\|\varphi\|_1 \le C_\varphi$. Obviously, $\overline{p - p(u_N)}\varphi \in C_0^\infty(I)$. In fact, $p - p(u_N) - \overline{p - p(u_N)}\varphi \in H_0^1(I)$. Then there hold

$$\begin{aligned} &\|p - p(u_N)\|_1^2 \\ &\quad \le \left(1 + \left(\frac{|I|}{2}\right)^2\right) |p - p(u_N)|_1^2 \\ &\quad = \left(1 + \left(\frac{|I|}{2}\right)^2\right) a(p - p(u_N),\ p - p(u_N)) \\ &\quad = \left(1 + \left(\frac{|I|}{2}\right)^2\right) \\ &\qquad \times \Big\{ a\left(\overline{p - p(u_N)}\varphi,\ p - p(u_N)\right) \\ &\qquad\quad + \left(y - y(u_N), p - p(u_N) - \overline{p - p(u_N)}\varphi\right) \Big\} \\ &\quad \le \left(1 + \left(\frac{|I|}{2}\right)^2\right) \\ &\qquad \times \Big\{ \left|\overline{p - p(u_N)}\right| \cdot \left(\frac{\epsilon_1}{2}|\varphi|_1^2 + \frac{1}{2\epsilon_1}|p - p(u_N)|_1^2\right) \\ &\qquad\quad + \frac{\epsilon_2}{2}\|y - y(u_N)\|_0^2 \\ &\qquad\quad + \frac{1}{\epsilon_2}\left(\|p - p(u_N)\|_0^2 + \left|\overline{p - p(u_N)}\right|^2 \cdot \|\varphi\|_0^2\right) \Big\}, \end{aligned} \tag{22}$$

where we used the generalized Schwarz inequality, continuous systems (5), and auxiliary equation (20). Let

$$\left(1 + \left(\frac{|I|}{2}\right)^2\right) \frac{\left|\overline{p - p(u_N)}\right|}{2\epsilon_1} = \left(1 + \left(\frac{|I|}{2}\right)^2\right) \frac{1}{\epsilon_2} = \frac{1}{2}. \tag{23}$$

Then

$$\epsilon_1 = \left(1 + \left(\frac{|I|}{2}\right)^2\right) \left|\overline{p - p(u_N)}\right|, \qquad \epsilon_2 = 2\left(1 + \left(\frac{|I|}{2}\right)^2\right). \tag{24}$$

It is clear that (22) reduces to

$$\begin{aligned}
&\|p - p(u_N)\|_1^2 \\
&\quad \le \left(1 + \left(\frac{|I|}{2}\right)^2\right) \\
&\qquad \times \left\{ \frac{1 + (|I|/2)^2}{2} \left|\overline{p - p(u_N)}\right|^2 \cdot |\varphi|_1^2 \right. \\
&\qquad\quad + \frac{\left|\overline{p - p(u_N)}\right|^2}{2\left(1 + (|I|/2)^2\right)} \|\varphi\|_0^2 \\
&\qquad\quad \left. + \left(1 + \left(\frac{|I|}{2}\right)^2\right) \|y - y(u_N)\|_0^2 \right\} \\
&\qquad + \frac{1}{2}\|p - p(u_N)\|_1^2.
\end{aligned} \tag{25}$$

Then

$$\begin{aligned}
&\|p - p(u_N)\|_1^2 \\
&\quad \le 2\left(1 + \left(\frac{|I|}{2}\right)^2\right) \\
&\qquad \times \left\{ \left(1 + \left(\frac{|I|}{2}\right)^2\right) \left|\overline{p - p(u_N)}\right|^2 \cdot \|\varphi\|_1^2 \right. \\
&\qquad\quad \left. + \left(1 + \left(\frac{|I|}{2}\right)^2\right) \|y - y(u_N)\|_0^2 \right\} \\
&\quad = 2\left(1 + \left(\frac{|I|}{2}\right)^2\right)^2 \left\{ \left|\overline{p - p(u_N)}\right|^2 \cdot \|\varphi\|_1^2 \right. \\
&\qquad\quad \left. + \|y - y(u_N)\|_0^2 \right\},
\end{aligned} \tag{26}$$

where we used $(1 + (|I|/2)^2)|\overline{p - p(u_N)}|^2 \ge (1/(1 + (|I|/2)^2))|\overline{p - p(u_N)}|^2$.

