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Research Article

Least Squares Estimation for α-Fractional Bridge with Discrete Observations

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We consider a fractional bridge defined as $dX_t = -\alpha(X_t/(T-t))dt + dB_t^H$, $0 \le t < T$, where B^H is a fractional Brownian motion of Hurst parameter $H > 1/2$ and parameter $\alpha > 0$ is unknown. We are interested in the problem of estimating the unknown parameter $\alpha > 0$. Assume that the process is observed at discrete time $t_i = i\Delta_n$, $i = 0, \ldots, n$, and $T_n = n\Delta_n$ denotes the length of the "observation window." We construct a least squares estimator $\widehat{\alpha}_n$ of α which is consistent; namely, $\widehat{\alpha}_n$ converges to α in probability as $n \to \infty$.

1. Introduction

Self-similar stochastic processes with long range dependence are of practical interest in various applications, including econometrics, internet traffic, and hydrology. These are processes $X = \{X_t : t \ge 0\}$ whose dependence on the time parameter t is self-similar, in the sense that there exists a (self-similarity) parameter $H \in (0,1)$ such that, for any constant $c \ge 0$, $\{X_{ct} : t \ge 0\}$ and $\{c^H X_t : t \ge 0\}$ have the same distribution. These processes are often endowed with other distinctive properties.

The fractional Brownian motion (fBm) is the usual candidate to model phenomena in which the self-similarity property can be observed from the empirical data. The fBm is a suitable generalization of the standard Brownian motion, which exhibits long-range dependence and self-similarity and has stationary increments. Some surveys and complete literatures could be found in Biagini et al. [1], Hu [2], Mishura [3], and Nualart [4].

Recently, Es-Sebaiy and Nourdin [5] study the asymptotic properties of a least squares estimator for the parameter α of a fractional bridge defined as

$$X_0 = 0, \qquad dX_t = -\alpha \frac{X_t}{T-t} dt + dB_t^H, \quad 0 \le t < T, \tag{1}$$

where B^H is a fBm with Hurst parameter $H > 1/2$ and the process X was observed continuously. In particular, when $H = 1/2$, Barczy and Pap [6, 7] study the various problems related to the α-Wiener bridge. The parametric estimation problems for fractional diffusion processes based on continuous-time observations have been studied, for example, in Tudor and Viens [8], Hu and Nualart [9], and Belfadli et al. [10].

In applications usually the process cannot be observed continuously. Only discrete-time observations are available. There exists a rich literature on the parameter estimation problem for diffusion processes driven by fBm based on discrete observations (see, e.g., Hu and Song [11], Es-Sebaiy [12]).

Motivated by all these results, in this paper, we will consider the α fractional bridge (1). Assume that the process X is observed equidistantly in time with the step size $t_i = i\Delta_n$, $i = 0, \ldots, n$, and $T_n = n\Delta_n$ denotes the length of the "observation window." We also assume that $T_n + \Delta_n = T$ and $\Delta_n \to 0$ when $n \to \infty$. Our goal is to study the asymptotic behavior of the least squares estimator (LSE for short) $\widehat{\alpha}_n$ of α based on the sampling data X_{t_i}, $i = 0, \ldots, n$. Our technics used in this work are inspired from Es-Sebaiy [12].

The least squares estimator $\widehat{\alpha}_n$ aims to minimize

$$\alpha \longmapsto \sum_{i=1}^{n} \int_{t_i-1}^{t_i} \left| \dot{X}_t + \alpha \frac{X_{t_{i-1}}}{T - t_{i-1}} \right|^2 dt. \tag{2}$$

This is a quadratic function of α. The minimum is achieved when

$$\widehat{\alpha}_n = -\frac{\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(X_{t_{i-1}} / (T - t_{i-1}) \right) \delta^H X_t}{\Delta_n \sum_{i=1}^{n} \left(X_{t_{i-1}}^2 / (T - t_{i-1})^2 \right)}. \quad (3)$$

By (1), we can get the following result

$$\widehat{\alpha}_n - \alpha = -\frac{\sum_{i=1}^{n} M_i}{\Delta_n \sum_{i=1}^{n} \left(X_{t_{i-1}}^2 / (T - t_{i-1})^2 \right)}, \quad (4)$$

where $M_i = \alpha(X_{t_{i-1}}/(T-t_{i-1})\int_{t_{i-1}}^{t_i}((X_{t_{i-1}}/(T-t_{i-1}))-(X_s/(T-s)))ds + \int_{t_{i-1}}^{t_i}(X_{t_{i-1}}/(T-t_{i-1})\delta^H B_t^H$, $i = 1, \ldots, n$.

The paper is organized as follows. In Section 2 some known results that we will use are recalled. The consistency of estimator is proved in Section 3.

