Research Article

Best Proximity Point Results in G-Metric Spaces

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G-metric spaces proved to be a rich source for fixed point theory; however, the best proximity point problem has not been considered in such spaces. The aim of this paper is to introduce certain new classes of proximal contraction mappings and establish the best proximity point theorems for such kind of mappings in G-metric spaces. As a consequence of these results, we deduce certain new best proximity and fixed point results. Moreover, we present an example to illustrate the usability of the obtained results.

1. Introduction and Preliminaries

The best approximation results provide an approximate solution to the fixed point equation $Tx = x$, when the non-self-mapping T has no fixed point. In particular, a well-known best approximation theorem, due to Fan [1], asserts the fact that if K is a nonempty compact convex subset of a Hausdorff locally convex topological vector space E and $T : K \rightarrow E$ is a continuous mapping, then there exists an element x satisfying the condition $d(x, Tx) = \inf\{d(y, Tx) : y \in K\}$, where d is a metric on E.

The best proximity point evolves as a generalization of the concept of the best approximation. The best approximation theorem guarantees the existence of an approximate solution; the best proximity point theorem is contemplated for solving the problem to find an approximate solution which is optimal. Given nonempty closed subsets A and B of E, when a non-self-mapping $T : A \rightarrow B$ has not a fixed point, it is quite natural to find an element x^* such that $d(x^*, Tx^*)$ is minimum. The best proximity point theorems guarantee the existence of an element x^* such that $d(x^*, Tx^*) = d(A, B) := \inf\{d(x, y) : x \in A, y \in B\}$; this element is called the best proximity point of T. Moreover, if the mapping under consideration is a self-mapping, the best proximity point theorem reduces to a fixed point result. For some results in this direction, we refer to [2–7] and references therein.

On the other hand, Mustafa and Sims introduced the notion of G-metric and investigated the topology of such spaces. The authors also characterized some celebrated fixed point results in the context of G-metric space. Following this initial paper, a number of authors have published so many fixed point results on the setting of G-metric space (see [8–14] and references therein). Samet et al. [15] and Jleli and Samet [16] reported that some published results can be considered a straight consequence of the existence theorem in the setting of usual metric space. More recently, Asadi et al. [17] proved some fixed point theorems in the framework of G-metric space that cannot be obtained from the existence results in the context of associated metric space. G-metric spaces proved to be rich for fixed point theory but the best proximity problem remains open. In this paper we prove certain best proximity point results and as consequence we deduce some recent fixed point results as corollaries.

First we recollect some necessary definitions and results in this direction. The notion of G-metric spaces is defined as follows.

Definition 1 (see [18]). Let X be a nonempty set and let $G : X \times X \times X \rightarrow \mathbb{R}^+$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$, if $x = y = z$,

(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables),

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ (rectangle inequality) for all $x, y, z, a \in X$.

Then the function G is called a generalized metric or, more specifically, a G-metric on X, and the pair (X, G) is called a G-metric space.

Note that every G-metric on X induces a metric d_G on X defined by

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \quad \forall x, y \in X. \quad (1)$$

For a better understanding of the subject we give the following examples of G-metrics.

Example 2. Let (X, d) be a metric space. The function $G : X \times X \times X \rightarrow [0, +\infty)$, defined by

$$G(x, y, z) = \max \{d(x, y), d(y, z), d(z, x)\}, \quad (2)$$

for all $x, y, z \in X$, is a G-metric on X.

Example 3 (see, e.g., [18]). Let $X = [0, \infty)$. The function $G : X \times X \times X \rightarrow [0, +\infty)$, defined by

$$G(x, y, z) = |x - y| + |y - z| + |z - x|, \quad (3)$$

for all $x, y, z \in X$, is a G-metric on X.

In their initial paper, Mustafa and Sims [18] also defined the basic topological concepts in G-metric spaces as follows.

Definition 4 (see [18]). Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points of X. We say that $\{x_n\}$ is G-convergent to $x \in X$ if

$$\lim_{n, m \rightarrow +\infty} G(x, x_n, x_m) = 0; \quad (4)$$

that is, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$. We call x the limit of the sequence and write $x_n \rightarrow x$ or $\lim_{n \rightarrow +\infty} x_n = x$.

Proposition 5 (see [18]). *Let (X, G) be a G-metric space. The following are equivalent:*

(1) *$\{x_n\}$ is G-convergent to x,*

(2) *$G(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow +\infty$,*

(3) *$G(x_n, x, x) \rightarrow 0$ as $n \rightarrow +\infty$,*

(4) *$G(x_n, x_m, x) \rightarrow 0$ as $n, m \rightarrow +\infty$.*

Definition 6 (see [18]). Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called a G-Cauchy sequence if, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $m, n, l \geq N$; that is, $G(x_n, x_m, x_l) \rightarrow 0$ as $n, m, l \rightarrow +\infty$.

Proposition 7 (see [18]). *Let (X, G) be a G-metric space. Then the following are equivalent:*

(1) *the sequence $\{x_n\}$ is G-Cauchy,*

(2) *for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $m, n \geq N$.*

Definition 8 (see [18]). A G-metric space (X, G) is called G-complete if every G-Cauchy sequence is G-convergent in (X, G).

