

## Research Article

# Complete Monotonicity of Functions Connected with the Exponential Function and Derivatives

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Some complete monotonicity results that the functions  $\pm 1/(e^{\pm t} - 1)$  are logarithmically completely monotonic, and that differences between consecutive derivatives of these two functions are completely monotonic, and that the ratios between consecutive derivatives of these two functions are decreasing on  $(0, \infty)$  are discovered. As applications of these newly discovered results, some complete monotonicity results concerning the polylogarithm are found. Finally a conjecture on the complete monotonicity of the above-mentioned ratios is posed.

## 1. Introduction and Main Results

Throughout this paper, we denote the set of all positive integers by  $\mathbb{N}$ .

Recently, the following problem was posed in [1, page 569]. For  $t \neq 0$  and  $k \in \mathbb{N}$ , determine the numbers  $a_{k,i-1}$  for  $1 \leq i \leq k$  such that

$$\frac{1}{(1 - e^{-t})^k} = 1 + \sum_{i=1}^k a_{k,i-1} \left( \frac{1}{e^t - 1} \right)^{(i-1)}. \quad (1)$$

This problem, among other things, was answered in [1] by eight identities. Two of the eight identities may be recited as follows.

**Theorem A** (see [1, Theorem 2.1]). For  $i \in \{0\} \cup \mathbb{N}$ , one has

$$\left( \frac{1}{e^t - 1} \right)^{(i)} = \sum_{m=1}^{i+1} \frac{\lambda_{i,m}}{(e^t - 1)^m}, \quad (2)$$

where

$$\begin{aligned} \lambda_{i,m} &= (-1)^i (m-1)! S(i+1, m), \\ S(i, m) &= \frac{1}{m!} \sum_{\ell=1}^m (-1)^{m-\ell} \binom{m}{\ell} \ell^i, \quad 1 \leq m \leq i, \end{aligned} \quad (3)$$

are Stirling numbers of the second kind.

**Theorem B** (see [1, Theorem 2.3]). For  $i, k \in \mathbb{N}$  with  $1 \leq i \leq k$ , the coefficients  $a_{k,i-1}$  defined in (1) may be calculated by

$$a_{k,i-1} = (-1)^{i^2+1} M_{k-i+1}(k, i), \quad (4)$$

where

$$M_j(k, i) = \begin{vmatrix} \frac{1}{(i-1)!} \binom{k}{i} & S(i+1, i) & \cdots & S(i+j-1, i) \\ \frac{1}{i!} \binom{k}{i+1} & S(i+1, i+1) & \cdots & S(i+j-1, i+1) \\ \frac{1}{(i+1)!} \binom{k}{i+2} & 0 & \cdots & S(i+j-1, i+2) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(i+j-2)!} \binom{k}{i+j-1} & 0 & \cdots & S(i+j-1, i+j-1) \end{vmatrix}. \quad (5)$$

For more information, please refer to [2, 3] and related references therein.

Stimulated by results obtained in [1], as mentioned above, we are interested in two functional sequences:

$$\begin{aligned} F_i(t) &= (-1)^i \left( \frac{1}{e^t - 1} \right)^{(i)}, \\ G_i(t) &= (-1)^i \left( \frac{1}{1 - e^{-t}} \right)^{(i)}, \end{aligned} \quad (6)$$

where  $t \in (0, \infty)$  and  $i \in \{0\} \cup \mathbb{N}$ , and we firstly discover in this paper the following results.

**Theorem 1.** For  $i \in \{0\} \cup \mathbb{N}$ , the functions  $F_i(t)$  and  $G_i(t)$  are completely monotonic on  $(0, \infty)$ . More strongly, the functions  $F_0(t)$  and  $G_0(t)$  are logarithmically completely monotonic on  $(0, \infty)$ .

For  $i, k \in \{0\} \cup \mathbb{N}$ , the functions  $F_i(t)$  and  $G_i(t)$  satisfy

$$\begin{aligned} F_i^{(k)}(t) &= (-1)^k F_{i+k}(t), \\ G_i^{(k)}(t) &= (-1)^k G_{i+k}(t). \end{aligned} \quad (7)$$

**Theorem 2.** For given  $i \in \{0\} \cup \mathbb{N}$ , the differences

$$\begin{aligned} \mathfrak{F}_i(t) &= F_{i+1}(t) - F_i(t), \\ \mathfrak{G}_0(t) &= G_1(t) - G_0(t) + 1, \\ \mathfrak{G}_{i+1}(t) &= G_{i+2}(t) - G_{i+1}(t) \end{aligned} \quad (8)$$

are completely monotonic functions on  $(0, \infty)$ . In particular, the sequences  $F_i(t)$  and  $G_i(t)$  are increasing with respect to  $i$  for  $t \in (0, \infty)$ ; that is, the inequalities

$$F_i(t) < F_{i+1}(t), \quad G_i(t) < G_{i+1}(t) \quad (9)$$

are valid for all  $i \in \{0\} \cup \mathbb{N}$  and  $t \in (0, \infty)$ .