2. Preliminaries

Recall that fBm B^H with index $H \in (0,1)$ is a mean zero Gaussian process $B^H = \{B_t^H, t \geq 0\}$ with $B_0^H = 0$ and the covariance

$$R^H(t,s) := E(B_t B_s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) \quad (5)$$

for all $s, t \geq 0$. For $H = 1/2$, B^H coincides with the standard Brownian motion B. B^H is neither a semimartingale nor a Markov process unless $H = 1/2$, so many of the powerful techniques from stochastic analysis are not available when dealing with B^H. It is possible to construct a stochastic calculus of variations with respect to the Gaussian process B^H, which will be related to the Malliavin calculus. Some surveys and complete literatures could be found in Alòs et al. [13], Nualart [4] and the reference. We recall here the basic definitions and results of this calculus. The crucial ingredient is the canonical Hilbert space $\mathscr{H}$ (it is also said to be reproducing kernel Hilbert space) associated with the fBm which is defined as the closure of the linear space $\mathscr{E}$ generated by the indicator functions $\{1_{[0,t]}, t \in [0,T]\}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathscr{H}} = R_H(t,s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right). \quad (6)$$

The mapping $1_{[0,s]} \to B_s^H$ can be extended to a linear isometry between $\mathscr{H}$ and the Gaussian space associated with B^H. We will denote the isometry by $\varphi \to B^H(\varphi)$. For $1/2 < H < 1$ we denote by $\mathscr{S}$ the set of smooth functionals of the form

$$F = f\left(B^H(\varphi_1), \ldots, B^H(\varphi_n)\right), \quad (7)$$

where $f \in C_b^\infty(\mathbb{R}^n)$ and $\varphi_i \in \mathscr{H}$. The Malliavin derivative of a functional F as above is given by

$$D^H F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\left(B^H(\varphi_1), \ldots, B^H(\varphi_n)\right)\varphi_i, \quad (8)$$

and this operator can be extended to the closure $\mathbb{D}^{m,2}$ $(m \geq 1)$ of $\mathscr{S}$ with respect to the norm

$$\|F\|_{m,2}^2 \equiv E|F|^2 + E\left\|D^H F\right\|_{\mathscr{H}}^2 + \cdots + E\left\|D^{H,m} F\right\|_{\mathscr{H}^{\widehat{\otimes} m}}^2, \quad (9)$$

where $\mathscr{H}^{\widehat{\otimes} m}$ denotes the m fold symmetric tensor product and the mth derivative $D^{H,m}$ is defined by iteration. The divergence integral δ^H is the adjoint operator of D^H. Concretely, a random variable $u \in L^2(\Omega, \mathscr{H})$ belongs to the domain of the divergence operator δ^H (in symbol $\mathrm{Dom}(\delta^H)$) if

$$E\left|\left\langle D^H F, u \right\rangle_{\mathscr{H}}\right| \leq c\|F\|_{L^2} \quad (10)$$

for every $F \in \mathscr{S}$. In this case $\delta^H(u)$ is given by the duality relationship

$$E\left(F\delta^H(u)\right) = E\left\langle D^H F, u\right\rangle_{\mathscr{H}} \quad (11)$$

for any $F \in \mathbb{D}^{1,2}$, and we have the following integration by parts:

$$F\delta^H(u) = \delta^H(Fu) + \left\langle D^H F, u\right\rangle_{\mathscr{H}} \quad (12)$$

for any $u \in \mathrm{Dom}(\delta^H)$, $F \in \mathbb{D}^{1,2}$ such that $Fu \in L^2(\Omega, \mathscr{H})$. It follows that

$$E\left[\delta^H(u)^2\right] = E\|u\|_{\mathscr{H}}^2 + E\left\langle D^H u, \left(D^H u\right)^*\right\rangle_{\mathscr{H}\otimes\mathscr{H}}, \quad (13)$$

where $(D^H u)^*$ is the adjoint of $D^H u$ in the Hilbert space $\mathscr{H} \otimes \mathscr{H}$, and

$$\|u\|_{\mathscr{H}}^2 = \iint_0^T u_s u_r \phi_H(s,r)\, ds\, dr, \quad (14)$$

where

$$\phi_H(s,r) = \frac{\partial^2 R_H}{\partial s \partial r}(s,r) = H(2H-1)|s-r|^{2H-2} \geq 0, \quad (15)$$

and, for $\varphi : [0,T]^2 \to \mathbb{R}$, we have

$$\|\varphi\|_{\mathscr{H}\otimes\mathscr{H}}^2 = \int_{[0,T]^4} \varphi(t,s)\varphi(t',s')\phi_H(t,t')\phi_H(s,s')\, dt\, ds\, dt'\, ds'. \quad (16)$$

We denote by $|\mathscr{H}|$ the subspace of $\mathscr{H}$, which is defined as the set of measurable functions f on $[0,T]$ with

$$\|f\|_{|\mathscr{H}|}^2 := \iint_0^T |f(s)||f(r)|\phi_H(s,r)\, ds\, dr < \infty. \quad (17)$$

Note that, if $\varphi, \psi \in |\mathscr{H}|$, then

$$EB^H(\varphi)B^H(\psi) = \iint_0^T \varphi(s)\psi(r)\phi_H(s,r)\, ds\, dr. \quad (18)$$

It follows actually from Pipiras and Taqqu [14] that the space $|\mathscr{H}| \subset \mathscr{H}$ is a Banach space for the norm $\|\cdot\|_{|\mathscr{H}|}$. Moreover,

$$L^2([0,T]) \subset L^{1/H}([0,T]) \subset |\mathscr{H}| \subset \mathscr{H}. \tag{19}$$

