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Research Article

Periodicity of the Positive Solutions of a Fuzzy Max-Difference Equation

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We investigate the periodic nature of the positive solutions of the fuzzy max-difference equation $x_{n+1} = \max\{A_n/x_{n-m}, x_{n-k}\}$, $n = 0, 1, \ldots$, where $k, m \in \{1, 2, \ldots\}$, A_n is a periodic sequence of fuzzy numbers, and $x_{-d}, x_{-d+1}, \ldots, x_0$ are positive fuzzy numbers with $d = \{m, k\}$. We show that every positive solution of this equation is eventually periodic with period $k + 1$.

1. Introduction

The max operator arises naturally in certain models in automatic control theory (see [1, 2]). In recent years, the discrete case involving difference equations with maximum has been receiving increasing attention (see [3–8]). Elsayed and Stević [9] considered the max-difference equation

$$x_{n+1} = \max\left\{\frac{B}{x_n}, x_{n-2}\right\}, \quad n = 0, 1, \ldots, \tag{1}$$

where $B \in \mathbf{R} \equiv (-\infty, +\infty)$ and the initial conditions $x_{-2}, x_{-1}, x_0 \in \mathbf{R}$ and showed that every well-defined solution of this equation is eventually periodic with period 3.

In [10], Iričanin and Elsayed investigated the max-difference equation

$$x_{n+1} = \max\left\{\frac{B}{x_n}, x_{n-3}\right\}, \quad n = 0, 1, \ldots, \tag{2}$$

where $B \in \mathbf{R}$ and the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0 \in \mathbf{R}$ and showed that every well-defined solution of this equation is eventually periodic with period 4.

Recently Xiao and Shi [11] studied the max-difference equation

$$x_{n+1} = \max\left\{\frac{B}{x_n}, x_{n-1}\right\}, \quad n = 0, 1, \ldots, \tag{3}$$

where $B \in \mathbf{R}$ and the initial conditions $x_{-1}, x_0 \in \mathbf{R}$ and showed that every well-defined solution of the above equation is eventually periodic with period 2.

In [12], we dealt with the max-difference equation

$$x_{n+1} = \max\left\{\frac{B}{x_n}, x_{n-k}\right\}, \quad n = 0, 1, \ldots, \tag{4}$$

where $B \in \mathbf{R}$, $k \in \{1, 2, \ldots\}$ and the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in \mathbf{R}$ and showed that every well-defined solution of the above equation is eventually periodic with period $k+1$, which extended the results of [9–11] to the general case.

Recently there has been an increase in interest in the study of fuzzy difference equations (see [13–15]). In [16], Stefanidou and Papaschinopoulos studied the periodicity of the positive solutions of the following fuzzy max-difference equation

$$x_n = \max\left\{\frac{A}{x_{n-k}}, \frac{B}{x_{n-m}}\right\}, \quad n = 0, 1, \ldots, \tag{5}$$

where A, B, and the initial conditions $x_{-d}, x_{-d+1}, \ldots, x_0$ with $d = \max\{k, m\}$ are positive fuzzy numbers.

In [17], Zhang et al. dealt with the existence, the boundedness, and the asymptotic behavior of the positive solutions to a first order fuzzy Ricatti difference equation

$$x_{n+1} = \frac{A + x_n}{B + x_n}, \quad n = 0, 1, \ldots, \tag{6}$$

where A, B, and the initial condition x_0 are positive fuzzy numbers.

In this note, our goal is to investigate the periodicity of the positive solutions of the fuzzy max-difference equation

$$x_{n+1} = \max\left\{\frac{A_n}{x_{n-m}}, x_{n-k}\right\}, \quad n = 0, 1, \ldots, \tag{7}$$

where $k, m \in \{1, 2, \ldots\}$, A_n is a periodic sequence of fuzzy numbers, and $x_{-d}, x_{-d+1}, \ldots, x_0$ are positive fuzzy numbers with $d = \{m, k\}$. Our main result is the following theorem.

Theorem 1. *Let $k, m \in \{1, 2, \ldots\}$ and A_n be a periodic sequence of fuzzy numbers. Then every positive solution of (7) is eventually periodic with period $k + 1$.*

2. Preliminaries

We need the following definitions. A function U from $\mathbf{R}^+ = (0, +\infty)$ into the interval $[0, 1]$ is called a fuzzy number if the following statements hold (see [18]).