Definition 9. Let (X, G) be a G-metric space. A mapping $F : X \times X \times X \rightarrow X$ is said to be continuous if, for any three G-convergent sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converging to x, y, and z, respectively, $\{F(x_n, y_n, z_n)\}$ is G-convergent to $F(x, y, z)$.

Mustafa [19] extended the well-known Banach contraction principle mapping in the framework of G-metric spaces as follows.

Theorem 10 (see [19]). *Let (X, G) be a complete G-metric space and let $T : X \rightarrow X$ be a mapping satisfying the following condition for all $x, y, z \in X$:*

$$G(Tx, Ty, Tz) \leq kG(x, y, z), \quad (5)$$

where $k \in [0, 1)$. Then T has a unique fixed point.

Theorem 11 (see [19]). *Let (X, G) be a complete G-metric space and let $T : X \rightarrow X$ be a mapping satisfying the following condition for all $x, y \in X$:*

$$G(Tx, Ty, Ty) \leq kG(x, y, y), \quad (6)$$

where $k \in [0, 1)$. Then T has a unique fixed point.

Remark 12. We notice that condition (5) implies condition (6). The converse is true only if $k \in [0, 1/2)$. For details see [19].

Lemma 13 (see [19]). *By the rectangle inequality (G5) together with the symmetry (G4), we have*

$$G(x, y, y) = G(y, y, x) \leq G(y, x, x) + G(x, y, x)$$
$$= 2G(y, x, x). \quad (7)$$

2. Main Results

At first we assume that

$$\Psi = \{\psi : [0, \infty) \longrightarrow [0, \infty) \text{ such that } \psi \text{ is}$$
$$\text{nondecreasing and continuous}\},$$
$$\Phi = \{\phi : [0, \infty) \longrightarrow [0, \infty) \text{ such that } \phi \text{ is}$$
$$\text{lower semicontinuous}\}, \quad (8)$$

where $\psi(t) = \phi(t) = 0$ if and only if $t = 0$.

Recall that every G-metric on X induces a metric d_G on X defined by

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \quad \forall x, y \in X. \quad (9)$$

Let (X, G) be a G-metric space. Suppose that A and B are nonempty subsets of a G-metric space (X, G). We define the following sets:

$$A_0 = \{x \in A : d_G(x, y) = d_G(A, B) \text{ for some } y \in B\},$$

$$B_0 = \{y \in B : d_G(x, y) = d_G(A, B) \text{ for some } x \in A\}, \tag{10}$$

where $d_G(A, B) = \inf\{d_G(x, y) : x \in A, y \in B\}$.

Definition 14. Let (X, G) be a G-metric space and let A and B be two nonempty subsets of X. Then B is said to be approximatively compact with respect to A if every sequence $\{y_n\}$ in B, satisfying the condition $d_G(x, y_n) \to d_G(x, B)$ for some x in A, has a convergent subsequence.

Definition 15. Let A and B be two nonempty subsets of a G-metric space (X, G). Let $T : A \to B$ be a non-self-mapping. We say T is a G-ψ-ϕ-proximal contractive mapping if, for $x, y, u, u^*, v \in A$,

$$d_G(u, Tx) = d_G(A, B)$$

$$d_G(u^*, Tu) = d_G(A, B)$$

$$d_G(v, Ty) = d_G(A, B) \tag{11}$$

$$\Downarrow$$

$$\psi(G(u, u^*, v)) \leq \psi(G(x, u, y)) - \phi(G(x, u, y))$$

holds where $\psi \in \Psi$ and $\phi \in \Phi$.

Theorem 16. *Let A, B be two nonempty subsets of a G-metric space (X, G) such that (A, G) is a complete G-metric space, A_0 is nonempty, and B is approximatively compact with respect to A. Assume that $T : A \to B$ is a G-ψ-ϕ-proximal contractive mapping such that $T(A_0) \subseteq B_0$. Then T has the unique best proximity point; that is, there exists unique $z \in A$ such that $d_G(z, Tz) = d_G(A, B)$.*

Proof. Since the subset A_0 is not empty, we take x_0 in A_0. Taking $Tx_0 \in T(A_0) \subseteq B_0$ into account, we can find $x_1 \in A_0$ such that $d_G(x_1, Tx_0) = d_G(A, B)$. Further, since $Tx_1 \in T(A_0) \subseteq B_0$, it follows that there is an element x_2 in A_0 such that $d_G(x_2, Tx_1) = d_G(A, B)$. Recursively, we obtain a sequence $\{x_n\}$ in A_0 satisfying

$$d_G(x_{n+1}, Tx_n) = d_G(A, B) \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{12}$$