If $u \in \mathbb{D}^{1,2}(|\mathscr{H}|)$, $u \in \mathrm{Dom}(\delta^H)$, then we have (Nualart [4])

$$E\left(\delta^H(u)\right)^2 \le C_H\left(E\|u\|^2_{|\mathscr{H}|} + E\left\|D^H(u)\right\|^2_{|\mathscr{H}|\otimes|\mathscr{H}|}\right) \tag{20}$$

and if $\varphi : [0,T]^2 \to \mathbb{R}$, then

$$\|\varphi\|^2_{|\mathscr{H}|\otimes|\mathscr{H}|} = \int_{[0,T]^4} |\varphi(t,s)|\,|\varphi(t',s')|\,\phi_H(t,t')\,\phi_H(s,s')\,dt\,ds\,dt'\,ds'. \tag{21}$$

As a consequence, we have

$$E\left(\delta^H(u)\right)^2 \le C_H\left(E\|u\|^2_{L^{1/H}([0,T])} + E\left\|D^H(u)\right\|^2_{L^{1/H}([0,T]^2)}\right). \tag{22}$$

For every $n \ge 1$, let $\mathscr{H}_n$ be the nth Wiener chaos of B^H, that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_n(B^H(h)), h \in \mathscr{H}, \|h\|_{\mathscr{H}} = 1\}$, where H_n is the nth Hermite polynomial. The mapping $I_n(h^{\otimes n}) = n! H_n(B^H(h))$ provides a linear isometry between the symmetric tensor product $\mathscr{H}^{\odot n}$ (equipped with the modified norm $\|\cdot\|_{\mathscr{H}^{\odot n}} = (1/\sqrt{n!})\|\cdot\|_{\mathscr{H}^{\otimes n}}$) and $\mathscr{H}_n$. For every $f, g \in \mathscr{H}^{\odot n}$ the following multiplication formula holds

$$E\left(I_n(f)\, I_n(g)\right) = n!\langle f, g\rangle_{\mathscr{H}^{\otimes n}}. \tag{23}$$

Let $f, g : [0,T] \to \mathbb{R}$ be Hölder continuous functions of orders $\alpha \in (0,1)$ and $\beta \in (0,1)$ with $\alpha + \beta > 1$. Young proved that the Riemann-Stieltjes integral (so-called Young integral) $\int_0^T f_s dg_s$ exists. Moreover, if $\alpha = \beta \in (1/2, 1)$ and $F : \mathbb{R}^2 \to \mathbb{R}$ is a function of class $\mathscr{C}^1$, the integrals $\int_0^{\cdot}(\partial F/\partial f)(f_u, g_u)df_u$ and $\int_0^{\cdot}(\partial F/\partial g)(f_u, g_u)dg_u$ exist in the Young sense and the following change of variables formula holds:

$$\begin{aligned} F(f_t, g_t) &= F(f_0, g_0) + \int_0^t \frac{\partial F}{\partial f}(f_u, g_u)\, df_u \\ &\quad + \int_0^t \frac{\partial F}{\partial g}(f_u, g_u)\, dg_u, \quad t \in [0,T]. \end{aligned} \tag{24}$$

As a consequence, if $H \in (1/2, 1)$ and $(u_t, t \in [0,T])$ is a process with Hölder paths of order $\alpha < (1-H, 1)$, the integral $\int_0^T u_s dB^H_s$ is well defined as Young integral. Suppose that, for any $t \in [0,T]$, $u_t \in \mathbb{D}^{1,2}(|\mathscr{H}|)$, and

$$\iint_0^T |D_s u_t|\,|t-s|^{2H-2} ds\, dt < \infty \quad \text{a.s.} \tag{25}$$

Then, following from Alòs and Nualart [15], we have

$$\begin{aligned} \int_0^t u_s dB^H_s &= \int_0^t u_s \delta^H B^H_s + H(2H-1) \\ &\quad \times \iint_0^t D_s u_r |r-s|^{2H-2} dr\, ds. \end{aligned} \tag{26}$$

In particular, when φ is a nonrandom Hölder continuous function of order $\alpha \in (1-H, 1)$, we have

$$\int_0^t \varphi_s dB^H_s = \int_0^t \varphi_s \delta^H B^H_s = B^H(\varphi). \tag{27}$$

In addition, for all $\varphi, \psi \in |\mathscr{H}|$,

$$\begin{aligned} E&\left(\int_0^T \varphi_s dB^H_s \int_0^T \psi_s dB^H_s\right) \\ &= H(2H-1)\iint_0^T \varphi_u \psi_v |u-v|^{2H-2} du\, dv. \end{aligned} \tag{28}$$

3. Asymptotic Behavior of the Least Squares Estimator

Throughout this paper we assume $H \in (1/2, 1)$. We will study (1) driven by a fractional Brownian motion B^H with Hurst parameter H and $\alpha > 0$ being the unknown parameter to be estimated for discretely observed X. It is readily checked that we have the following explicit expression for X_t:

$$X_t = (T-t)^{\alpha}\int_0^t (T-s)^{-\alpha} dB^H_s, \quad 0 \le t < T, \tag{29}$$

where the integral can be understood as Young integral. In order to study the asymptotic behavior of the least squares estimator, let us introduce the following processes:

$$A_t := \int_0^t (T-s)^{-\alpha} dB^H_s, \quad 0 \le t < T. \tag{30}$$

Hence, we have

$$X_t = (T-t)^{\alpha} A_t, \quad 0 \le t < T. \tag{31}$$

For simplicity, we assume that the notation $a_n \trianglerighteq b_n$ means that there exists positive constants $C = C_{H,\alpha} > 0$ (depending only on H, α and its value may differ from line to line) so that

$$\sup_{n\ge 1}\frac{|a_n|}{|b_n|} < C < \infty. \tag{32}$$

We firstly give the following lemmas.