(1) U is normal (i.e., $U(x) = 1$ for some $x \in \mathbf{R}^+$).

(2) U is a convex fuzzy set (i.e., $U(\lambda x + (1 - \lambda) y) \geq \min\{U(x), U(y)\}$ for any $\lambda \in [0, 1]$ and any $x, y \in \mathbf{R}^+$).

(3) U is upper semicontinuous.

(4) The support $\operatorname{supp} U = \overline{\bigcup_{a \in (0,1]} [U]_a} = \overline{\{x : U(x) > 0\}}$ is compact,

where $[U]_a = \{x \in \mathbf{R}^+ : U(x) \geq a\}$ (for any $a \in (0, 1]$) (which are said to be the a-cuts of the fuzzy number U) and $\overline{M}$ is the closure of set M. We see from [19, Theorem 3.1.5 and Theorem 3.1.8] the a-cuts of the fuzzy number U are closed intervals.

A fuzzy number U is said to be positive if $\min(\operatorname{supp} U) > 0$. If $U \in \mathbf{R}^+$, then U is a positive fuzzy number (it is called a trivial fuzzy number also) with $[U]_a = [U, U]$ for any $a \in (0, 1]$.

For some positive integer k, let $U_1, U_2, \ldots, U_k$ be fuzzy numbers and $a \in (0, 1]$ with

$$[U_i]_a = [U_{i,l,a}, U_{i,r,a}] \quad \text{for } 0 \leq i \leq k. \tag{8}$$

Write

$$V_{l,a} = \max\{U_{i,l,a} : 0 \leq i \leq k\},$$
$$V_{r,a} = \max\{U_{i,r,a} : 0 \leq i \leq k\}. \tag{9}$$

Then we know from [20, Theorem 2.1] that there exists a fuzzy number V such that

$$[V]_a = [V_{l,a}, V_{r,a}] \quad \text{for any } a \in (0, 1]. \tag{10}$$

By [21] and [22, Lemma 2.3] one can define

$$V = \max\{U_i : 0 \leq i \leq k\}. \tag{11}$$

A sequence of positive fuzzy numbers $\{x_n\}_{n=-d}^{\infty}$ is said to be a solution of (7) if it satisfies (7). If there exists a positive integer M and p such that, for all $n \geq M$,

$$x_{n+p} = x_n, \tag{12}$$

then $\{x_n\}_{n=-d}^{\infty}$ is said to be eventually periodic with period p.

Proposition 2. *Let $x_{-d}, x_{-d+1}, \ldots, x_0$ be a sequence of positive fuzzy numbers. Then there exists a unique positive solution $\{x_n\}_{n=-d}^{\infty}$ of (7) with initial values $x_{-d}, x_{-d+1}, \ldots, x_0$.*

Proof. Assume that $[A_n]_a = [A_{n,l,a}, A_{n,r,a}]$ (for any $a \in (0, 1]$) and $n \geq 0$. Let $x_{-d}, x_{-d+1}, \ldots, x_0$ be positive fuzzy numbers such that

$$[x_i]_a = [P_{i,a}, Q_{i,a}] \quad \text{for } -d \leq i \leq 0, \ a \in (0, 1], \tag{13}$$

and let $\{(P_{n,a}, Q_{n,a})\}_{n=-d}^{\infty} (a \in (0, 1])$ be the unique positive solution of the following system of difference equations:

$$P_{n+1,a} = \max\left\{\frac{A_{n,l,a}}{Q_{n-m,a}}, P_{n-k,a}\right\},$$
$$Q_{n+1,a} = \max\left\{\frac{A_{n,r,a}}{P_{n-m,a}}, Q_{n-k,a}\right\}, \tag{14}$$

with initial values $(P_{i,a}, Q_{i,a}) (-d \leq i \leq 0)$. Arguing as in Proposition 3.1 of [23] we may show that $\{(P_{n,a}, Q_{n,a})\}_{n=-d}^{\infty} (a \in (0, 1])$ determines a sequence of positive fuzzy numbers $\{x_n\}_{n=-d}^{\infty}$ with

$$[x_n]_a = [P_{n,a}, Q_{n,a}], \quad n \geq -d, a \in (0, 1], \tag{15}$$

and that $\{x_n\}_{n=-d}^{\infty}$ is the unique positive solution of (7) with initial values $x_{-d}, x_{-d+1}, \ldots, x_0$. This completes the proof of the proposition. □

3. Proof of Theorem 1

Lemma 3. *Consider the system of difference equations*

$$y_{n+1} = \max\left\{\frac{C_n}{z_{n-m}}, y_{n-k}\right\},$$
$$z_{n+1} = \max\left\{\frac{B_n}{y_{n-m}}, z_{n-k}\right\}, \quad n = 0, 1, \ldots, \tag{16}$$

where B_n, C_n are two periodic sequences of positive real numbers and the initial values $y_{-d}, z_{-d}, \ldots, y_0, z_0$ are positive real numbers. Then every positive solution of (16) is eventually periodic of period $k + 1$.