This shows that

$$d_G(u, Tx) = d_G(A, B),$$

$$d_G(u^*, Tu) = d_G(A, B), \tag{13}$$

$$d_G(v, Ty) = d_G(A, B),$$

where $x = x_{n-1}$, $u = x_n$, $u^* = x_{n+1}$, $y = x_n$, and $v = x_{n+1}$. Therefore from (11) we have

$$\psi(G(x_n, x_{n+1}, x_{n+1})) \leq \psi(G(x_{n-1}, x_n, x_n))$$

$$- \phi(G(x_{n-1}, x_n, x_n)) \tag{14}$$

$$\leq \psi(G(x_{n-1}, x_n, x_n))$$

which implies $G(x_n, x_{n+1}, x_{n+1}) \leq G(x_{n-1}, x_n, x_n)$. So the sequence $\{G(x_n, x_{n+1}, x_{n+1})\}$ is decreasing sequence in $\mathbb{R}^+$ and thus it is convergent to $t \in \mathbb{R}^+$. We claim that $t = 0$. Suppose, on the contrary, that $t > 0$. Taking limit as $n \to \infty$ in (14) we get

$$\psi(t) \leq \psi(t) - \phi(t) \tag{15}$$

which implies $\phi(t) = 0$. That is, $t = 0$ which is a contrary. Hence, $t = 0$. That is,

$$\lim_{n \to \infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{16}$$

We will show that $\{x_n\}_{n=0}^{\infty}$ is a G-Cauchy sequence.

Suppose, on the contrary, that there exists $\varepsilon > 0$ and a sequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$G(x_{m(k)}, x_{m(k)+1}, x_{n(k)}) \geq \varepsilon \tag{17}$$

with $n(k) \geq m(k) > k$. Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (17). Hence,

$$G(x_{m(k)}, x_{m(k)+1}, x_{n(k)-1}) < \varepsilon. \tag{18}$$

By Proposition 5(iii) and (G5) we have

$$\varepsilon \leq G(x_{m(k)}, x_{m(k)+1}, x_{n(k)}) = G(x_{n(k)}, x_{m(k)}, x_{m(k)+1})$$

$$\leq G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{m(k)+1}, x_{m(k)})$$

$$\leq G(x_{m(k)}, x_{m(k)+1}, x_{n(k)-1}) + 2s_{n(k)-1}$$

$$\leq \varepsilon + 2s_{n(k)-1}. \tag{19}$$

Letting $k \to \infty$ in (19) we derive that

$$\lim_{k \to \infty} G(x_{m(k)}, x_{m(k)+1}, x_{n(k)}) = \varepsilon. \tag{20}$$

Also, by Proposition 5(iii) and (G5) we obtain the following inequalities:

$$
\begin{aligned}
G\left(x_{m(k)}, x_{m(k)+1}, x_{n(k)}\right) &\leq G\left(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}\right) \\
&\quad + G\left(x_{m(k)-1}, x_{m(k)+1}, x_{n(k)}\right) \\
&= G\left(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}\right) \\
&\quad + G\left(x_{n(k)}, x_{m(k)-1}, x_{m(k)+1}\right) \\
&\leq G\left(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}\right) \\
&\quad + G\left(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}\right) \\
&\quad + G\left(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)+1}\right) \\
&\leq 2 s_{m(k)-1} + 2 s_{n(k)-1} \\
&\quad + G\left(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)+1}\right),
\end{aligned}
$$

$$
\begin{aligned}
G\left(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)+1}\right) &\leq G\left(x_{n(k)-1}, x_n, x_n\right) \\
&\quad + G\left(x_{n(k)}, x_{m(k)-1}, x_{m(k)+1}\right) \\
&= G\left(x_{n(k)-1}, x_n, x_n\right) \\
&\quad + G\left(x_{m(k)-1}, x_{m(k)+1}, x_{n(k)}\right) \\
&\leq G\left(x_{n(k)-1}, x_n, x_n\right) \\
&\quad + G\left(x_{m(k)-1}, x_{m(k)}, x_{m(k)}\right) \\
&\quad + G\left(x_{m(k)}, x_{m(k)+1}, x_{n(k)}\right) \\
&= s_{n(k)-1} + s_{m(k)-1} \\
&\quad + G\left(x_{m(k)}, x_{m(k)+1}, x_{n(k)}\right).
\end{aligned}
\tag{21}
$$

Letting $k \rightarrow \infty$ in (21) and applying (20) we find that

$$
\lim_{k \rightarrow \infty} G\left(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)+1}\right) = \varepsilon.
\tag{22}
$$

Again by Proposition 5(iii) and (G5) we have

$$
\begin{aligned}
G\left(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)+1}\right) &= G\left(x_{m(k)+1}, x_{m(k)-1}, x_{n(k)-1}\right) \\
&= G\left(x_{m(k)+1}, x_{m(k)-1}, x_{n(k)-1}\right) \\
&\leq G\left(x_{m(k)+1}, x_{m(k)}, x_{m(k)}\right) \\
&\quad + G\left(x_{m(k)}, x_{m(k)-1}, x_{n(k)-1}\right) \\
&= G\left(x_{m(k)+1}, x_{m(k)}, x_{m(k)}\right) \\
&\quad + G\left(x_{m(k)-1}, x_{m(k)}, x_{n(k)-1}\right) \\
&\leq 2 s_{m(k)} \\
&\quad + G\left(x_{m(k)-1}, x_{m(k)}, x_{n(k)-1}\right),
\end{aligned}
$$