**Theorem 3.** For given  $i \in \{0\} \cup \mathbb{N}$ , the ratios

$$\mathcal{F}_i(t) = \frac{F_{i+1}(t)}{F_i(t)}, \quad \mathcal{G}_i(t) = \frac{G_{i+1}(t)}{G_i(t)} \quad (10)$$

are decreasing on  $(0, \infty)$ , with

$$\lim_{t \rightarrow \infty} \mathcal{F}_i(t) = 1, \quad \lim_{t \rightarrow \infty} \mathcal{G}_0(t) = 0, \quad \lim_{t \rightarrow \infty} \mathcal{G}_{i+1}(t) = 1. \quad (11)$$

In order to further show the importance and significance of the two functional sequences in (6), we secondly give several applications of Theorems 1 to 3 in Section 4.

Finally, we pose a conjecture on the complete monotonicity of the functions  $\mathcal{F}_i(t)$  and  $\mathcal{G}_i(t)$  defined in (10).

## 2. Two Definitions and a Lemma

Now we list definitions of the completely monotonic and the logarithmically completely monotonic functions, which just now appeared in Theorems 1 and 2, and recite a lemma, which is needed to prove Theorem 3.

**Definition 4** (see [4, 5]). A function  $q(x)$  is said to be completely monotonic on an interval  $I$  if  $q(x)$  has derivatives of all orders on  $I$  and  $(-1)^n q^{(n)}(x) \geq 0$  for  $x \in I$  and  $n \geq 0$ .

The noted Hausdorff-Bernstein-Widder theorem [5, page 161, Theorem 12b] says that a necessary and sufficient condition that  $f(x)$  should be completely monotonic for  $0 < x < \infty$  is that

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t), \quad (12)$$

where  $\alpha(t)$  is nondecreasing and the integral converges for  $0 < x < \infty$ . In other words, a function  $f$  defined on  $(0, \infty)$  is completely monotonic on  $(0, \infty)$  if and only if it is a Laplace transform. For more information on the theory of completely monotonic functions, please refer to [4, Chapter XIII], [5, Chapter IV], and the newly published monograph [6]. This means that it is useful to confirm the complete monotonicity of functions.

**Definition 5** (see [7, 8]). A positive function  $f(x)$  is said to be logarithmically completely monotonic on an interval  $I \subseteq \mathbb{R}$  if it has derivatives of all orders on  $I$  and its logarithm  $\ln f(x)$  satisfies  $(-1)^k [\ln f(x)]^{(k)} \geq 0$  for  $k \in \mathbb{N}$  on  $I$ .

It has been proved that any logarithmically completely monotonic function on  $I$  is also completely monotonic on  $I$ ,

but not conversely. For more information on this class of functions, please refer to [7–9] and [10, Section 1.3] and closely related references therein. This shows that it is helpful to prove the logarithmically complete monotonicity of functions.

**Lemma 6** (see [11, Lemma 2.2]). *Suppose that  $a_k, b_k > 0$  and that  $\{u_k(t)\}$  is a sequence of positive and differentiable functions such that the series*

$$\sum_{k=0}^{\infty} a_k u_k^{(\ell)}(t), \quad \sum_{k=0}^{\infty} b_k u_k^{(\ell)}(t), \quad \ell = 0, 1, \quad (13)$$

*converge absolutely and uniformly over compact subsets of  $[0, \infty)$ .*

- (1) *If the logarithmic derivative  $u'_k(t)/u_k(t)$  forms an increasing sequence of functions and if  $a_k/b_k$  decreases (resp., increases), then*

$$f(t) = \frac{\sum_{k=0}^{\infty} a_k u_k(t)}{\sum_{k=0}^{\infty} b_k u_k(t)} \quad (14)$$

*decreases (resp., increases) for  $t \geq 0$ .*

- (2) *If the logarithmic derivative  $u'_k(t)/u_k(t)$  forms a decreasing sequence of functions and if  $a_k/b_k$  decreases (resp., increases), then the function  $f(t)$  increases (resp., decreases) for  $t \geq 0$ .*

### 3. Proofs of Main Results

Now we start out to prove our theorems.

