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Research Article

A Hilbert-Type Integral Inequality with Multiparameters and a Nonhomogeneous Kernel

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We first introduce Γ-function and Riemann ζ-function to characterize the constant factor jointly. A Hilbert-type integral inequality with multiparameters and a nonhomogeneous kernel is given using the way of weight function and the technique of real analysis. The equivalent form is considered and its constant factors are proved to be the best possible. Some meaningful results are obtained by taking the special parameter values.

1. Introduction

If $\theta(x)(>0)$ is a measurable function and ρ is a positive number, set the function spaces as follows:

$$L^2(0,\infty) := \left\{h \geq 0; \|h\|_2 := \left\{\int_0^\infty h^2(x)\,dx\right\}^{1/2} < \infty\right\},$$
$$L_\theta^\rho(0,\infty) := \left\{h \geq 0; \|h\|_{\rho,\theta} := \left\{\int_0^\infty \theta(x) h^\rho(x)\,dx\right\}^{1/\rho} < \infty\right\}. \tag{1}$$

If $f, g \in L^2(0,\infty)$, $\|f\|_2, \|g\|_2 > 0$, then we have [1]

$$\iint_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \pi\|f\|_2\|g\|_2, \tag{2}$$

where the constant factor π is the best possible. Inequality (2) is the famous Hilbert integral inequality, which is important in analysis and its applications [1, 2]. We note that the kernel of (2) is homogeneous of -1-degree, and the methods and results about Hilbert-type inequality with the negative homogeneous kernel and multiparameters are summarized systematically in the paper in [3]. In recent years, Hilbert-type inequality was researched from homogeneous kernel turn to nonhomogeneous kernel [4–10], such that two integral inequalities with the best constant factor are obtained in [6, 7] as follows:

$$\iint_0^\infty e^{-xy} f(x)g(y)\,dx\,dy < \sqrt{\pi}\|f\|_2\|g\|_2. \tag{3}$$

If $\varphi(x) = e^{-x}$, $\psi(y) = e^{-y}$, $f \in L_\varphi^2(0,\infty)$, $g \in L_\psi^2(0,\infty)$, $\|f\|_{2,\varphi}, \|g\|_{2,\psi} > 0$, then

$$\iint_0^\infty \frac{f(x)g(y)}{1+e^{x+y}}\,dx\,dy < \pi\|f\|_{2,\varphi}\|g\|_{2,\psi}. \tag{4}$$

In this paper, a Hilbert-type integral inequality with nonhomogeneous kernel is given using the way of weight function and the technique of real analysis as follows.

If $\varphi(x) = x^{-3}$, $\psi(y) = y^{-3}$, $f \in L_\varphi^2(0,\infty)$, $g \in L_\psi^2(0,\infty)$, $\|f\|_{2,\varphi}, \|g\|_{2,\psi} > 0$, then

$$\iint_0^\infty e^{-xy}\coth(xy) f(x)g(y)\,dx\,dy < \left(\frac{\pi^2}{4} - 1\right)\|f\|_{2,\varphi}\|g\|_{2,\psi}. \tag{5}$$

2. Some Lemmas

Lemma 1. *Let $a > -1$, $\mathrm{Re}(s) > 0$; then, the Laplace integral transform of the power function x^a is as follows [11]:*

$$F(s) = \int_0^\infty x^a e^{-sx} dx = \frac{\Gamma(a+1)}{s^{a+1}}, \quad (6)$$

where $\Gamma(z)$ is the Γ-function ($\Gamma(z) = \int_0^\infty e^{-u} u^{z-1} du (z > 0)$).

Lemma 2. *Let m be a positive integer; then, one has the summation formula [12]*

$$S = \sum_{k=1}^{\infty} \frac{1}{k^{2m}} = \frac{2^{2m-1}\pi^{2m}}{(2m)!} B_m, \quad (7)$$

and [8]

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2m}} = \frac{\left(2^{2m}-1\right)\pi^{2m} B_m}{2(2m)!}, \quad (8)$$

where the $B_{m's}$ are the Bernoulli numbers, namely, $B_1 = 1/6$, $B_2 = 1/30$, $B_3 = 1/42$, $B_4 = 1/30$, $B_5 = 5/66$, $B_6 = 691/2730$, $B_7 = 7/6$, and so forth.

Lemma 3. *If $p > 1$, $1/p + 1/q = 1$, $\alpha, \beta > 0$, define the weight function as follows:*

$$\omega(\alpha,\beta,x) := \int_0^\infty e^{-\alpha xy} \coth(\beta xy) \frac{y^\beta}{x^{p\beta/q}} dy, \quad x \in (0,+\infty),$$
$$\omega(\alpha,\beta,y) := \int_0^\infty e^{-\alpha xy} \coth(\beta xy) \frac{x^\beta}{y^{q\beta/p}} dx, \quad y \in (0,+\infty); \quad (9)$$

then, one has

$$\omega(\alpha,\beta,x) = C(\alpha,\beta)\, x^{-p\beta-1},$$
$$\omega(\alpha,\beta,y) = C(\alpha,\beta)\, y^{-q\beta-1}, \quad (10)$$

where

$$C(\alpha,\beta) = \frac{1}{2^\beta \beta^{\beta+1}} \Gamma(\beta+1) \times \left[\sum_{k=0}^{\infty} \frac{1}{(k+(\alpha/2\beta))^{\beta+1}} - \frac{1}{2}\left(\frac{2\beta}{\alpha}\right)^{\beta+1}\right], \quad (11)$$

particularly.