$$
\begin{aligned}
G\left(x_{m(k)-1}, x_{m(k)}, x_{n(k)-1}\right) &= G\left(x_{m(k)-1}, x_{m(k)}, x_{n(k)-1}\right) \\
&\leq G\left(x_{m(k)-1}, x_{m(k)+1}, x_{m(k)+1}\right) \\
&\quad + G\left(x_{m(k)+1}, x_{m(k)}, x_{n(k)-1}\right) \\
&\leq G\left(x_{m(k)-1}, x_{m(k)}, x_{m(k)}\right) \\
&\quad + G\left(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}\right),
\end{aligned}
$$

$$
\begin{aligned}
G\left(x_{m(k)+1}, x_{m(k)}, x_{n(k)-1}\right) &= s_{m(k)-1} + s_{m(k)} \\
&\quad + G\left(x_{m(k)+1}, x_{m(k)}, x_{n(k)-1}\right) \\
&= s_{m(k)-1} + s_{m(k)} \\
&\quad + G\left(x_{m(k)}, x_{m(k)+1}, x_{n(k)-1}\right) \\
&< s_{m(k)-1} + s_{m(k)} + \varepsilon.
\end{aligned}
\tag{23}
$$

Taking limit as $k \rightarrow \infty$ in (23) and applying (22) we have

$$
\lim_{k \rightarrow \infty} G\left(x_{m(k)-1}, x_{m(k)}, x_{n(k)-1}\right) = \varepsilon.
\tag{24}
$$

By (11) with $x = x_{m(k)-1}$, $u = x_{m(k)-1}$, $u^* = x_{m(k)}$, $y = x_{n(k)-1}$, and $v = x_{n(k)}$ we have

$$
\begin{aligned}
&\psi\left(G\left(x_{m(k)}, x_{m(k)+1}, x_{n(k)}\right)\right) \\
&\quad \leq \psi\left(G\left(x_{m(k)-1}, x_{m(k)}, x_{n(k)-1}\right)\right) \\
&\quad\quad - \phi\left(G\left(x_{m(k)-1}, x_{m(k)}, x_{n(k)-1}\right)\right).
\end{aligned}
\tag{25}
$$

Taking limit as $k \rightarrow \infty$ in the above inequality we have

$$
\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon)
\tag{26}
$$

which implies $\varepsilon = 0$ which is a contradiction. Thus,

$$
\lim_{m,n \rightarrow \infty} G\left(x_m, x_{m+1}, x_n\right) = 0.
\tag{27}
$$

That is, $\{x_n\}_0^\infty$ is a Cauchy sequence. Since (A, G) is a complete G-metric space, so there exists $z \in A$ such that $x_n \rightarrow z$ as $n \rightarrow \infty$. On the other hand, for all $n \in \mathbb{N}$, we can write

$$
\begin{aligned}
d_G(z, B) &\leq d_G(z, Tx_n) \\
&\leq d_G(z, x_{n+1}) + d_G(x_{n+1}, Tx_n) \\
&= d_G(z, x_{n+1}) + d(A, B).
\end{aligned}
\tag{28}
$$

Taking the limit as $n \rightarrow +\infty$ in the above inequality, we get

$$
\lim_{n \rightarrow +\infty} d_G(z, Tx_n) = d_G(z, B) = d_G(A, B).
\tag{29}
$$

Since B is approximatively compact with respect to A, so the sequence $\{Tx_n\}$ has a subsequence $\{Tx_{n_k}\}$ that converges to some $y^* \in B$. Hence,

$$
d_G(z, y^*) = \lim_{n \rightarrow \infty} d_G\left(x_{n_k+1}, Tx_{n_k}\right) = d_G(A, B)
\tag{30}
$$

and so $z \in A_0$. Now, since $Tz \in T(A_0) \subseteq B_0$, there exists $w \in A_0$ such that $d_G(w, Tz) = d_G(A, B)$.

From (11) with $x = x_n$, $u = x_{n+1}$, $u^* = x_{n+2}$, $y = z$, and $v = w$ we have

$$\psi\left(G\left(x_{n+1}, x_{n+2}, w\right)\right) \leq \psi\left(G\left(x_n, x_{n+1}, z\right)\right)$$
$$-\phi\left(G\left(x_n, x_{n+1}, z\right)\right). \tag{31}$$