*Proof of Theorem 1.* Since

$$\begin{aligned} F_0(t) &= G_0(t) - 1 = \sum_{\ell=1}^{\infty} e^{-\ell t}, \\ F_i(t) &= G_i(t) = \sum_{\ell=1}^{\infty} \ell^i e^{-\ell t} \end{aligned} \quad (15)$$

for  $i \geq 1$ , the functions  $F_i(t)$  and  $G_i(t)$  for  $i \in \{0\} \cup \mathbb{N}$  are completely monotonic on  $(0, \infty)$ .

Taking the logarithms of the functions  $F_0(t)$  and  $G_0(t)$  and differentiating yield

$$\begin{aligned} [\ln F_0(t)]' &= \frac{d}{dt} \left( \ln \frac{1}{e^t - 1} \right) = -\frac{e^t}{e^t - 1} = -[F_0(t) + 1], \\ [\ln G_0(t)]' &= \frac{d}{dt} \left( \ln \frac{1}{1 - e^{-t}} \right) = -F_0(t). \end{aligned} \quad (16)$$

Consequently, by definition of logarithmically completely monotonic functions and the above obtained complete monotonicity of the function  $F_0(t)$ , it is ready to deduce the logarithmically complete monotonicity of  $F_0(t)$  and  $G_0(t)$  on  $(0, \infty)$ .

The formulas in (7) can be straightforwardly verified. The proof of Theorem 1 is complete.  $\square$

*Proof of Theorem 2.* A simple computation yields

$$\begin{aligned} F_{i+1}(t) - F_i(t) &= (-1)^{i+1} \left[ \left( \frac{1}{e^t - 1} \right)^{(i+1)} + \left( \frac{1}{e^t - 1} \right)^{(i)} \right] \\ &= (-1)^{i+1} \left[ \left( \sum_{\ell=1}^{\infty} e^{-\ell t} \right)^{(i+1)} + \left( \sum_{\ell=1}^{\infty} e^{-\ell t} \right)^{(i)} \right] \\ &= (-1)^{i+1} \left[ \sum_{\ell=1}^{\infty} (-1)^{i+1} \ell^{i+1} e^{-\ell t} + \sum_{\ell=1}^{\infty} (-1)^i \ell^i e^{-\ell t} \right] \\ &= \sum_{\ell=2}^{\infty} (\ell - 1) \ell^i e^{-\ell t} \\ &> 0, \end{aligned} \quad (17)$$

$$(-1)^n [F_{i+1}(t) - F_i(t)]^{(n)} = \sum_{\ell=2}^{\infty} (\ell - 1) \ell^{n+i} e^{-\ell t} > 0$$

for  $n \in \mathbb{N}$ . By definition, the difference  $\mathfrak{F}_i(t) = F_{i+1}(t) - F_i(t)$  for  $i \in \{0\} \cup \mathbb{N}$  is completely monotonic on  $(0, \infty)$ .

From the first equalities in (15), respectively, it follows that the functions

$$\begin{aligned} \mathfrak{G}_0(t) &= G_1(t) - G_0(t) + 1 = F_1(t) - F_0(t), \\ \mathfrak{G}_{i+1}(t) &= G_{i+2}(t) - G_{i+1}(t) = F_{i+2}(t) - F_{i+1}(t) \end{aligned} \quad (18)$$

are all completely monotonic on  $(0, \infty)$ .

The inequalities in (9) follow from the positivity of  $\mathfrak{F}_i(t)$  and  $\mathfrak{G}_i(t)$  for  $i \in \mathbb{N}$ . The proof of Theorem 2 is complete.  $\square$

*Proof of Theorem 3.* It is easy to see that

$$\begin{aligned} \mathcal{F}_i(t) &= -\frac{[1/(e^t - 1)]^{(i+1)}}{[1/(e^t - 1)]^{(i)}} = -\frac{(\sum_{k=1}^{\infty} e^{-kt})^{(i+1)}}{(\sum_{k=1}^{\infty} e^{-kt})^{(i)}} \\ &= -\frac{\sum_{k=1}^{\infty} (-1)^{i+1} k^{i+1} e^{-kt}}{\sum_{k=1}^{\infty} (-1)^i k^i e^{-kt}} = \frac{\sum_{k=1}^{\infty} k^{i+1} e^{-kt}}{\sum_{k=1}^{\infty} k^i e^{-kt}}. \end{aligned} \quad (19)$$

Let  $a_k = k^{i+1}$ ,  $b_k = k^i$ , and  $u_k(t) = e^{-kt}$ . Then  $a_k/b_k = k$  is an increasing sequence and  $u'_k(t)/u_k(t) = -k$  form a decreasing sequence. By Lemma 6, it follows that the function  $\mathcal{F}_i(t)$  is decreasing on  $(0, \infty)$ .