(1) When $\alpha = 2\beta$, by (11), we find that

$$C(\alpha,\beta) = C(\beta) = \frac{1}{2^\beta \beta^{\beta+1}} \Gamma(\beta+1)\left[\zeta(\beta+1) - \frac{1}{2}\right], \quad (12)$$

where $\zeta(x)$ is Riemann ζ-function ($\zeta(x) = \sum_{k=1}^{\infty}(1/k^x)$ $(x > 1)$).

(2) When $\alpha = 2\beta$, $\beta = 2m-1$ $(m = 1, 2, \ldots)$, from (8) and (12), we find that

$$C(\alpha,\beta) = \frac{\pi^{2m} B_m}{2m(2m-1)^{2m}} - \frac{(2m-1)!}{[2(2m-1)]^{2m}}, \quad (13)$$

where the $B_{m's}$ are the Bernoulli numbers, namely, $B_1 = 1/6$, $B_2 = 1/30$, $B_3 = 1/42$, $B_4 = 1/30$, $B_5 = 5/66$, $B_6 = 691/2730$, $B_7 = 7/6$, and so forth.

Proof. Setting $\beta xy = u$, then, by (6), we have

$$\begin{aligned}
\omega(\alpha,\beta,x) &= \int_0^\infty e^{-\alpha xy} \coth(\beta xy) \frac{y^\beta}{x^{p\beta/q}} dy \\
&= \frac{1}{\beta^{\beta+1}} x^{-p\beta-1} \int_0^\infty \frac{e^{-(\alpha/\beta)u} + e^{-(2+\alpha/\beta)u}}{1-e^{-2u}} u^\beta du \\
&= \frac{1}{\beta^{\beta+1}} x^{-p\beta-1} \\
&\quad \times \sum_{k=0}^{\infty} \left\{ \frac{\Gamma(\beta+1)}{(2k+\alpha/\beta)^{\beta+1}} + \frac{\Gamma(\beta+1)}{[2(k+1)+\alpha/\beta]^{\beta+1}} \right\} \\
&= \frac{1}{2^\beta \beta^{\beta+1}} x^{-p\beta-1} \Gamma(\beta+1) \\
&\quad \times \left[\sum_{k=0}^{\infty} \frac{1}{(k+\alpha/2\beta)^{\beta+1}} - \frac{1}{2}\left(\frac{2\beta}{\alpha}\right)^{\beta+1}\right] \\
&= C(\alpha,\beta)\, x^{-p\beta-1}.
\end{aligned} \quad (14)$$

By the same way, we obtain $\omega(\alpha,\beta,y) = C(\alpha,\beta) y^{-q\beta-1}$. □

Lemma 4. *If $p > 1$, $1/p + 1/q = 1$, $\alpha, \beta > 0$, $0 < \varepsilon < \min\{q\beta, p\beta\}$, and ε is small enough, define the real functions as follows:*

$$\tilde{f}(x) = \begin{cases} 0, & x \in (0,1), \\ x^{(p\beta-\varepsilon)/p}, & x \in [1,\infty), \end{cases}$$
$$\tilde{g}(y) = \begin{cases} 0, & y \in (1,\infty), \\ y^{(q\beta+\varepsilon)/q}, & y \in (0,1]; \end{cases} \quad (15)$$

then, one finds that

$$\tilde{J}\varepsilon = \left[\int_0^\infty x^{-p\beta-1} \tilde{f}^p(x)\, dx\right]^{1/p} \times \left[\int_0^\infty y^{-q\beta-1} \tilde{g}^q(y)\, dy\right]^{1/q} \varepsilon = 1, \quad (16)$$

$$\tilde{I}\varepsilon = \iint_0^\infty e^{-\alpha xy} \coth(\beta xy) \tilde{f}(x)\, \tilde{g}(y)\, dx\, dy > C(\alpha,\beta)(1-o(1)) \quad (\varepsilon \longrightarrow 0^+). \quad (17)$$

Proof. We easily get

$$\widetilde{J}\varepsilon = \left[\int_0^\infty x^{-p\beta-1}\widetilde{f}^p(x)\,dx\right]^{1/p}\left[\int_0^\infty y^{-q\beta-1}\widetilde{g}^q(y)\,dy\right]^{1/q}\varepsilon$$
$$= \left[\int_1^\infty x^{-(1+\varepsilon)}dx\right]^{1/p}\left[\int_0^1 y^{-1+\varepsilon}dy\right]^{1/q}\varepsilon = 1. \tag{18}$$