Taking limit as $n \to \infty$ we get

$$\psi\left(G\left(z, z, w\right)\right) \leq \psi(0) - \phi(0) = 0. \tag{32}$$

Then $G(z, z, w) = 0$. That is, $w = z$. Thus $d_G(z, Tz) = d_G(A, B)$. Therefore T has the best proximity point. To prove uniqueness, suppose that $p \neq q$, such that $d_G(p, Tp) = d_G(A, B)$ and $d_G(q, Tq) = d_G(A, B)$. Now by (65) with $x = u = u^* = p$ and $y = v = q$ we get

$$\psi\left(G\left(p, p, q\right)\right) \leq \psi\left(G\left(p, p, q\right)\right) - \phi\left(G\left(p, p, q\right)\right) \tag{33}$$

which implies $\phi(G(p, p, q)) = 0$; that is, $p = q$. □

Example 17. Let $X = [0, \infty)$ and $G(x, y, z) = (1/4)(|x - y| + |y - z| + |x - z|)$ be a G-metric on X. Then $d_G(x, y) = |x - y|$. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{6, 7, 8, 9, 10\}$. Define $T : A \to B$ by

$$T(x) = \begin{cases} 6, & \text{if } x = 5 \\ x + 5, & \text{otherwise.} \end{cases} \tag{34}$$

Also define $\psi, \phi : [0, \infty) \to [0, \infty)$ by $\psi(t) = t$ and $\phi(t) = (1/2)t$. Clearly, $d_G(A, B) = 1$, $A_0 = \{5\}$, $B_0 = \{6\}$, and $TA_0 \subseteq B_0$. Let $d_G(u, Tx) = d_G(A, B)$ and $d_G(v, Ty) = d_G(A, B) = 1$. Then $(u, x), (v, y) \in \{(5, 5), (5, 1)\}$. Also, if $d_G(u^*, Tu) = d_G(A, B) = 1$, then $u^* = 5$. Therefore, if

$$d_G(u, Tx) = d_G(A, B),$$
$$d_G\left(u^*, Tu\right) = d_G(A, B), \tag{35}$$
$$d_G(v, Ty) = d_G(A, B),$$

then

$$(u, u^*, v, x, y) \in \{(5, 5, 5, 5, 5), (5, 5, 5, 1, 1),$$
$$(5, 5, 5, 1, 5), (5, 5, 5, 5, 1)\}. \tag{36}$$

Now since $u = u^* = v = 5$ so, $\psi(G(u, u^*, v)) = 0$. Hence,

$$\psi\left(G\left(u, u^*, v\right)\right) = 0 \leq \frac{1}{2}G\left(x, u, y\right)$$
$$= \psi\left(G\left(x, u, y\right)\right) - \phi\left(G\left(x, u, y\right)\right). \tag{37}$$

That is,

$$d_G(u, Tx) = d_G(A, B),$$
$$d_G\left(u^*, Tu\right) = d_G(A, B),$$
$$d_G(v, Ty) = d_G(A, B) \tag{38}$$
$$\Downarrow$$
$$\psi\left(G\left(u, u^*, v\right)\right) \leq \psi\left(G\left(x, u, y\right)\right) - \phi\left(G\left(x, u, y\right)\right).$$

Thus T is a G-ψ-ϕ-proximal contractive mapping. All conditions of Theorem 16 hold true and T has the unique best proximity point. Here, $z = 5$ is the unique best proximity point of T.

If in Theorem 16 we take $\psi(t) = t$ and $\phi(t) = (1 - r)t$, where $0 \leq r < 1$, then we deduce the following corollary.

Corollary 18. *Let A, B be two nonempty subsets of a G-metric space (X, G) such that (A, G) is a complete G-metric space, A_0 is nonempty, and B is approximatively compact with respect to A. Assume that $T : A \to B$ is a non-self-mapping such that $T(A_0) \subseteq B_0$ and, for $x, y, u, u^*, v \in A$,*

$$d_G(u, Tx) = d_G(A, B),$$
$$d_G\left(u^*, Tu\right) = d_G(A, B),$$
$$d_G(v, Ty) = d_G(A, B) \tag{39}$$
$$\Downarrow$$
$$G\left(u, u^*, v\right) \leq rG\left(x, u, y\right)$$

holds where $0 \leq r < 1$. Then T has the unique best proximity point. That is, there exists unique $z \in A$ such that $d_G(z, Tz) = d_G(A, B)$.

Theorem 19. *Let A, B be two nonempty subsets of a G-metric space (X, G) such that (A, G) is a complete G-metric space, A_0 is nonempty, and B is approximatively compact with respect to A. Assume that $T : A \to B$ is a non-self-mapping such that $T(A_0) \subseteq B_0$ and, for $x, y, u, u^*, u^{**}, v, v^* \in A$,*

$$d_G(u, Tx) = d_G(A, B),$$
$$d_G\left(u^*, Tu\right) = d_G(A, B),$$
$$d_G\left(u^{**}, Tu^*\right) = d_G(A, B),$$
$$d_G(v, Ty) = d_G(A, B),$$
$$d_G\left(v^*, Tv\right) = d_G(A, B) \tag{40}$$
$$\Downarrow$$
$$G\left(u, v, v^*\right) \leq aG\left(x, u, u^*\right) + bG\left(y, v, v^*\right)$$
$$+ cG\left(x, u, v\right) + dG\left(y, v, u^{**}\right)$$

holds where $a, b, c, d \geq 0$ and $a + b + c + d < 1$. Then T has the best proximity point. Moreover, if $c < 1/2$, then the best proximity point of T is unique.