Hence

$$\begin{aligned}
&\|p - p(u_N)\|_1^2 \\
&\quad \le 2\left(1 + \left(\frac{|I|}{2}\right)^2\right)^2 \\
&\qquad \times \left\{ \|\varphi\|_1^2 \cdot \frac{1}{|I|} \left(\alpha\|u - u_N\|_0 + \|p_N - p(u_N)\|_0\right)^2 \right. \\
&\qquad\quad \left. + \left(1 + \left(\frac{|I|}{2}\right)^2\right) \left(\frac{|I|}{2}\right)^2 \|u - u_N\|_0^2 \right\} \\
&\quad = 2\left(1 + \left(\frac{|I|}{2}\right)^2\right)^2 \\
&\qquad \times \left\{ \left[\frac{2\|\varphi\|_1^2 \alpha^2}{|I|} + \left(1 + \left(\frac{|I|}{2}\right)^2\right)\left(\frac{|I|}{2}\right)^2 \right] \right. \\
&\qquad\quad \left. \times \|u - u_N\|_0^2 + \frac{2\|\varphi\|_1^2}{|I|} \|p_N - p(u_N)\|_0^2 \right\} \\
&\quad = 2\left(1 + \left(\frac{|I|}{2}\right)^2\right)^2 \max\left\{ \frac{2\|\varphi\|_1^2 \alpha^2}{|I|} + c_1^2, \frac{2\|\varphi\|_1^2}{|I|} \right\} \\
&\qquad \times \left\{ \|u - u_N\|_0^2 + \|p_N - p(u_N)\|_0^2 \right\},
\end{aligned} \tag{27}$$

which means that

$$\begin{aligned}
&\|p - p(u_N)\|_1 \\
&\le \left(2\left(1 + \left(\frac{|I|}{2}\right)^2\right)^2 \right. \\
&\qquad \left. \times \max\left\{ \frac{2C_\varphi^2 \alpha^2}{|I|} + \left(1 + \left(\frac{|I|}{2}\right)^2\right)\left(\frac{|I|}{2}\right)^2, \frac{2C_\varphi^2}{|I|} \right\} \right)^{1/2} \\
&\quad \times \left\{ \|u - u_N\|_0 + \|p_N - p(u_N)\|_0 \right\}.
\end{aligned} \tag{28}$$

Denote by c_2 the constant in (28). With simple calculation, we have

$$c_2 = \sqrt{2\left(1 + \left(\frac{|I|}{2}\right)^2\right)^2 \max\left\{ \frac{2C_\varphi^2 \alpha^2}{|I|} + c_1^2, \frac{2C_\varphi^2}{|I|} \right\}}. \tag{29}$$

We select $\varphi \in C_0^\infty(I)$ which satisfies $\overline{\varphi} = 1$ and $\|\varphi\|_1 \le C_\varphi$. For instance, $\varphi = (3/2)(1 - x^2)$, which satisfies

$$\|\varphi\|_1 = \frac{3}{2}\sqrt{\frac{74}{15}} \triangleq C_\varphi. \tag{30}$$

Meanwhile,

$$\begin{aligned}
&(\lambda - \lambda_N, (\lambda - \lambda_N)\varphi) \\
&\quad = (\lambda - \lambda_N)^2 \int_I \varphi = (\lambda - \lambda_N)^2 |I| = \|\lambda - \lambda_N\|_0^2.
\end{aligned} \tag{31}$$