Taking  $t \rightarrow \infty$  at the very ends of (19) shows that

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathcal{F}_i(t) &= \lim_{t \rightarrow \infty} \frac{\sum_{k=1}^{\infty} k^{i+1} e^{-kt}}{\sum_{k=1}^{\infty} k^i e^{-kt}} \\ &= \lim_{t \rightarrow \infty} \frac{1 + \sum_{k=2}^{\infty} k^{i+1} e^{(1-k)t}}{1 + \sum_{k=2}^{\infty} k^i e^{(1-k)t}} = 1. \end{aligned} \quad (20)$$

The first limit in (11) thus follows.

Making use of the second relation in (15) yields  $\mathcal{G}_i(t) = \mathcal{F}_i(t)$  for  $i \in \mathbb{N}$ . Accordingly, the function  $\mathcal{G}_i(t)$  has the same monotonicity and the same limit for  $t \rightarrow \infty$  as  $\mathcal{F}_i(t)$  does for

$i \in \mathbb{N}$ . As a result, the function  $\mathcal{G}_i(t)$  is decreasing on  $(0, \infty)$  and the third limit in (11) is valid for  $i \in \mathbb{N}$ .

A straightforward computation gives

$$\mathcal{G}_0(t) = \frac{1}{e^t - 1} = F_0(t) \quad (21)$$

which obviously tends to 0 as  $t \rightarrow \infty$  and apparently decreases on  $(0, \infty)$ . The proof of Theorem 3 is complete.  $\square$

#### 4. Some Applications

We recall from [12] that the polylogarithm  $\text{Li}_s(z)$  is the function

$$\text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} \quad (22)$$

defined for all  $s \in \mathbb{C}$  and over the open unit disk  $|z| < 1$  in the complex plane  $\mathbb{C}$ . Its definition on the whole complex plane then follows uniquely via analytic continuation.

For given  $i \in \{0\} \cup \mathbb{N}$ , since  $F_i(t) = \sum_{\ell=1}^{\infty} \ell^i e^{-\ell t}$ , we observe that

$$F_i(t) = \text{Li}_{-i}(e^{-t}), \quad G_{i+1}(t) = \text{Li}_{-(i+1)}(e^{-t}). \quad (23)$$

Therefore, considering (15), Theorems 1 to 3 can be used to find some properties of the polylogarithm  $\text{Li}_s(z)$  as follows.

**Theorem 7.** For  $i \in \{0\} \cup \mathbb{N}$ , the polylogarithm  $\text{Li}_{-i}(e^{-t})$  is completely monotonic with respect to  $t \in (0, \infty)$ . More strongly, the polylogarithm  $\text{Li}_0(e^{-t})$  is logarithmically completely monotonic with respect to  $t \in (0, \infty)$ .

For  $i, k \in \{0\} \cup \mathbb{N}$ , the polylogarithm  $\text{Li}_{-i}(e^{-t})$  satisfies

$$[\text{Li}_{-i}(e^{-t})]^{(k)} = (-1)^k \text{Li}_{-(i+k)}(e^{-t}). \quad (24)$$

**Theorem 8.** For given  $i \in \{0\} \cup \mathbb{N}$ , the differences

$$\begin{aligned} \mathfrak{F}_i(t) &= \text{Li}_{-(i+1)}(e^{-t}) - \text{Li}_{-i}(e^{-t}), \\ \mathfrak{G}_i(t) &= \begin{cases} \text{Li}_{-i}(e^{-t}) - \text{Li}_0(e^{-t}) + 1, & i = 0, \\ \mathfrak{F}_{i+1}(t), & i \geq 1, \end{cases} \end{aligned} \quad (25)$$

are completely monotonic functions on  $(0, \infty)$ . In particular, the polylogarithm sequence  $\text{Li}_{-i}(e^{-t})$  is increasing with respect to  $i$  for given  $t \in (0, \infty)$ ; that is, the inequality

$$\text{Li}_{-i}(e^{-t}) < \text{Li}_{-(i+1)}(e^{-t}) \quad (26)$$

is valid for all  $i \in \{0\} \cup \mathbb{N}$  and  $t \in (0, \infty)$ .