Lemma 1. *Let $\alpha > 0$, $1/2 < H < 1$. Then*

$$\int_0^{T_n} \frac{X_s}{T-s} dB^H_s = \int_0^{T_n} \frac{X_s}{T-s}\delta^H B^H_s + \beta_n, \tag{33}$$

where

$$\begin{aligned} \beta_n &= H(2H-1)\int_0^{T_n}\int_0^r (T-r)^{\alpha-1}(T-s)^{-\alpha}(r-s)^{2H-2} ds\, dr, \\ \lim_{n\to\infty}\beta_n &= HB(\alpha, 2H-1)\, T^{2H-1}. \end{aligned} \tag{34}$$

Proof. By (26), we have

$$\int_0^{T_n} \frac{X_s}{T-s} dB_s^H = \int_0^{T_n} \frac{X_s}{T-s} \delta^H B_s^H + H(2H-1) \times \iint_0^{T_n} D_s^H \frac{X_r}{T-r} |s-r|^{2H-2} dr\, ds$$
$$= \int_0^{T_n} \frac{X_s}{T-s} \delta^H B_s^H + H(2H-1) \times \int_0^{T_n} \int_0^r (T-r)^{\alpha-1} (T-s)^{-\alpha} (r-s)^{2H-2} ds\, dr$$
$$= \int_0^{T_n} \frac{X_s}{T-s} \delta^H B_s^H + \beta_n. \quad (35)$$

On the other hand,

$$\lim_{n\to\infty} \beta_n = H(2H-1) \times \lim_{n\to\infty} \int_0^{T_n} \int_0^r (T-r)^{\alpha-1} (T-s)^{-\alpha} (r-s)^{2H-2} ds\, dr$$
$$= H(2H-1) \lim_{n\to\infty} \int_{T-T_n}^{T} \int_r^T r^{\alpha-1} s^{-\alpha} (s-r)^{2H-2} ds\, dr$$
$$= H(2H-1) \lim_{n\to\infty} \int_0^{T_n} \int_r^{T_n} (T-T_n+r)^{\alpha-1} \times (T-T_n+s)^{-\alpha} (s-r)^{2H-2} ds\, dr$$
$$= H(2H-1) \int_0^T \int_r^T r^{\alpha-1} s^{-\alpha} (s-r)^{2H-2} ds\, dr$$
$$= H(2H-1) \int_0^T s^{-\alpha} \int_0^s r^{\alpha-1} (s-r)^{2H-2} dr\, ds$$
$$= HB(\alpha, 2H-1) T^{2H-1}. \quad (36)$$

This completes the proof. □

The following Lemma 2 comes from Lemma 3.2 of Es-Sebaiy and Nourdin [5].

Lemma 2. *Letting* $0 < \alpha < H$, $1/2 < H < 1$, *one has*

$$E\left(\frac{X_t}{T-t}\right)^2 \le \frac{H(2H-1)}{H-\alpha} B(1-\alpha, 2H-1) (T-t)^{2\alpha-2} T^{2H-2\alpha}, \quad 0 \le t < T. \quad (37)$$

Lemma 3. *Assume* $1-H < \alpha < H$, $1/2 < H < 1$, *and let* $F_{T_n} = \int_0^{T_n} (X_t/(T-t)) \delta^H B_t^H$. *Then*

$$\lim_{n\to\infty} E\left(F_{T_n}^2\right) = \frac{H^2(2H-1)^2 B(\alpha, 2H-1) B(1-\alpha, 2H-1)}{2(H+\alpha-1)(H-\alpha)} T^{4H-2}. \quad (38)$$

Proof. By the isometry property of the double stochastic integral I_2, the variance of F_{T_n} is given by

$$E\left(F_{T_n}^2\right) = \frac{H^2(2H-1)^2}{2} I_{T_n}, \quad (39)$$

where

$$I_{T_n} = \int_{[0,T_n]^4} (T-t_1)^{\alpha-1} (T-s_1)^{-\alpha} (T-t_2)^{\alpha-1} (T-s_2)^{-\alpha} \times |s_1-s_2|^{2H-2} |t_1-t_2|^{2H-2} ds_1 ds_2 dt_1 dt_2. \quad (40)$$

Now, we study I_{T_n}, by setting

$$I_1 = \int_{[0,T_n]^2} (T-s_1)^{-\alpha} (T-s_2)^{-\alpha} |s_1-s_2|^{2H-2} ds_1 ds_2,$$
$$I_2 = \int_{[0,T_n]^2} (T-t_1)^{\alpha-1} (T-t_2)^{\alpha-1} |t_1-t_2|^{2H-2} dt_1 dt_2. \quad (41)$$