Hence

$$\begin{aligned}\|\lambda-\lambda_N\|_0^2 &= (\lambda-\lambda_N,(\lambda-\lambda_N)\varphi)\\ &= a((\lambda-\lambda_N)\varphi, p-p(u_N))\\ &\quad -(y-y(u_N),(\lambda-\lambda_N)\varphi)\\ &\le |\lambda-\lambda_N|\left\{\|\varphi'\|_0\cdot\|(p-p(u_N))'\|_0\right.\\ &\qquad \left.+\|\varphi\|_0\cdot\|y-y(u_N)\|_0\right\}\\ &\le \frac{\epsilon_1}{2}|\lambda-\lambda_N|^2\cdot\|\varphi'\|_0^2+\frac{1}{2\epsilon_1}\|(p-p(u_N))'\|_0^2\\ &\quad +\frac{\epsilon_2}{2}|\lambda-\lambda_N|^2\cdot\|\varphi\|_0^2+\frac{1}{2\epsilon_2}\|y-y(u_N)\|_0^2\\ &= \frac{\epsilon_1}{2|I|}\|\lambda-\lambda_N\|_0^2\|\varphi\|_1^2\\ &\quad +\frac{1}{2\epsilon_1}\left(\|(p-p(u_N))'\|_0^2+\|y-y(u_N)\|_0^2\right),\end{aligned} \tag{32}$$

where $\epsilon_1=\epsilon_2=|I|/\|\varphi\|_1^2$.

Thus

$$\begin{aligned}\|\lambda-\lambda_N\|_0^2 &\le \frac{\|\varphi\|_1^2}{|I|}\left\{\|(p-p(u_N))'\|_0^2+\|y-y(u_N)\|_0^2\right\}\\ &\le \frac{\|\varphi\|_1^2}{|I|}\left\{\|p-p(u_N)\|_1^2+\|y-y(u_N)\|_0^2\right\}.\end{aligned} \tag{33}$$

With the constant c_2, we infer that

$$\begin{aligned}&\|\lambda-\lambda_N\|_0^2\\ &\le \frac{C_\varphi^2}{|I|}\left\{4\left(1+\left(\frac{|I|}{2}\right)^2\right)^2\right.\\ &\qquad\times\max\left\{\frac{2C_\varphi^2\alpha^2}{|I|}+\left(1+\left(\frac{|I|}{2}\right)^2\right)\left(\frac{|I|}{2}\right)^2,\frac{2C_\varphi^2}{|I|}\right\}\\ &\qquad\left.+\left(1+\left(\frac{|I|}{2}\right)^2\right)\left(\frac{|I|}{2}\right)^2\right\}\\ &\quad\times\left\{\|p_N-p(u_N)\|_0^2+\|u-u_N\|_0^2\right\}\\ &=\frac{C_\varphi^2}{|I|}\left\{2c_2^2+c_1^2\right\}\left\{\|p_N-p(u_N)\|_0^2+\|u-u_N\|_0^2\right\}.\end{aligned} \tag{34}$$

Then

$$\begin{aligned}&\|\lambda-\lambda_N\|_0\\ &\le \frac{C_\varphi^2}{|I|}\left\{2c_2^2+c_1^2\right\}\left\{\|p_N-p(u_N)\|_0+\|u-u_N\|_0\right\}.\end{aligned} \tag{35}$$

We denote by c_3 the constant in (35); that is,

$$c_3=\sqrt{\frac{C_\varphi^2}{|I|}\left\{2c_2^2+c_1^2\right\}}. \tag{36}$$

We calculate the error of u in L^2-norm as follows:

$$\begin{aligned}&\|u-u_N\|_0^2\\ &\le (p_N-p(u_N),u-u_N)\\ &\quad -(\lambda-\lambda_N,y_N-y(u_N))\\ &\le \frac{\epsilon_1}{2}\|p_N-p(u_N)\|_0^2+\frac{1}{2\epsilon_1}\|u-u_N\|_0^2\\ &\quad +\frac{1}{2\epsilon_2}\|y_N-y(u_N)\|_0^2\\ &\quad +\epsilon_2\cdot c_3^2\left\{\|u-u_N\|_0^2+\|p_N-p(u_N)\|_0^2\right\}\\ &=\left(\frac{\epsilon_1}{2}+\epsilon_2\cdot c_3^2\right)\|p_N-p(u_N)\|_0^2\\ &\quad +\left(\frac{1}{2\epsilon_1}+\epsilon_2\cdot c_3^2\right)\|u-u_N\|_0^2\\ &\quad +\frac{1}{2\epsilon_2}\|y_N-y(u_N)\|_0^2.\end{aligned} \tag{37}$$