**Theorem 9.** For given  $i \in \{0\} \cup \mathbb{N}$ , the ratios

$$\begin{aligned} \mathcal{F}_i(t) &= \frac{\text{Li}_{-(i+1)}(e^{-t})}{\text{Li}_{-i}(e^{-t})}, \\ \mathcal{G}_i(t) &= \begin{cases} \mathcal{F}_i(t), & i \geq 1, \\ \frac{\text{Li}_{-i}(e^{-t})}{1 + \text{Li}_0(e^{-t})}, & i = 0, \end{cases} \end{aligned} \quad (27)$$

are decreasing on  $(0, \infty)$ , with

$$\lim_{t \rightarrow \infty} \mathcal{F}_i(t) = 1, \quad \lim_{t \rightarrow \infty} \mathcal{G}_0(t) = 0, \quad \lim_{t \rightarrow \infty} \mathcal{G}_{i+1}(t) = 1. \quad (28)$$

Furthermore, by the above (logarithmically) complete monotonicity in Theorems 7 and 8 and by some complete monotonicity properties of composite functions, we can obtain the complete monotonicity of functions involving the polylogarithm  $\text{Li}_{-i}(1/t)$  for  $i \in \{0\} \cup \mathbb{N}$  as follows.

**Theorem 10.** The following complete monotonicity is valid.

- (1) The polylogarithm  $\text{Li}_0(1/t) = 1/(t-1)$  is logarithmically completely monotonic with respect to  $t \in (1, \infty)$ .
- (2) For  $i \in \{0\} \cup \mathbb{N}$ , the polylogarithm  $\text{Li}_{-i}(1/t)$  is completely monotonic with respect to  $t \in (1, \infty)$ .
- (3) For given  $i \in \{0\} \cup \mathbb{N}$ , the differences

$$\begin{aligned} \mathfrak{P}_i(t) &= \text{Li}_{-(i+1)}\left(\frac{1}{t}\right) - \text{Li}_{-i}\left(\frac{1}{t}\right), \\ \mathfrak{Q}_i(t) &= \begin{cases} \text{Li}_{-i}\left(\frac{1}{t}\right) - \text{Li}_0\left(\frac{1}{t}\right) + 1, & i = 0, \\ \mathfrak{P}_{i+1}(t), & i \geq 1, \end{cases} \end{aligned} \quad (29)$$

are completely monotonic functions on  $(1, \infty)$ .

*Proof.* The second item of [13, Theorem 5] tells us that if  $h'(x)$  is completely monotonic on an interval  $I$  and  $f(x)$  is logarithmically completely monotonic on the domain  $h(I)$ , then the composite function  $f \circ h(x) = f(h(x))$  is logarithmically completely monotonic on  $I$ . It is easy to verify that the derivative of  $\ln t$  is  $1/t$  and completely monotonic on  $(0, \infty)$ . Combining these conclusions with the logarithmically complete monotonicity of  $\text{Li}_0(e^{-t})$  in Theorem 7 yields that the polylogarithm  $\text{Li}_0(1/t) = 1/(t-1)$  is logarithmically completely monotonic with respect to  $t \in (1, \infty)$ .

In [14, page 83], it was given that if  $f$  and  $g$  are functions such that  $f(g(x))$  is defined on  $(0, \infty)$  and if  $f$  and  $g'$  are completely monotonic, then  $x \mapsto f(g(x))$  is also completely monotonic on  $(0, \infty)$ . Replacing  $f(t)$  by  $\text{Li}_{-i}(e^{-t})$  and  $g(t)$  by  $\ln t$  and making use of the complete monotonicity of  $\text{Li}_{-i}(e^{-t})$  in Theorem 7 lead to the complete monotonicity of the polylogarithm  $\text{Li}_{-i}(1/t)$ .

The leftover proofs are similar to the above arguments. The proof of Theorem 10 is complete.  $\square$

**Remark 11.** To the best of our knowledge, the above (logarithmically) complete monotonicity results concerning the polylogarithm are new. This shows us the importance of the two functional sequences in (6) and the significance of Theorems 1 to 3.

## 5. A Conjecture

By Theorem 1 and the fact that any logarithmically completely monotonic function on  $I$  is also completely monotonic on  $I$ , it is clear that the functions

$$\begin{aligned}\mathcal{G}_0(t) &= \mathcal{F}_0(t) - 1 = \frac{1}{e^t - 1} = F_0(t), \\ \mathcal{G}_1(t) &= \mathcal{F}_1(t) = 1 + \frac{2}{e^t - 1} = 1 + 2F_0(t)\end{aligned}\quad (30)$$

are all completely monotonic on  $(0, \infty)$ . This motivates us to pose the following conjecture.

**Conjecture 12.** For given  $i \in \{0\} \cup \mathbb{N}$ , the functions  $\mathcal{F}_i(t)$  and  $\mathcal{G}_i(t)$  defined in (10) or (27) are completely monotonic on  $(0, \infty)$ .

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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