We have $I_{T_n} = I_1 I_2$. By (17.40) of Es-Sebaiy and Nourdin [5], we have

$$I_1 = \int_{[0,T_n]^2} (T-s_1)^{-\alpha} (T-s_2)^{-\alpha} |s_1-s_2|^{2H-2} ds_1 ds_2 = \frac{B(1-\alpha, 2H-1)}{H-\alpha} T_n^{2H-2\alpha} \longrightarrow \frac{B(1-\alpha, 2H-1)}{H-\alpha} T^{2H-2\alpha}, \quad n \longrightarrow \infty. \quad (42)$$

Similarly

$$I_2 \longrightarrow \frac{B(\alpha, 2H-1)}{H+\alpha-1} T^{2H+2\alpha-2}. \quad (43)$$

Thus, the proof is finished. □

The following theorem gives the consistency of the least squares estimator $\hat{\alpha}_n$ of α.

Theorem 4. *Let* $1/2 < \alpha < H < 1$. *If* $\Delta_n \to 0$, $T_n = n\Delta_n \to T$ *as* $n \to \infty$, *and* $T_n + \Delta_n = T$, *then, one has*

$$\hat{\alpha}_n \xrightarrow{P} \alpha, \quad n \longrightarrow \infty, \quad (44)$$

where $\xrightarrow{P}$ *means convergence in probability.*

Proof. By (4), we have

$$\widehat{\alpha}_n - \alpha = -\frac{(\alpha/n)\sum_{i=1}^{n} M_i}{(\alpha\Delta_n/n)\sum_{i=1}^{n}\left(X_{t_{i-1}}^2/(T-t_{i-1})^2\right)}. \quad (45)$$

Letting $0 < \varepsilon < 1$, we obtain

$$\begin{aligned} P\left(\left|\widehat{\alpha}_n - \alpha\right| > \varepsilon\right) &= P\left(\left|\frac{(\alpha/n)\sum_{i=1}^{n} M_i}{(\alpha\Delta_n/n)\sum_{i=1}^{n}\left(X_{t_{i-1}}^2/(T-t_{i-1})^2\right)}\right| > \varepsilon\right) \\ &\leq P\left(\left|\frac{\alpha}{n}\sum_{i=1}^{n} M_i\right| > \varepsilon(1-\varepsilon)\right) \\ &\quad + P\left(\left|\frac{\alpha\Delta_n}{n}\sum_{i=1}^{n}\frac{X_{t_{i-1}}^2}{(T-t_{i-1})^2} - 1\right| > \varepsilon\right) \\ &:= B_1(n) + B_2(n). \end{aligned} \quad (46)$$

First, we considering the term $B_1(n)$, we have

$$\begin{aligned} B_1(n) &= P\left(\left|\frac{\alpha}{n}\sum_{i=1}^{n} M_i\right| > \varepsilon(1-\varepsilon)\right) \\ &\leq P\left(\left|\frac{\alpha}{n}\sum_{i=1}^{n}\left[M_i - \int_{t_{i-1}}^{t_i}\frac{X_{t_{i-1}}}{T-t_{i-1}}\delta^H B_t^H\right]\right| > \frac{1}{3}\varepsilon(1-\varepsilon)\right) \\ &\quad + P\left(\left|\frac{\alpha}{n}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(\frac{X_{t_{i-1}}}{T-t_{i-1}} - \frac{X_t}{T-t}\right)\delta^H B_t^H\right| > \frac{1}{3}\varepsilon(1-\varepsilon)\right) \\ &\quad + P\left(\left|\frac{\alpha}{n}\int_0^{T_n}\frac{X_t}{T-t}\delta^H B_t^H\right| > \frac{1}{3}\varepsilon(1-\varepsilon)\right) \\ &:= B_{1,1}(n) + B_{1,2}(n) + B_{1,3}(n). \end{aligned} \quad (47)$$

For the term $B_{1,1}(n)$, using Lemma 2, we obtain

$$\begin{aligned} &\sum_{i=1}^{n} E\left|\left[M_i - \int_{t_{i-1}}^{t_i}\frac{X_{t_{i-1}}}{T-t_{i-1}}\delta^H B_t^H\right]\right| \\ &\leq \alpha\sum_{i=1}^{n}\left(E\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}\right)^2\right)^{1/2} \\ &\quad \times \int_{t_{i-1}}^{t_i}\left(E\left(\frac{X_{t_{i-1}}}{T-t_{i-1}} - \frac{X_t}{T-t}\right)^2\right)^{1/2} dt \\ &\trianglelefteq \sum_{i=1}^{n}(T-t_{i-1})^{\alpha-1}\int_{t_{i-1}}^{t_i}\left(E\left(\frac{X_{t_{i-1}}}{T-t_{i-1}} - \frac{X_t}{T-t}\right)^2\right)^{1/2} dt \\ &\leq \Delta_n^{\alpha-1}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(E\left(\frac{X_{t_{i-1}}}{T-t_{i-1}} - \frac{X_t}{T-t}\right)^2\right)^{1/2} dt \\ &\leq \Delta_n^{\alpha-1}\left[\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(E\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}\right)^2\right)^{1/2} dt + \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(E\left(\frac{X_t}{T-t}\right)^2\right)^{1/2} dt\right] \\ &\leq \Delta_n^{\alpha-1}\left[\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(E\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}\right)^2\right)^{1/2} dt + \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(E\left(\frac{X_{t_i}}{T-t_i}\right)^2\right)^{1/2} dt\right] \\ &\trianglelefteq n\Delta_n^{2\alpha-1}. \end{aligned} \quad (48)$$

