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Research Article

Global Behavior of the Difference Equation $x_{n+1} = x_{n-1}g(x_n)$

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We consider the following difference equation $x_{n+1} = x_{n-1}g(x_n)$, $n = 0, 1, \ldots$, where initial values $x_{-1}, x_0 \in [0, +\infty)$ and $g : [0, +\infty) \to (0, 1]$ is a strictly decreasing continuous surjective function. We show the following. (1) Every positive solution of this equation converges to $a, 0, a, 0, \ldots$, or $0, a, 0, a, \ldots$ for some $a \in [0, +\infty)$. (2) Assume $a \in (0, +\infty)$. Then the set of initial conditions $(x_{-1}, x_0) \in (0, +\infty) \times (0, +\infty)$ such that the positive solutions of this equation converge to $a, 0, a, 0, \ldots$, or $0, a, 0, a, \ldots$ is a unique strictly increasing continuous function or an empty set.

1. Introduction

Recently there have been published quite a lot of works concerning global behavior of the difference equations [1–8]. These results are not only valuable in their own right, but they can provide insight into their differential counterparts.

In [9], Kulenović and Ladas considered the positive solutions for difference equation

$$x_{n+1} = \frac{x_{n-1}}{1 + Ax_n} \tag{1}$$

with $A > 0$. They gave some partial results on the convergence of this equation.

Kalikow et al. [10] studied the following difference equation:

$$x_{n+1} = \frac{x_{n-1}}{1 + f(x_n)}, \quad n = 0, 1, 2, \ldots, \tag{E1}$$

where initial values $x_{-1}, x_0 \in [0, +\infty)$ and f is in a certain class of increasing continuous functions. They showed that the set of initial conditions (x_{-1}, x_0) of $(E1)$ in the first quadrant that converge to any given boundary point of the first quadrant forms a unique strictly increasing continuous function.

Motivated by the above studies, in this paper, we consider the following difference equation:

$$x_{n+1} = x_{n-1}g(x_n), \quad n = 0, 1, \ldots, \tag{2}$$

where initial values $x_{-1}, x_0 \in [0, +\infty)$ and $g : [0, +\infty) \to (0, 1]$ is a strictly decreasing continuous surjective function. Our main result is the following theorem.

Theorem 1. (1) *Every positive solution of* (2) *converges to*

$$a, 0, a, 0, \ldots, \quad or \quad 0, a, 0, a, \ldots \tag{3}$$

for some $a \in [0, +\infty)$.

(2) *Assume* $a \in (0, +\infty)$. *Then the set of initial conditions* $(x_{-1}, x_0) \in (0, +\infty) \times (0, +\infty)$ *such that the positive solutions of* (2) *converge to*

$$a, 0, a, 0, \ldots, \quad or \quad 0, a, 0, a, \ldots \tag{4}$$

is a unique strictly increasing continuous function or an empty set.

2. The Main Result

Proof of Theorem 1(1). Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of (2). Then x_{2n} and x_{2n+1} are decreasing sequences since $g(x) \leq 1$. Let $\lim_{n\to\infty} x_{2n} = p$ and $\lim_{n\to\infty} x_{2n-1} = q$. Then we have

$$p = pg(q), \tag{5}$$

which implies $p = 0$ or $g(q) = 1$. If $g(q) = 1$, then $q = 0$ since $g : [0, +\infty) \to (0, 1]$ is a strictly decreasing continuous surjective function with $g(0) = 1$. This completes proof of Theorem 1(1). □

Write $D = [0, +\infty) \times [0, +\infty)$ and define $f : D \to D$ by

$$f(x, y) = (y, xg(y)), \tag{6}$$

for all $(x, y) \in D$. It is easy to see that if $\{x_n\}_{n=-1}^{\infty}$ is a solution of (2), then $f^n(x_{-1}, x_0) = (x_{n-1}, x_n)$ for any $n \geq 0$. In the following, let

$$\begin{aligned} L_0 &= \{a\} \times [0, +\infty), & L_1 &= [0, +\infty) \times \{a\}, \\ R_0 &= [a, +\infty) \times \{0\}, & R_1 &= \{0\} \times [a, +\infty) \end{aligned} \tag{7}$$

for some $a \in (0, +\infty)$.

Lemma 2. *The following statements are true:*

(i) *f is a homeomorphism;*

(ii) $f(L_1) = L_0$*;*

(iii) $f(R_0) = R_1$ *and* $f(R_1) = R_0$.