Proof. Following the same lines in the proof of Theorem 16, we can construct a sequence $\{x_n\}$ in A_0 satisfying

$$d_G\left(x_{n+1}, Tx_n\right) = d_G(A, B) \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{41}$$

From (40) with $x = x_{n-1}$, $u = x_n$, $u^* = x_{n+1}$, $u^{**} = x_{n+2}$, $y = x_n$, $v = x_{n+1}$, and $v^* = x_{n+2}$ we have

$$
\begin{aligned}
G\left(x_n, x_{n+1}, x_{n+2}\right) \leq\; & aG\left(x_{n-1}, x_n, x_{n+1}\right) + bG\left(x_n, x_{n+1}, x_{n+2}\right) \\
& + cG\left(x_{n-1}, x_n, x_{n+1}\right) \\
& + dG\left(x_n, x_{n+1}, x_{n+2}\right)
\end{aligned}
\tag{42}
$$

which implies

$$
G\left(x_n, x_{n+1}, x_{n+2}\right) \leq hG\left(x_{n-1}, x_n, x_{n+1}\right) \tag{43}
$$

where $k = (a + b)/(1 - b - c) < 1$. Thus,

$$
G\left(x_n, x_{n+1}, x_{n+2}\right) \leq k^n G\left(x_0, x_1, x_2\right) \tag{44}
$$

for all $n \in \mathbb{N}$. From (G3) we know that

$$
G\left(x_n, x_n, x_{n+1}\right) \leq G\left(x_n, x_{n+1}, x_{n+2}\right) \tag{45}
$$

with $x_n \neq x_{n+1}$ and by Proposition 5(iii) we know that

$$
G\left(x_{n+1}, x_{n+1}, x_n\right) \leq 2G\left(x_n, x_n, x_{n+1}\right). \tag{46}
$$

Thus using (44) we obtain

$$
G\left(x_{n+1}, x_{n+1}, x_n\right) \leq 2k^n G\left(x_0, x_1, x_2\right). \tag{47}
$$

Moreover, for all $n, m \in \mathbb{N}$, $n < m$, we have by rectangle inequality

$$
\begin{aligned}
G\left(x_m, x_n, x_n\right) \leq\; & G\left(x_m, x_{m-1}, x_{m-1}\right) + G\left(x_{n+1}, x_{n+2}, x_{n+2}\right) \\
& + G\left(x_{n+2}, x_{n+3}, x_{n+3}\right) + \cdots \\
& + G\left(x_{m-1}, x_m, x_m\right) \\
\leq\; & 2\left(k^n + k^{n+1} + k^{n+2} + \cdots + k^{m-1}\right) \\
& \times G\left(x_0, x_1, x_2\right) \\
\leq\; & \frac{2k^n}{1 - k} G\left(x_0, x_1, x_2\right),
\end{aligned}
\tag{48}
$$

and so $\lim_{m,n \to \infty} G(x_n, x_m, x_m) = 0$. Thus $\{x_n\}$ is a Cauchy sequence. Due to the completeness of (A, G), there exists $z \in A$ such that $\{x_n\}$ converges to z. As in proof of Theorem 16 we have $d_G(w, Tz) = d_G(A, B)$ for some $w \in A_0$. Again, since $Tw \in T(A_0) \subseteq B_0$, so there exists $w^* \in A_0$ such that $d_G(w^*, Tw) = d_G(A, B)$.

From (40) with $x = x_{n-1}$, $u = x_n$, $u^* = x_{n+1}$, $u^{**} = x_{n+2}$, $y = z$, $v = w$, and $v^* = w^*$ we have

$$
\begin{aligned}
G\left(x_n, w, w^*\right) \leq\; & aG\left(x_{n-1}, x_n, x_{n+1}\right) + bG\left(z, w, w^*\right) \\
& + cG\left(x_{n-1}, x_n, w\right) + dG\left(z, w, x_{n+2}\right).
\end{aligned}
\tag{49}
$$

By taking limit as $n \to \infty$ in the above inequality we get

$$
\begin{aligned}
G\left(z, w, w^*\right) \leq\; & aG\left(z, z, z\right) + bG\left(z, w, w^*\right) \\
& + cG\left(z, z, w\right) + dG\left(z, w, z\right)
\end{aligned}
\tag{50}
$$

which implies

$$
G\left(z, w, w^*\right) \leq \frac{c + d}{1 - b} G\left(z, z, w\right). \tag{51}
$$