So, we get

$$\frac{\alpha}{n}\sum_{i=1}^{n} E\left|\left[M_i - \int_{t_{i-1}}^{t_i}\frac{X_{t_{i-1}}}{T-t_{i-1}}\delta^H B_t^H\right]\right| \trianglelefteq \Delta_n^{2\alpha-1}. \quad (49)$$

Hence,

$$B_{1,1}(n) \trianglelefteq \frac{\Delta_n^{2\alpha-1}}{\varepsilon(1-\varepsilon)}. \quad (50)$$

For the term $B_{1,2}(n)$, it follows the fact that, for $0 \leq t < \mathrm{T}$,

$$\begin{aligned} \frac{X_{t_{i-1}}}{T-t_{i-1}} - \frac{X_t}{T-t} = -&\left[\left((T-t)^{\alpha-1} - (T-t_{i-1})^{\alpha-1}\right)A_{t_{i-1}}\right. \\ &\left. + (T-t)^{\alpha-1}\left(A_t - A_{t_{i-1}}\right)\right]. \end{aligned} \quad (51)$$

We have

$$\begin{aligned} &E\left|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(\frac{X_{t_{i-1}}}{T-t_{i-1}} - \frac{X_t}{T-t}\right)\delta^H B_t^H\right| \\ &\leq E\left|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left((T-t)^{\alpha-1} - (T-t_{i-1})^{\alpha-1}\right)A_{t_{i-1}}\delta^H B_t^H\right| \\ &\quad + E\left|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(T-t)^{\alpha-1}\left(A_t - A_{t_{i-1}}\right)\delta^H B_t^H\right|. \end{aligned} \quad (52)$$

Using inequality (22) and $EA_t = 0$, $D_s^H A_t = (T-s)^{-\alpha} 1_{[0,t]}(s)$, we have

$$
\begin{aligned}
&E\left|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left((T-t)^{\alpha-1}-(T-t_{i-1})^{\alpha-1}\right)A_{t_{i-1}}\delta^H B_t^H\right| \\
&= E\left|\int_0^{T_n}\sum_{i=1}^{n}\left((T-t)^{\alpha-1}-(T-t_{i-1})^{\alpha-1}\right)A_{t_{i-1}}1_{(t_{i-1},t_i]}(t)\,\delta^H B_t^H\right| \\
&\leq \left(E\left|\int_0^{T_n}\sum_{i=1}^{n}\left((T-t)^{\alpha-1}-(T-t_{i-1})^{\alpha-1}\right)\right.\right. \\
&\qquad\left.\left.\times A_{t_{i-1}}1_{(t_{i-1},t_i]}(t)\,\delta^H B_t^H\right|^2\right)^{1/2} \\
&\leq C_H\left(\iint_0^{T_n}\left|\sum_{i=1}^{n}\left((T-t)^{\alpha-1}-(T-t_{i-1})^{\alpha-1}\right)\right.\right. \\
&\qquad\left.\left.\times D_s^H A_{t_{i-1}}1_{(t_{i-1},t_i]}(t)\right|^{1/H}ds\,dt\right)^H \\
&= C_H\left(\iint_0^{T_n}\sum_{i=1}^{n}\left|\left((T-t)^{\alpha-1}-(T-t_{i-1})^{\alpha-1}\right)(T-s)^{-\alpha}\right|^{1/H}\right. \\
&\qquad\left.\times 1_{(t_{i-1},t_i]}(t)\,1_{[0,t_{i-1})}(s)\,ds\,dt\right)^H \\
&= C_H\left(\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left((T-t)^{\alpha-1}-(T-t_{i-1})^{\alpha-1}\right)^{1/H}dt\right. \\
&\qquad\left.\times\int_0^{t_{i-1}}(T-s)^{-\alpha/H}ds\right)^H \\
&\leq C_H T^{H-\alpha}\left(\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\Delta_n^{(\alpha-1)/H}dt\right)^H \leq C_H T^{H-\alpha} n\Delta_n^{H+\alpha-1}.
\end{aligned}
\tag{53}
$$