Proof. Let $\{(y_n, z_n)\}_{n=-d}^{\infty}$ be a positive solution of (16). We have from (16) that, for any $n \geq 0$ and any $i \geq 0$,

$$y_{(n+1)(k+1)+i} = \max\left\{\frac{C_{(n+1)(k+1)+i-1}}{z_{(n+1)(k+1)+i-m-1}}, y_{n(k+1)+i}\right\} \geq y_{n(k+1)+i}, \tag{17}$$

$$z_{(n+1)(k+1)+i} = \max\left\{\frac{B_{(n+1)(k+1)+i-1}}{y_{(n+1)(k+1)+i-m-1}}, z_{n(k+1)+i}\right\} \geq z_{n(k+1)+i}. \tag{18}$$

Then $\{y_{n(k+1)+i}\}_{n=0}^{\infty}$ and $\{z_{n(k+1)+i}\}_{n=0}^{\infty}$ are increasing for every $0 \leq i \leq k$.

Now we show that $\{y_{n(k+1)+i}\}_{n=0}^{+\infty}$ is a constant sequence eventually for every $0 \leq i \leq k$. Indeed, if $\{y_{n(k+1)+r}\}_{n=0}^{+\infty}$ is not constant sequence eventually for some $0 \leq r \leq k$, then there exist $km < n_1 < n_2 < \cdots$ such that $y_{n_i(k+1)+r} > y_{(n_i-1)(k+1)+r}$ and $C_{n_i(k+1)+r-1}$ is a constant sequence for all $i \geq 1$ since C_n is a periodic sequence. Thus we have

$$\begin{aligned} yn_{i+1}\,(k+1)+r &= \max\left\{\frac{C_{n_{i+1}(k+1)+r-1}}{z_{n_{i+1}(k+1)+r-m-1}}, y_{(n_{i+1}-1)(k+1)+r}\right\} \\ &= \frac{C_{n_{i+1}(k+1)+r-1}}{z_{n_{i+1}(k+1)+r-m-1}} > y_{(n_{i+1}-1)(k+1)+r} \\ &\geq y_{n_i(k+1)+r} = \max\left\{\frac{C_{n_i(k+1)+r-1}}{z_{n_i(k+1)+r-m-1}}, y_{(n_i-1)(k+1)+r}\right\} \\ &= \frac{C_{n_i(k+1)+r-1}}{z_{n_i(k+1)+r-m-1}}. \end{aligned} \tag{19}$$

From this we obtain that, for all $i \geq 1$,

$$z_{n_i(k+1)+r-m-1} > z_{n_{i+1}(k+1)+r-m-1}. \tag{20}$$

This is a contradiction.

In a similar fashion, we can show that $\{z_{n(k+1)+i}\}_{n=0}^{+\infty}$ is also a constant sequence eventually for every $0 \leq i \leq k$.

From the above we see that $\{(y_n, z_n)\}_{n=-d}^{\infty}$ is eventually periodic with period $k+1$. This completes the proof of Lemma 3. □

Proof of Theorem 1. Let $\{x_n\}_{n=-d}^{\infty}$ be a positive solution of (7) with initial values $x_{-d}, x_{-d+1}, \ldots, x_0$ satisfying (13) and let (15) hold. We see from Proposition 2 that $\{(P_{n,a}, Q_{n,a})\}_{n=-d}^{\infty}(a \in (0,1])$ satisfies system (14). Using Lemma 3 we know that $\{(P_{n,a}, Q_{n,a})\}_{n=-d}^{\infty}$ is eventually periodic with period $k+1$. Therefore, it follows from (14) and Lemma 3 that $\{x_n\}_{n=-d}^{\infty}$ is eventually periodic of period $k+1$. This completes the proof of Theorem 1. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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