Assume to contrary, $w \neq w^*$. Therefor from (G3) we have $G(z, z, w) \leq G(z, w, w^*)$. Then from (51) we deduce

$$
\begin{aligned}
G\left(z, w, w^*\right) & \leq \frac{c + d}{1 - b} G\left(z, z, w\right) \\
& \leq \frac{c + d}{1 - b} G\left(z, w, w^*\right) < G\left(z, w, w^*\right)
\end{aligned}
\tag{52}
$$

which is a contradiction. Hence, $w = w^*$; that is, $d_G(w, Tw) = d_G(w^*, Tw) = d_G(A, B)$. That is, T has the best proximity point. To prove uniqueness, suppose that $p \neq q$, $d_G(p, Tp) = d_G(A, B)$, and $d_G(q, Tq) = d_G(A, B)$. Now by (40) with $x = u = u^* = u^{**} = p$ and $y = v = v^* = q$ we have

$$
\begin{aligned}
G\left(p, q, q\right) \leq\; & aG\left(p, p, p\right) + bG\left(q, q, q\right) + cG\left(p, p, q\right) \\
& + dG\left(q, q, p\right)
\end{aligned}
\tag{53}
$$

which implies

$$
\begin{aligned}
G\left(p, q, q\right) & \leq \frac{c}{1 - d} G\left(p, p, q\right) \\
& \leq \frac{2c}{1 - d} G\left(p, q, q\right) < G\left(p, q, q\right)
\end{aligned}
\tag{54}
$$

which is a contradiction. Hence, $p = q$. That is, T has the unique best proximity point. □

Theorem 20. *Let A, B be two nonempty subsets of a G-metric space (X, G) such that (A, G) is a complete G-metric space, A_0 is nonempty, and B is approximatively compact with respect to A. Assume that $T : A \to B$ is a non-self-mapping such that $T(A_0) \subseteq B_0$ and, for $x, y, u, v \in A$,*

$$
\begin{aligned}
d_G\left(u, Tx\right) &= d_G\left(A, B\right), \\
d_G\left(v, Ty\right) &= d_G\left(A, B\right)
\end{aligned}
$$

$$
\Downarrow
$$

$$
G\left(u, v, v\right) \leq \alpha G\left(x, u, y\right) \tag{55}
$$

$$
+ \beta \frac{\sqrt{G\left(x, y, y\right) G\left(x, u, u\right)}}{1 + G\left(u, v, v\right)}
$$

$$
+ \gamma G\left(x, y, y\right) + \delta G\left(x, u, u\right)
$$

holds where $\alpha, \beta, \gamma, \delta \geq 0$ and $\alpha + \beta + \gamma + \delta < 1$. Then T has the best proximity point. Moreover, if $2\alpha + \gamma < 1$, then T has the unique best proximity point.

Proof. Following the same lines in the proof of Theorem 16, we can construct a sequence $\{x_n\}$ in A_0 satisfying

$$
d_G\left(x_{n+1}, Tx_n\right) = d_G\left(A, B\right) \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{56}
$$

From (55) with $x = x_{n-1}$, $u = x_n$, $y = x_n$, and $v = x_{n+1}$ we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq \alpha G(x_{n-1}, x_n, x_n)$$

$$+ \beta \frac{\sqrt{G(x_{n-1}, x_n, x_n) G(x_{n-1}, x_n, x_n)}}{1 + G(x_n, x_{n+1}, x_{n+1})}$$

$$+ \gamma G(x_{n-1}, x_n, x_n) + \delta G(x_{n-1}, x_n, x_n)$$

$$\leq (\alpha + \beta + \gamma + \delta) G(x_{n-1}, x_n, x_n)$$

(57)

for all $n \in \mathbb{N} \cup \{0\}$. This implies

$$G(x_n, x_{n+1}, x_{n+1}) \leq k^n G(x_0, x_1, x_1),$$ (58)

where $k = \alpha + \beta + \gamma + \delta < 1$. Now, for all $n, m \in \mathbb{N}$, $n < m$, we have by rectangle inequality

$$G(x_n, x_m, x_m) \leq G(x_n, x_{n+1}, x_{n+1})$$

$$+ G(x_{n+1}, x_{n+2}, x_{n+2})$$

$$+ G(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots$$

$$+ G(x_{m-1}, x_m, x_m)$$

$$\leq \left(k^n + k^{n+1} + k^{n+2} + \cdots + k^{m-1}\right)$$

$$\times G(x_0, x_1, x_1)$$

$$\leq \frac{k^n}{1 - k} G(x_0, x_1, x_1),$$

(59)

which implies $\lim_{n,m \to \infty} G(x_n, x_m, x_m) = 0$. Thus $\{x_n\}$ is a Cauchy sequence. Due to the completeness of (A, G), there exists $z \in A$ such that $\{x_n\}$ converges to z. Now as in proof of Theorem 16 there exists $w \in A_0$ such that $d_G(w, Tz) = d_G(A, B)$. Now from (55) with $x = x_{n-1}$, $u = x_n$, $y = z$, and $v = w$ we deduce

$$G(x_n, w, w) \leq \alpha G(x_{n-1}, x_n, z)$$

$$+ \beta \frac{\sqrt{G(x_{n-1}, z, z) G(x_{n-1}, x_n, x_n)}}{1 + G(x_n, w, w)}$$

$$+ \gamma G(x_{n-1}, z, z) + \delta G(x_{n-1}, x_n, x_n).$$

(60)