On the other hand,

$$
\begin{aligned}
&E\left|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(T-t)^{\alpha-1}\left(A_t-A_{t_{i-1}}\right)\delta^H B_t^H\right| \\
&= E\left|\int_0^{T_n}\sum_{i=1}^{n}(T-t)^{\alpha-1}\left(A_t-A_{t_{i-1}}\right)1_{(t_{i-1},t_i]}(t)\,\delta^H B_t^H\right| \\
&\leq C_H\left(\iint_0^{T_n}\left|\sum_{i=1}^{n}(T-t)^{\alpha-1}D_s^H\left(A_t-A_{t_{i-1}}\right)\right.\right. \\
&\qquad\left.\left.\times 1_{(t_{i-1},t_i]}(t)\right|^{1/H}ds\,dt\right)^H \\
&\leq C_H\left(\iint_0^{T_n}\sum_{i=1}^{n}\left|(T-t)^{\alpha-1}D_s^H\left(A_t-A_{t_{i-1}}\right)\right|^{1/H}\right. \\
&\qquad\left.\times 1_{(t_{i-1},t_i]}(t)\,ds\,dt\right)^H \\
&= C_H\left(\iint_0^{T_n}\sum_{i=1}^{n}\left((T-t)^{\alpha-1}(T-s)^{-\alpha}\right)^H 1_{[t_{i-1},t]}(s)\right. \\
&\qquad\left.\times 1_{(t_{i-1},t_i]}(t)\,ds\,dt\right)^H \\
&= C_H\left(\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(T-t)^{(\alpha-1)/H}dt\int_{t_{i-1}}^{t}(T-s)^{-\alpha/H}ds\right)^H \\
&\leq C_H\left(\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(T-t)^{(\alpha-1)/H}dt\int_{t_{i-1}}^{t}(T-t_n)^{-\alpha/H}ds\right)^H \\
&\leq C_H\left(n\Delta_n^{(2H-1)/H}\right)^H \leq C_H n\Delta_n^{2H-1}.
\end{aligned}
\tag{54}
$$

So, we get

$$
E\left|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}-\frac{X_t}{T-t}\right)\delta^H B_t^H\right| \trianglerighteq n\Delta_n^{H+\alpha-1}. \tag{55}
$$

Thus,

$$
\frac{\alpha}{n}E\left|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}-\frac{X_t}{T-t}\right)\delta^H B_t^H\right| \trianglerighteq \Delta_n^{H+\alpha-1}. \tag{56}
$$

Hence

$$
B_{1,2}(n) \trianglerighteq \frac{\Delta_n^{H+\alpha-1}}{\varepsilon(1-\varepsilon)}. \tag{57}
$$

For the term $B_{1,3}(n)$, by setting $F_{T_n} = \int_0^{T_n}(X_t/(T-t))\delta^H B_t^H$ and by using Lemma 3, we get

$$
\begin{aligned}
B_{1,3}(n) &= P\left(\left|\frac{\alpha}{n}\int_0^{T_n}\frac{X_t}{T-t}\delta^H B_t^H\right| > \frac{1}{3}\varepsilon(1-\varepsilon)\right) \\
&\leq \left[\frac{3\alpha}{\varepsilon(1-\varepsilon)n}\right]^2 E\left(F_{T_n}^2\right) \trianglerighteq \frac{1}{\varepsilon^2(1-\varepsilon)^2 n^2}.
\end{aligned}
\tag{58}
$$

As a consequence,

$$
B_1(n) \trianglerighteq \frac{\Delta_n^{2\alpha-1}}{\varepsilon(1-\varepsilon)} + \frac{\Delta_n^{H+\alpha-1}}{\varepsilon(1-\varepsilon)} + \frac{1}{\varepsilon^2(1-\varepsilon)^2 n^2}. \tag{59}
$$

Second, we estimate the term $B_2(n)$:

$$\begin{aligned}
B_2(n) &= P\left(\left|\frac{\alpha\Delta_n}{n}\sum_{i=1}^{n}\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}\right)^2 - 1\right| > \varepsilon\right) \\
&\le P\left(\left|\frac{\alpha}{n}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left[\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}\right)^2 - \left(\frac{X_t}{T-t}\right)^2\right]dt\right| > \varepsilon/2\right) \\
&\quad + P\left(\left|\frac{\alpha}{n}\int_0^{T_n}\left(\frac{X_t}{T-t}\right)^2 dt - 1\right| > \varepsilon/2\right) \\
&:= B_{2,1}(n) + B_{2,2}(n).
\end{aligned} \tag{60}$$

We firstly consider $B_{2,1}(n)$, since

$$\begin{aligned}
E&\left|\frac{\alpha}{n}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left[\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}\right)^2 - \left(\frac{X_t}{T-t}\right)^2\right]\right| dt \\
&\le \frac{\alpha}{n}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} E\left|\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}\right)^2 - \left(\frac{X_t}{T-t}\right)^2\right| dt \\
&\le \frac{\alpha}{n}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(E\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}\right)^2 + E\left(\frac{X_t}{T-\mathrm{t}}\right)^2\right) dt \\
&\le \frac{\alpha}{n}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(E\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}\right)^2 + E\left(\frac{X_{t_i}}{T-t_i}\right)^2\right) dt \\
&\le \frac{2\alpha}{n}\sum_{i=1}^{n}\Delta_n^{2\alpha-1} \trianglerighteq \Delta_n^{2\alpha-1}.
\end{aligned} \tag{61}$$

By Markov inequality, we obtain

$$B_{2,1}(n) \trianglerighteq \frac{\Delta_n^{2\alpha-1}}{\varepsilon}. \tag{62}$$