Provided that $1/2\epsilon_1+\epsilon_2\cdot c_3^2=1/2$, we get

$$\begin{aligned}&\|u-u_N\|_0^2\\ &\le 2\left(\frac{\epsilon_1}{2}+\epsilon_2\cdot c_3^2\right)\|p_N-p(u_N)\|_0^2\\ &\quad +\frac{1}{\epsilon_2}\|y_N-y(u_N)\|_0^2\\ &=\max\left\{\epsilon_1+2\epsilon_2\cdot c_3^2,\frac{1}{\epsilon_2}\right\}\left\{\|p_N-p(u_N)\|_0^2\right.\\ &\qquad\left.+\|y_N-y(u_N)\|_0^2\right\}.\end{aligned} \tag{38}$$

Considering the item $\max\{\epsilon_1+2\epsilon_2\cdot c_3^2,1/\epsilon_2\}$ with the constraint $1/2\epsilon_1+\epsilon_2\cdot c_3^2=1/2$, we get

$$F(\epsilon_1)=\max\left\{\epsilon_1+1-\frac{1}{\epsilon_1},\frac{2c_3^2\epsilon_1}{\epsilon_1-1}\right\}. \tag{39}$$

In fact, for $\forall\epsilon_1>1$, the derivation of the following function

$$f(\epsilon_1)=\epsilon_1+1-\frac{1}{\epsilon_1}-\frac{2c_3^2\epsilon_1}{\epsilon_1-1} \tag{40}$$

is

$$f'(\epsilon_1)=1+\frac{1}{\epsilon_1^2}+\frac{2c_3^2}{(\epsilon_1-1)^2}\ge 0. \tag{41}$$

Then we have $\epsilon_1^0 = 2c_3^2 + \epsilon > 1$ and

$$\lim_{\epsilon \to 0} f\left(\epsilon_1^0\right) = 0. \tag{42}$$

Now, we are at the point to investigate

$$\min F\left(\epsilon_1\right). \tag{43}$$

If $\epsilon_1 = \epsilon_1^0$, we get

$$\min F\left(\epsilon_1\right) = \epsilon_1^0 + 1 - \frac{1}{\epsilon_1^0} \leq 2c_3^2 + \epsilon + 1. \tag{44}$$

If $1 < \epsilon_1 < \epsilon_1^0$, $f(\epsilon_1) < 0$, we obtain

$$F\left(\epsilon_1\right) = \frac{2c_3^2 \epsilon_1}{\epsilon_1 - 1}, \qquad F'\left(\epsilon_1\right) = -\frac{2c_3^2}{\left(\epsilon_1 - 1\right)^2} < 0. \tag{45}$$

Then

$$\lim_{\epsilon' \to 0} \min F\left(\epsilon_1\right) = 2c_3^2 \left(1 + \frac{1}{2c_3^2 + \epsilon - \epsilon' - 1}\right) < 4c_3^2. \tag{46}$$

If $\epsilon_1 > \epsilon_1^0$, $f(\epsilon_1) > 0$, we infer that

$$F\left(\epsilon_1\right) = \epsilon_1 + 1 - \frac{1}{\epsilon_1}, \qquad F'\left(\epsilon_1\right) = 1 + \frac{1}{\epsilon_1^2} > 0. \tag{47}$$

Then there hold

$$\lim_{\epsilon'' \to 0} \min F\left(\epsilon_1\right) = 2c_3^2 + \epsilon + \epsilon'' + 1 - \frac{1}{2c_3^2 + \epsilon + \epsilon''} < 2c_3^2 + 1. \tag{48}$$

Combining the above discussions, we deduce that

$$\min\left\{\max\left\{\epsilon_1 + 2\epsilon_2 \cdot c_3^2, \frac{1}{\epsilon_2}\right\}\right\} < 2c_3^2 + 1. \tag{49}$$

Obviously,

$$\begin{aligned} &\left\|u - u_N\right\|_0 \\ &\quad \leq \sqrt{2c_3^2 + 1}\,\left\{\left\|p_N - p\left(u_N\right)\right\|_0 + \left\|y_N - y\left(u_N\right)\right\|_0\right\}. \end{aligned} \tag{50}$$