Proof. (i) Since $f(x_1, y_1) \neq f(x_2, y_2)$ for any $(x_1, y_1), (x_2, y_2) \in D$ with $(x_1, y_1) \neq (x_2, y_2)$ and $f^{-1}(u, v) = (v/g(u), u)$ is continuous for any $(u, v) \in D$, f is a homeomorphism.

(ii) Let $(x, y) \in L_1$ and $(u, v) = f(x, y) = (y, xg(y))$. Then $y = a$, $x \geq 0$, and

$$u = y = a, \qquad v = xg(y) = xg(a) \geq 0, \tag{8}$$

which implies $f(L_1) \subset L_0$.

Let $(u, v) \in L_0$. Then $u = a$ and $v \geq 0$. Choose $(x, y) = (v/g(a), a) \in L_1$. Then $f(x, y) = (u, v)$. Thus $f(L_1) = L_0$.

The proof of (iii) is similar to that of (ii). This completes the proof of Lemma 2. □

In order to show Theorem 1(2), we will construct two families of strictly increasing functions $y = h_{2n}(x)$ and $x = g_{2n+1}(y)$ $(n \geq 1)$ as follows. Set

$$x = g_2(y) = \frac{a}{g(y)} \quad (y \geq 0). \tag{9}$$

Then $y = h_2(x) = g_2^{-1}(x) = g^{-1}(a/x)$ is a strictly increasing function which maps $[a, +\infty)$ onto $[0, +\infty)$. Set

$$x = g_3(y) = \frac{h_2(y)}{g(y)} \quad (y \geq a). \tag{10}$$

Then $x = g_3(y)$ is a strictly increasing function which maps $[a, +\infty)$ onto $[0, +\infty)$.

Assume that, for some positive integer n, we already define strictly increasing functions $y = h_{2n}(x)$ and $x = g_{2n+1}(y)$ such that both h_{2n} and g_{2n+1} map $[a, +\infty)$ onto $[0, +\infty)$. Set

$$x = g_{2n+2}(y) = \frac{g_{2n+1}^{-1}(y)}{g(y)} \quad (y \geq 0). \tag{11}$$

Then both $y = h_{2n+2}(x) = g_{2n+2}^{-1}(x)$ and $x = g_{2n+3}(y) = h_{2n+2}(y)/g(y)$ are strictly increasing functions which map $[a, +\infty)$ onto $[0, +\infty)$. In such a way, we construct two families of strictly increasing functions $y = h_{2n}(x)$ and $x = g_{2n+1}(y)$ $(n \geq 1)$.

Set $P_0 = [a, +\infty) \times [0, +\infty)$ and $Q_0 = [0, +\infty) \times [a, +\infty)$. For any $n \geq 1$, write

$$\begin{gathered} P_n = f^{-2}(P_{n-1}), \qquad Q_n = f^{-2}(Q_{n-1}), \\ L_n = f^{-1}(L_{n-1}). \end{gathered} \tag{12}$$

Let $(x, y) \in L_2$. Since $f(L_2) = L_1$ and $(u, v) = f(x, y) = (y, xg(y)) \in L_1$, it follows that

$$xg(y) = v = a, \quad y = u \geq 0. \tag{13}$$

Thus $x = g_2(y) = a/g(y)$ and $L_2 = \{(x, y) : y = h_2(x), x \geq a\}$.

Let $(x, y) \in L_3$. Since $f(L_3) = L_2$ and $(u, v) = f(x, y) = (y, xg(y)) \in L_2$, it follows that

$$xg(y) = v = h_2(u) = h_2(y), \quad y = u \geq a. \tag{14}$$

Thus $x = g_3(y) = h_2(y)/g(y)$ $(y \geq a)$ and $L_3 = \{(x, y) : x = g_3(y), y \geq a\}$. Using induction, one can easily show that, for any $n \geq 1$,

$$\begin{aligned} L_{2n} &= \{(x, y) : y = h_{2n}(x), x \geq a\}, \\ L_{2n+1} &= \{(x, y) : x = g_{2n+1}(y), y \geq a\}. \end{aligned} \tag{15}$$