Now, we estimate the term $B_{2,2}(n)$. Applying the change of variable formula (24), we get

$$\begin{aligned}
\frac{\alpha}{n}\int_0^{T_n}\left(\frac{X_t}{T-t}\right)^2 dt - 1 &= \frac{1}{n(\alpha-(1/2))} \\
&\quad \times \left(\frac{X_{T_n}}{2\Delta_n} - \int_0^{T_n}\frac{X_t}{T-t}\delta^H B_t^H - \beta_n\right).
\end{aligned} \tag{63}$$

Hence,

$$\begin{aligned}
B_{2,2}(n) &\le P\left(\left|\frac{X_{T_n}}{T_n(2\alpha-1)}\right| > \frac{\varepsilon}{6}\right) \\
&\quad + P\left(\left|\frac{1}{n(\alpha-(1/2))}\int_0^{T_n}\frac{X_t}{T-t}\delta B_t^H\right| > \frac{\varepsilon}{6}\right) \\
&\quad + P\left(\left|\frac{\beta_n}{n(\alpha-(1/2))}\right| > \frac{\varepsilon}{6}\right).
\end{aligned} \tag{64}$$

By Markov inequality and Lemma 2, we obtain

$$B_{2,2}(n) \trianglerighteq \frac{\Delta_n^{2\alpha}}{\varepsilon^2 T_n^2} + \frac{1}{\varepsilon n^2} + \frac{1}{\varepsilon n}. \tag{65}$$

Therefore

$$B_2(n) \trianglerighteq \frac{\Delta_n^{2\alpha-1}}{\varepsilon} + \frac{\Delta_n^{2\alpha}}{\varepsilon^2 T_n^2} + \frac{1}{\varepsilon n^2} + \frac{1}{\varepsilon n} \le \frac{\Delta_n^{2\alpha-1}}{\varepsilon} + \frac{\Delta_n^{2\alpha}}{\varepsilon^2 T_n^2} + \frac{1}{\varepsilon n}. \tag{66}$$

Combining (59) and (66), this completes the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] F. Biagini, Y. Hu, B. Øksendal, and T. Zhang, *Stochastic Calculus for fBm and Applications. Probability and Its Application*, Springer, Berlin, Germany, 2008.

[2] Y. Hu, "Integral transformations and anticipative calculus for fractional Brownian motions," *Memoirs of the American Mathematical Society*, vol. 175, no. 825, 2005.

[3] Y. S. Mishura, *Stochastic Calculus for Fractional Brownian Motion and Related Processes*, vol. 1929 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2008.

[4] D. Nualart, *The Malliavin Calculus and Related Topics*, Springer, Berlin, Germany, 2nd edition, 2006.

[5] K. Es-Sebaiy and I. Nourdin, "Parameter estimation for α fractional bridges," in *Malliavin Calculus and Stochastic Analysis: A Festschrift in Honor of David Nualart*, F. Viens, J. Feng, Y. Hu, and E. Nualart, Eds., vol. 34 of *Springer Proceedings in Mathematics and Statistics*, pp. 385–412, 2013.

[6] M. Barczy and G. Pap, "Asymptotic behavior of maximum likelihood estimator for time inhomogeneous diffusion processes," *Journal of Statistical Planning and Inference*, vol. 140, no. 6, pp. 1576–1593, 2010.

[7] M. Barczy and G. Pap, "α-Wiener bridges: singularity of induced measures and sample path properties," *Stochastic Analysis and Applications*, vol. 28, no. 3, pp. 447–466, 2010.

[8] C. A. Tudor and F. G. Viens, "Statistical aspects of the fractional stochastic calculus," *The Annals of Statistics*, vol. 35, no. 3, pp. 1183–1212, 2007.

[9] Y. Hu and D. Nualart, "Parameter estimation for fractional Ornstein-Uhlenbeck processes," *Statistics & Probability Letters*, vol. 80, no. 11-12, pp. 1030–1038, 2010.

[10] R. Belfadli, K. Es-Sebaiy, and Y. Ouknine, "Parameter estimation for fractional Ornstein-Uhlenbeck processes: non-ergodic case," *Frontiers in Science and Engineering*, vol. 1, pp. 1–16, 2011.

[11] Y. Hu and J. Song, "Parameter estimation for fractional Ornstein-Uhlenbeck processes with discrete observations," in *Malliavin Calculus and Stochastic Analysis: A Festschrift in Honor of David Nualart*, F. Viens, J. Feng, Y. Hu, and E. Nualart, Eds., vol. 34 of *Springer Proceedings in Mathematics and Statistics*, pp. 427–442, 2013.

[12] K. Es-Sebaiy, "Berry-Esséen bounds for the least squares estimator for discretely observed fractional Ornstein-Uhlenbeck processes," *Statistics & Probability Letters*, vol. 83, no. 10, pp. 2372–2385, 2013.

[13] E. Alòs, O. Mazet, and D. Nualart, "Stochastic calculus with respect to Gaussian processes," *The Annals of Probability*, vol. 29, no. 2, pp. 766–801, 2001.

[14] V. Pipiras and M. S. Taqqu, "Integration questions related to fractional Brownian motion," *Probability Theory and Related Fields*, vol. 118, no. 2, pp. 251–291, 2000.

[15] E. Alòs and D. Nualart, "Stochastic integration with respect to the fractional Brownian motion," *Stochastics and Stochastics Reports*, vol. 75, no. 3, pp. 129–152, 2003.