We denote by c_4 the constant in (50); that is,

$$c_4 = \sqrt{2c_3^2 + 1}. \tag{51}$$

For any $v \in L^2(I)$, we define a projection operator $\mathbb{P}_N : L^2(I) \to V_N$, which satisfies

$$\left(\mathbb{P}_N v - v, w_N\right) = 0, \quad \forall w_N \in U_N. \tag{52}$$

Lemma 3. *For all $v \in H^\sigma(I)$ ($\sigma \geq 0$), one has*

$$\left\|\mathbb{P}_N v - v\right\|_0 \leq c_5 N^{-\sigma} \|v\|_\sigma, \tag{53}$$

where $c_5 = 2\sqrt{2}$.

Proof. Firstly, assuming that $\sigma = 2p$ ($p \geq 1$) is integer, we define a differential operator as

$$A = \frac{d}{dx}\left(\left(1 - x^2\right)\frac{d}{dx}\right). \tag{54}$$

From the fact that

$$\frac{d}{dx}\left(\left(1 - x^2\right)\frac{dL_k}{dx}\right) + k\left(k + 1\right)L_k = 0, \tag{55}$$

it is easy to get

$$\begin{aligned} \widehat{v}_k &= \left(k + \frac{1}{2}\right)\left(v, L_k\right) \\ &= \frac{k + 1/2}{k\left(k + 1\right)}\int_{-1}^{1} AL_k\left(x\right)v\left(x\right)dx \\ &= -\frac{k + 1/2}{k\left(k + 1\right)}\int_{-1}^{1} Av\left(x\right)L_k\left(x\right)dx \\ &= -\frac{k + 1/2}{k\left(k + 1\right)}\left(Av\left(x\right), L_k\left(x\right)\right). \end{aligned} \tag{56}$$

By iterations, we obtain

$$\widehat{v}_k = \left(\frac{-1}{k\left(k + 1\right)}\right)^p \left(k + \frac{1}{2}\right)\left(A^p v, L_k\right). \tag{57}$$

Secondly, for all $v \in H^{2p}(I)$, we note that $A^p v = \sum_{i=0}^{\infty} \alpha_i L_i(x)$ and

$$\begin{aligned} \left(A^p v\left(x\right), L_k\left(x\right)\right) &= \alpha_i\left(k + \frac{1}{2}\right)^{-1}, \\ \left\|A^p v\right\|_0^2 &= \sum_{k=0}^{\infty}\left|\alpha_k\right|^2\left(k + \frac{1}{2}\right)^{-1}. \end{aligned} \tag{58}$$

Hence

$$\begin{aligned} \left\|\mathbb{P}_N v - v\right\|_0^2 &= \sum_{k=N+1}^{\infty}\left(k + \frac{1}{2}\right)^{-1}\left|\widehat{v}_k\right|^2 \\ &= \sum_{k=N+1}^{\infty}\left(\frac{1}{k\left(k + 1\right)}\right)^{2p}\left(k + \frac{1}{2}\right)\left|A^p v, L_k\right|^2 \\ &\leq N^{-4p}\sum_{k=N+1}^{\infty}\left(k + \frac{1}{2}\right)\left|\alpha_k\right|^2\left(k + \frac{1}{2}\right)^{-2} \\ &\leq N^{-4p}\left\|A^p v\right\|_0^2. \end{aligned} \tag{59}$$