Since f is a homeomorphism and $P_n = f^{-2}(P_{n-1})$ with $L_{2n} \cup R_0$ is the boundary of P_n, we have that, for any $n \geq 1$,

$$P_n = \{(x, y) : 0 \leq y \leq h_{2n}(x), x \geq a\}. \tag{16}$$

In a similar fashion, we may show that

$$Q_n = \{(x, y) : 0 \leq x \leq g_{2n+1}(y), y \geq a\}. \tag{17}$$

Since $L_2 \subset P_0$, $L_3 \subset Q_0$, and f is a homeomorphism, we have that $P_1 \subset P_0$ and $Q_1 \subset Q_0$, which implies that, for any $n \geq 1$,

$$\begin{aligned} L_{2n} &\subset P_{n-1}, & L_{2n+1} &\subset Q_{n-1}, \\ P_n &\subset P_{n-1}, & Q_n &\subset Q_{n-1}. \end{aligned} \tag{18}$$

It follows from (12) and (18) that, for $x \geq a$,

$$0 \leq \cdots \leq h_4(x) \leq h_2(x) \tag{19}$$

and for $y \geq a$,

$$0 \leq \cdots \leq g_5(y) \leq g_3(y). \tag{20}$$

Noting (19) and (20), we may assume that, for every $x \geq a$,

$$H(x) = \lim_{n \to \infty} h_{2n}(x) \tag{21}$$

and for every $y \geq a$,

$$G(y) = \lim_{n \to \infty} g_{2n+1}(y). \tag{22}$$

Set

$$\begin{aligned} L &= \{(x, y) : y = H(x), x \geq a\}, \\ M &= \{(x, y) : x = G(y), y \geq a\}. \end{aligned} \tag{23}$$

Lemma 3. *The following statements are true:*

(i) $f(L) = M$ *and* $f(M) = L$;

(ii) *both* $y = H(x)$ *and* $x = G(y)$ *are increasing continuous functions which map* $[a, +\infty)$ *onto* $[0, +\infty)$.

Proof. (i) Let $(x_0, y_0) \in L$. Then we have $y_0 = \lim_{n\to\infty} h_{2n}(x_0)$, which follows that

$$f(x_0, y_0) = f\left(x_0, \lim_{n\to\infty} h_{2n}(x_0)\right) = \lim_{n\to\infty} f(x_0, h_{2n}(x_0)). \tag{24}$$

Since $f(L_{2n}) = L_{2n-1}$, we have

$$\begin{aligned} f(x_0, h_{2n}(x_0)) &= (h_{2n}(x_0), x_0 g(h_{2n}(x_0))) \\ &= (g_{2n-1}(x_0 g(h_{2n}(x_0))), x_0 g(h_{2n}(x_0))). \end{aligned} \tag{25}$$

Let $y_n = x_0 g(h_{2n}(x_0))$. It follows from (24) and (25) that

$$f(x_0, y_0) = \lim_{n\to\infty} (g_{2n-1}(y_n), y_n) = (y_0, x_0 g(y_0)), \tag{26}$$

so we have

$$\lim_{n\to\infty} y_n = x_0 g(y_0), \qquad \lim_{n\to\infty} g_{2n-1}(y_n) = G(x_0 g(y_0)). \tag{27}$$

It follows from (25) and (27) that

$$f(x_0, y_0) = (G(x_0 g(y_0)), x_0 g(y_0)) \in M. \tag{28}$$

Thus we have $f(L) \subset M$.

Let $(x_0, y_0) \in M$. Then we have $x_0 = \lim_{n\to\infty} g_{2n+1}(y_0)$, which follows that

$$\begin{aligned} f^{-1}(x_0, y_0) &= f^{-1}\left(\lim_{n\to\infty} g_{2n+1}(y_0), y_0\right) \\ &= \lim_{n\to\infty} f^{-1}(g_{2n+1}(y_0), y_0). \end{aligned} \tag{29}$$

Since $f^{-1}(L_{2n+1}) = L_{2n+2}$, we have

$$\begin{aligned} &f^{-1}(g_{2n+1}(y_0), y_0) \\ &\quad = \left(\frac{y_0}{g(g_{2n+1}(y_0))}, g_{2n+1}(y_0)\right) \\ &\quad = \left(\frac{y_0}{g(g_{2n+1}(y_0))}, h_{2n+2}\left(\frac{y_0}{g(g_{2n+1}(y_0))}\right)\right). \end{aligned} \tag{30}$$