By taking limit as $n \to \infty$ in the above inequality we get $G(z, w, w) = 0$; that is, $z = w$. Hence, $d_G(z, Tz) = d_G(w, Tz) = d_G(A, B)$; that is, T has the best proximity point. To prove uniqueness, assume that $p \neq q$, such that $d_G(p, Tp) = d_G(A, B)$ and $d_G(q, Tq) = d_G(A, B)$. Now by (55) with $x = u = p$ and $y = v = q$ we have

$$G(p, q, q) \leq \alpha G(p, p, q)$$

$$+ \beta \frac{\sqrt{G(p, q, q) G(p, p, p)}}{1 + G(p, q, q)} + \gamma G(p, q, q)$$

$$+ \delta G(p, p, p)$$

(61)

which implies

$$G(p, q, q) \leq \frac{\alpha}{1 - \gamma} G(p, p, q)$$

$$\leq \frac{2\alpha}{1 - \gamma} G(p, q, q) < G(p, q, q)$$

(62)

which is a contradiction. Hence, $p = q$. $\qquad\square$

By taking $\beta = \gamma = \delta = 0$, in Theorem 20, we obtain the following Corollary.

Corollary 21. *Let A, B be two nonempty subsets of a G-metric space (X, G) such that (A, G) is a complete G-metric space, A_0 is nonempty, and B is approximatively compact with respect to A. Assume that $T : A \to B$ is a non-self-mapping such that $T(A_0) \subseteq B_0$ and, for $x, y, u, v \in A$,*

$$d_G(u, Tx) = d_G(A, B),$$

$$d_G(v, Ty) = d_G(A, B)$$

$$\Downarrow$$

$$G(u, v, v) \leq \alpha G(x, u, y)$$

(63)

holds where $0 \leq \alpha < 1$. Then T has the best proximity point. Moreover, if $\alpha < 1/2$, then T has the unique best proximity point.

3. Application to Fixed Point Theory

In this section, as an application of our best proximity results, we will derive certain new fixed point results.

Note that if

$$d_G(u, Tx) = d_G(A, B)$$

$$d_G(u^*, Tu) = d_G(A, B)$$

$$d_G(v, Ty) = d_G(A, B)$$

(64)

and $A = B = X$, then $u = Tx$, $u^* = Tu$, and $v = Ty$. That is, $u^* = T^2 x$. Therefore, if in Theorem 16 we take $A = B = X$, we deduce the following recent result.

Theorem 22 (Theorem 2.3 of [17]). *Let (X, G) be a complete G-metric space and let $T : X \to X$ be a mapping satisfying the following condition, for all $x, y \in X$, where $\psi \in \Psi$ and $\phi \in \Phi$:*

$$\psi\left(G\left(Tx, T^2 x, Ty\right)\right) \leq \psi\left(G\left(x, Tx, y\right)\right) - \phi\left(G\left(x, Tx, y\right)\right).$$

(65)

Then T has a unique fixed point.

Corollary 23. *Let (X, G) be a complete G-metric space and let $T : X \to X$ be a mapping satisfying the following condition for all $x, y \in X$, where $0 \leq r < 1$:*

$$G\left(Tx, T^2 x, Ty\right) \leq r G(x, Tx, y).$$

(66)

Then T has a unique fixed point.

Similarly we can deduce the following fixed point result from Theorem 19.

Theorem 24 (Theorem 2.2 of [17]). *Let (X, G) be a complete G-metric space and let $T : X \to X$ be a mapping satisfying the following condition for all $x, y \in X$, where $a, b, c, d \geq 0$ with $a + b + c + d < 1$:*

$$G\left(Tx, Ty, T^2 y\right) \leq aG\left(x, Tx, T^2 x\right) + bG\left(y, Ty, T^2 y\right)$$
$$+ cG\left(x, Tx, Ty\right) + cG\left(y, Ty, T^3 x\right). \tag{67}$$

Then T has a fixed point. Moreover, if $c < 1/2$, then T has a unique fixed point.

Finally, we can deduce the following fixed point result from Theorem 20.

Theorem 25. *Let (X, G) be a complete G-metric space and $T : X \to X$ be a mapping satisfying the following condition for all $x, y \in X$, where $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha + \beta + \gamma + \delta < 1$:*

$$G\left(Tx, Ty, Ty\right) \leq \alpha G\left(x, Tx, y\right)$$
$$+ \beta \frac{\sqrt{G\left(x, y, y\right) G\left(x, Tx, Tx\right)}}{1 + G\left(Tx, Ty, Ty\right)} \tag{68}$$
$$+ \gamma G\left(x, y, y\right) + \delta G\left(x, Tx, Tx\right).$$

Then T has a fixed point. Moreover, if $2\alpha + \gamma < 1$, then T has a unique fixed point.

By taking $\beta = \gamma = \delta = 0$ in the above theorem we have the following corollary.

Corollary 26 (Theorem 2.1 of [17]). *Let (X, G) be a complete G-metric space and let $T : X \to X$ be a mapping satisfying the following condition for all $x, y \in X$, where $0 \leq \alpha < 1$:*

$$G\left(Tx, Ty, Ty\right) \leq \alpha G\left(x, Tx, y\right). \tag{69}$$

Then T has a fixed point. Moreover, if $2\alpha + \gamma < 1$, then T has a unique fixed point.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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