Finally, there hold

$$\begin{aligned} \left|Av\right|^2 &= \left|\frac{d}{dx}\left(\left(1 - x^2\right)\frac{dv}{dx}\right)\right|^2 \\ &= \left|\left(1 - x^2\right)v'' - 2xv'\right|^2 \\ &\leq \left(\left|v''\right| + 2\left|v'\right|\right)^2 \\ &\leq 2v''^2 + 8v'^2 \\ &\leq 8\left\{v''^2 + v'^2 + v^2\right\}, \end{aligned} \tag{60}$$

which means that

$$\|Av\|_0^2 \le 8\|v\|_2^2. \tag{61}$$

Let $p = 1, \sigma = 2$. It is clear that

$$\|\mathbb{P}_N v - v\|_0 \le \sqrt{8}\, N^{-2}\|v\|_2. \tag{62}$$

This completes the proof. □

Now, we are at the point to calculate the constant for $\|y_N - y(u_N)\|_1 + \|p_N - p(u_N)\|_1$. Similarly, let $E^p = p_N - p(u_N)$ and let $E_I^p = \mathbb{P}_N E^p \in V_N$. Then

$$\begin{aligned}
&\|p_N - p(u_N)\|_1^2 \\
&\quad\le \left(1+\left(\frac{|I|}{2}\right)^2\right)\left(a\left(E^p, E^p - E_I^p\right) + \left(y(u_N) - y_N, E_I^p\right)\right) \\
&\quad= \left(1+\left(\frac{|I|}{2}\right)^2\right)\left(a\left(p(u_N) - p_N, E^p - E_I^p\right) \right.\\
&\qquad\left. + \left(y(u_N) - y_N, E_I^p\right)\right) \\
&\quad= \left(1+\left(\frac{|I|}{2}\right)^2\right) \\
&\qquad\times\left(\left(-p''(u_N), E^p - E_I^p\right)\right.\\
&\qquad\left. + \left(p_N'', E^p - E_I^p\right) + \left(y(u_N) - y_N, E_I^p\right)\right) \\
&\quad= \left(1+\left(\frac{|I|}{2}\right)^2\right)\left(\left(y_N - y_d + \lambda_N + p_N'', E^p - E_I^p\right)\right.\\
&\qquad\left. + \left(y(u_N) - y_N, E^p\right)\right) \\
&\quad\le \left(1+\left(\frac{|I|}{2}\right)^2\right) \|E^p\|_1 \left\{c_5 N^{-1}\left\|y_N - y_d + \lambda_N + p_N''\right\|_0\right.\\
&\qquad\left. + \|y_N - y(u_N)\|_0\right\},
\end{aligned} \tag{63}$$

which means that

$$\begin{aligned}
&\|p_N - p(u_N)\|_1 \\
&\quad\le \left(1+\left(\frac{|I|}{2}\right)^2\right)\left\{c_5 N^{-1}\left\|y_N - y_d + \lambda_N + p_N''\right\|_0\right. \\
&\qquad\left. + \|y_N - y(u_N)\|_0\right\}.
\end{aligned} \tag{64}$$

Likewise, let $E^y = y_N - y(u_N)$ and let $E_I^y = \mathbb{P}_{1,N}^0 E^y \in V_N$. Then

$$\begin{aligned}
&\|y_N - y(u_N)\|_1^2 \\
&\quad= \|E^y\|_1^2 \le \left(1+\left(\frac{|I|}{2}\right)^2\right) a\left(E^y, E^y\right) \\
&\quad= \left(1+\left(\frac{|I|}{2}\right)^2\right) a\left(E^y - E_I^y, E^y\right) \\
&\quad= \left(1+\left(\frac{|I|}{2}\right)^2\right)\left(-\left(u_N + y_N''\right), E^y - E_I^y\right) \\
&\quad\le \left(1+\left(\frac{|I|}{2}\right)^2\right) c_5 N^{-1}\left\|u_N + y_N''\right\|_0 \cdot \|E^y\|_1,
\end{aligned} \tag{65}$$

which is equivalent to

$$\|y_N - y(u_N)\|_1 \le \left(1+\left(\frac{|I|}{2}\right)^2\right) c_5 N^{-1}\left\|u_N + y_N''\right\|_0. \tag{66}$$

Hence

$$\|y_N - y(u_N)\|_1 + \|p_N - p(u_N)\|_1 \le c_6 \eta, \tag{67}$$

where

$$\begin{aligned}
c_6 &= \left(1+\left(\frac{|I|}{2}\right)^2\right)^2 c_5, \\
\eta &= N^{-1}\left\|y_N - y_d + \lambda_N + p_N''\right\|_0 + N^{-1}\left\|u_N + y_N''\right\|_0.
\end{aligned} \tag{68}$$