Let $z_n = y_0/g(g_{2n+1}(y_0))$. It follows from (29) and (30) that

$$f^{-1}(x_0, y_0) = \lim_{n\to\infty} (z_n, h_{2n+2}(z_n)) = \left(\frac{y_0}{g(x_0)}, x_0\right), \tag{31}$$

so we have

$$\lim_{n\to\infty} z_n = \frac{y_0}{g(x_0)}, \qquad \lim_{n\to\infty} h_{2n+2}(z_n) = H\left(\frac{y_0}{g(x_0)}\right). \tag{32}$$

It follows from (31) and (32) that

$$f^{-1}(x_0, y_0) = \left(\frac{y_0}{g(x_0)}, H\left(\frac{y_0}{g(x_0)}\right)\right) \in L. \tag{33}$$

Thus we have $f(L) = M$. In a similar fashion, we can show that $f(M) = L$.

(ii) Since $y = h_{2n}(x)$ $(n \geq 1)$ are strictly increasing functions, we have that $y = H(x)$ is an increasing function. For any $x_0 > a$, let

$$\lim_{x\to x_0^+} H(x) = y_0^+, \qquad \lim_{x\to x_0^-} H(x) = y_0^-; \tag{34}$$

then $y_0^+ \geq H(x_0) \geq y_0^-$.

Now we claim that $y_0^+ = y_0^-$. Indeed, if $y_0^+ > y_0^-$, then it follows from (6) that

$$\begin{aligned} f^2(x_0, y_0^+) &= (x_0 g(y_0^+), y_0^+ g[x_0 g(y_0^+)]), \\ f^2(x_0, y_0^-) &= (x_0 g(y_0^-), y_0^- g[x_0 g(y_0^-)]). \end{aligned} \tag{35}$$

So we have that

$$\begin{aligned} &x_0 g(y_0^+) < x_0 g(y_0^-), \\ &y_0^+ g[x_0 g(y_0^+)] > y_0^- g[x_0 g(y_0^-)]. \end{aligned} \tag{36}$$

It follows from (34) and (36) that there exist (x_1, y_1), $(x_2, y_2) \in L$ such that

$$\begin{aligned} f^2(x_1, y_1) &= (x_1 g(y_1), y_1 g[x_1 g(y_1)]), \\ f^2(x_2, y_2) &= (x_2 g(y_2), y_2 g[x_2 g(y_2)]), \end{aligned} \tag{37}$$

$$x_1 g(y_1) < x_2 g(y_2), \qquad y_1 g[x_1 g(y_1)] > y_2 g[x_2 g(y_2)]. \tag{38}$$

It follows from Lemma 3(i) and (37) that

$$(x_1 g(y_1), y_1 g[x_1 g(y_1)]), (x_2 g(y_2), y_2 g[x_2 g(y_2)]) \in L, \tag{39}$$

and this is a contradiction. The claim is proven.

In a similar fashion, we may show that $\lim_{x\to a^+} H(x) = H(a) = 0$. Thus $y = H(x)$ $(x \geq a)$ is an increasing continuous function. In a similar fashion, we may show that $x = G(y)$ $(y \geq a)$ is an increasing continuous function. Lemma 3 is proven. □

Let

$$\begin{gathered} L^1 = \{(x, y) : y = H(x) = 0, x \in [a, b]\}, \\ f(L^1) = M^1, \\ L^2 = \{(x, y) : y = H(x) > 0, x \in (b, +\infty)\}, \\ f(L^2) = M^2, \end{gathered} \tag{40}$$

where $a \in (0, +\infty)$ and $a \leq b$. It follows from Lemma 2(iii) and Lemma 3(ii) that

$$\begin{aligned} M^1 &= \{(x, y) : x = G(y) = 0, y \in [a, b]\}, \\ f(M^1) &= L^1, \\ M^2 &= \{(x, y) : x = G(y) > 0, y \in (b, +\infty)\}, \\ f(M^2) &= L^2. \end{aligned} \tag{41}$$

Proof of Theorem 1(2). Noting (40), we consider the following two cases.