Combining the above analyses, we get that

$$\begin{aligned}
&\|u - u_N\|_0 + \|y - y_N\|_1 + \|p - p_N\|_1 + \|\lambda - \lambda_N\|_0 \\
&\quad\le \|u - u_N\|_0 + \|y - y(u_N)\|_1 + \|y_N - y(u_N)\|_1 \\
&\qquad + \|p - p(u_N)\|_1 + \|p_N - p(u_N)\|_1 + \|\lambda - \lambda_N\|_0 \\
&\quad\le \|y_N - y(u_N)\|_1 + \|p_N - p(u_N)\|_1 \\
&\qquad + c_4\left\{\|p_N - p(u_N)\|_0 + \|y_N - y(u_N)\|_0\right\} \\
&\qquad + c_1 c_4\left\{\|p_N - p(u_N)\|_0 + \|y_N - y(u_N)\|_0\right\} \\
&\qquad + c_2\left\{\|p_N - p(u_N)\|_0\right. \\
&\qquad\quad\left. + c_4\left(\|p_N - p(u_N)\|_0 + \|y_N - y(u_N)\|_0\right)\right\} \\
&\qquad + c_3\left\{\|p_N - p(u_N)\|_{0,I}\right. \\
&\qquad\quad\left. + c_4\left(\|p_N - p(u_N)\|_0 + \|y_N - y(u_N)\|_0\right)\right\} \\
&\quad\le \|p_N - p(u_N)\|_1\left\{c_4 + 1 + c_1c_4 + c_2(c_4+1) + c_3(c_4+1)\right\} \\
&\qquad + \|y_N - y(u_N)\|_1\left\{c_4 + 1 + c_1c_4 + c_2c_4 + c_3c_4\right\} \\
&\quad\le \left\{c_4 + 1 + c_1c_4 + c_2(c_4+1) + c_3(c_4+1)\right\} \\
&\qquad\times\left\{\|p_N - p(u_N)\|_1 + \|y_N - y(u_N)\|_1\right\} \\
&\quad\le \left\{c_4 + 1 + c_1c_4 + c_2(c_4+1) + c_3(c_4+1)\right\} c_6 \eta,
\end{aligned} \tag{69}$$

which means that

$$\|u - u_N\|_0 + \|y - y_N\|_1 + \|p - p_N\|_1 + \|\lambda - \lambda_N\|_0 \le C\eta, \tag{70}$$

where

$$\begin{aligned} C &= \{1 + c_4 + c_1 c_4 + c_2 (c_4 + 1) + c_3 (c_4 + 1)\} c_6, \\ c_1 &= \left(1 + \left(\frac{|I|}{2}\right)^2\right)^{1/2} \frac{|I|}{2}, \\ c_2 &= \sqrt{2\left(1 + \left(\frac{|I|}{2}\right)^2\right)^2 \max\left\{\frac{2C_\varphi^2 \alpha^2}{|I|} + c_1^2, \frac{2C_\varphi^2}{|I|}\right\}}, \\ c_3 &= \sqrt{\frac{C_\varphi^2}{|I|} \{2c_2^2 + c_1^2\}}, \\ c_4 &= \sqrt{2c_3^2 + 1}, \\ c_5 &= \sqrt{2}, \\ c_6 &= \left(1 + \left(\frac{|I|}{2}\right)^2\right)^2 c_5. \end{aligned} \tag{71}$$

4. Conclusions

This paper discusses the explicit formulae of constants within upper bound of the a posteriori error estimate for optimal control problems with Legendre-Galerkin spectral methods in one dimension. Thus, with those formulae, it is easy to choose a suitable degree of polynomials to obtain an acceptable accuracy. In the future, we will study the corresponding constants in lower bound of the a posteriori error indicator. Meanwhile, the corresponding constants in a two-dimensional domain will be investigated.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work is partially supported by National Natural Science Foundation of China (no. 11201212), Promotive Research Fund for Excellent Young and Middle-aged Scientists of Shandong Province (no. BS2012DX004), AMEP, and the Special Funds for Doctoral Authorities of Linyi University.

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