Case 1 ($a = b$). It follows from (40) that

$$L = L^2 \cup \{(a, 0)\}. \tag{42}$$

Let $(x_{-1}, x_0) \in L^2$ and $\{x_n\}_{n=-1}^{\infty}$ be a solution of (2) with initial value (x_{-1}, x_0); it follows from Lemma 3(i) that

$$(x_{2n-1}, x_{2n}) = f^{2n}(x_{-1}, x_0) \in L, \tag{43}$$

which implies that $\lim_{n\to\infty}(x_{2n-1}, x_{2n}) \in L$. It follows from (42) and Theorem 1(1) that

$$\lim_{n\to\infty}(x_{2n-1}, x_{2n}) = (a, 0). \tag{44}$$

Next we claim that $y = H(x)$ $(x \geq a)$ is a strictly increasing function. Indeed, if there exists (x_{-1}, x_0), $(y_{-1}, y_0) \in L$ such that $y_{-1} > x_{-1}$ and $x_0 = y_0$, then there exist $r \in (1, +\infty)$ such that $y_{-1} = rx_{-1}$. Set

$$f^n(x_{-1}, x_0) = (x_{n-1}, x_n), \qquad f^n(y_{-1}, y_0) = (y_{n-1}, y_n), \qquad n = 1, 2, \ldots. \tag{45}$$

Then we have

$$\begin{aligned} y_1 &= y_{-1}g(y_0) \geq rx_{-1}g(x_0) = rx_1, \\ y_2 &= y_0 g(y_1) \leq x_0 g(x_1) = x_2. \end{aligned} \tag{46}$$

Using induction, one can show that, for any $n \geq 0$,

$$y_{2n-1} \geq rx_{2n-1}, \qquad y_{2n} \leq x_{2n}. \tag{47}$$

It follows from (44) and (47) that

$$(a, 0) = \lim_{n\to\infty}(y_{2n-1}, y_{2n}) \neq \lim_{n\to\infty}(x_{2n-1}, x_{2n}) = (a, 0). \tag{48}$$

This is a contradiction. The claim is proven.

Now let $(x_{-1}, x_0) \in D - L$ with $x_0 \neq 0$ and $\{x_n\}_{n=-1}^{\infty}$ be a solution of (2) with initial value (x_{-1}, x_0).

If $x_{-1} < a$, then it follows from Theorem 1(1) and (2) that $\lim_{n\to\infty}x_{2n-1} < a$ which implies $\lim_{n\to\infty}(x_{2n-1}, x_{2n}) \neq (a, 0)$.

If $x_{-1} \geq a$ and $x_0 > H(x_{-1})$, then there exists $n \geq 0$ such that

$$(x_{-1}, x_0) \in P_n - P_{n+1}, \tag{49}$$

from which it follows that

$$f^{2n}(x_{-1}, x_0) = (x_{2n-1}, x_{2n}) \in P_0 - P_1. \tag{50}$$

Then we have $x_{2n+1} < a$, which implies $\lim_{n\to\infty}(x_{2n-1}, x_{2n}) \neq (a, 0)$.

If $x_{-1} \geq a$ and $x_0 < H(x_{-1})$, then let $y_{-1} = x_{-1}$ and $y_0 = H(x_{-1})$, and there exists $r \in (1, +\infty)$ such that $y_0 = rx_0$. We can show that, for any $n \geq 1$,

$$\begin{aligned} & y_{2n} \geq rx_{2n}, \qquad x_{2n-1} \geq y_{2n-1}, \\ & \frac{x_{2n+1}}{y_{2n+1}} = \frac{x_{2n-1}g(x_{2n})}{y_{2n-1}g(y_{2n})} > \frac{x_{2n-1}}{y_{2n-1}} > \cdots > \frac{x_1}{y_1} \\ & \qquad = \frac{x_{-1}g(x_0)}{y_{-1}g(y_0)} = \frac{g(x_0)}{g(y_0)} > 1, \end{aligned} \tag{51}$$

which implies

$$\lim_{n\to\infty}(x_{2n-1}, x_{2n}) \neq \lim_{n\to\infty}(y_{2n-1}, y_{2n}) = (a, 0). \tag{52}$$

From all abovementioned, the set of initial conditions (x_{-1}, x_0) such that the positive solutions of (2) converge to

$$a, 0, a, 0, \ldots \tag{53}$$

is $y = H(x)$ $(x > a)$.

In a similar fashion, we also may show that the set of initial conditions (x_{-1}, x_0) such that the positive solutions of (2) converge to

$$0, a, 0, a, \ldots \tag{54}$$

is $x = G(y)$ $(y > a)$.

Case 2 ($a < b$). It follows from (41) and Case 1 that the set of initial conditions such that the positive solutions of (2) converge to

$$a, 0, a, 0, \ldots, \quad \text{or} \quad 0, a, 0, a, \ldots \tag{55}$$

is an empty set. This completes the proof of Theorem 1(2). $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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