Since $F(u) = u^{\beta+2}e^{-(\alpha/\beta)u}\coth u$ is continuous on $(0,\infty)$ and $\lim_{u\to 0}F(u) = 0$, $\lim_{u\to\infty}F(u) = 0$, there exists $M > 0$, such that $F(u) \le M$. From Fubini theorem [13], we have

$$\widetilde{I}\varepsilon = \iint_0^\infty e^{-\alpha xy}\coth(\beta xy)\,\widetilde{f}(x)\,\widetilde{g}(y)\,dx\,dy$$
$$= \varepsilon\int_1^\infty x^{(p\beta-\varepsilon)/p}dx\left[\int_0^1 e^{-\alpha xy}\coth(\beta xy)\,y^{(q\beta+\varepsilon)/q}dy\right]$$
$$= \frac{\varepsilon}{\beta^{\beta+1+\varepsilon/q}}$$
$$\times\int_1^\infty x^{-1-\varepsilon}dx\left[\int_0^{\beta x} e^{-(\alpha/\beta)u}(\coth u)\,u^{\beta+\varepsilon/q}du\right]$$
$$= \frac{\varepsilon}{\beta^{\beta+1+\varepsilon/q}}\int_0^\infty u^{\beta+\varepsilon/q}e^{-(\alpha/\beta)u}\coth u\,du$$
$$- \frac{\varepsilon}{\beta^{\beta+1+\varepsilon/q}}\int_1^\infty x^{-1-\varepsilon}dx\int_{ax}^\infty u^{\beta+\varepsilon/q}e^{-(\alpha/\beta)u}\coth u\,du$$
$$= \frac{1}{\beta^{\beta+1+\varepsilon/q}}\Gamma\left(\beta+1+\frac{\varepsilon}{q}\right)$$
$$\times\sum_{k=0}^\infty\left\{\frac{1}{(2k+\alpha/\beta)^{\beta+1}}+\frac{1}{[2(k+1)+\alpha/\beta]^{\beta+1}}\right\}$$
$$- \frac{1}{\beta^{\beta+1+\varepsilon/q}}\int_1^\infty x^{-1-\varepsilon}dx\int_{ax}^\infty u^{\beta+\varepsilon/q}e^{-(\alpha/\beta)u}\coth u\,du$$
$$> \frac{1}{2^\beta\beta^{\beta+1+\varepsilon/q}}\Gamma\left(\beta+1+\frac{\varepsilon}{q}\right)$$
$$\times\left[\sum_{k=0}^\infty\frac{1}{(k+\alpha/2\beta)^{\beta+1}}-\frac{1}{2}\left(\frac{2\beta}{\alpha}\right)^{\beta+1}\right]$$
$$- \frac{M\varepsilon}{\beta^{\beta+1+\varepsilon/q}}\int_1^\infty x^{-1}dx\int_{ax}^\infty u^{-2+\varepsilon/q}du$$
$$= \frac{1}{2^\beta\beta^{\beta+1+\varepsilon/q}}\Gamma\left(\beta+1+\frac{\varepsilon}{q}\right)$$
$$\times\left[\sum_{k=0}^\infty\frac{1}{(k+\alpha/2\beta)^{\beta+1}}-\frac{1}{2}\left(\frac{2\beta}{\alpha}\right)^{\beta+1}\right]$$
$$- \frac{M\varepsilon}{\beta^{\beta+1}}\frac{1}{(1-\varepsilon/q)^2}$$
$$= C(\alpha,\beta)(1-o(1))\quad(\varepsilon\longrightarrow 0^+). \tag{19}$$

□

3. Main Results and Applications

Theorem 5. *If $p > 1$, $1/p + 1/q = 1$, $\alpha, \beta > 0$, $\varphi(x) = x^{-p\beta-1}$, $\psi(y) = y^{-q\beta-1}$, $f \in L^p_\varphi(0,\infty)$, $g \in L^q_\psi(0,\infty)$, $\|f\|_{p,\varphi}, \|g\|_{q,\psi} > 0$, then one has*

$$\iint_0^\infty e^{-\alpha xy}\coth(\beta xy)\,f(x)\,g(y)\,dx\,dy < C(\alpha,\beta)\,\|f\|_{p,\varphi}\|g\|_{q,\varphi}, \tag{20}$$

where the constant factor $C(\alpha,\beta)$ (with (11)) is the best possible.

Proof. By Hölder's inequality [14], Fubini theorem, and Lemma 4, we obtain

$$I := \iint_0^\infty e^{-\alpha xy}\coth(\beta xy)\,f(x)\,g(y)\,dx\,dy$$
$$= \iint_0^\infty e^{-\alpha xy}\coth(\beta xy)\,f(x)\,g(y)$$
$$\times\left[\frac{y^{\beta/p}}{x^{\beta/q}}\right]\left[\frac{x^{\beta/q}}{y^{\beta/p}}\right]dx\,dy$$
$$\le\left[\iint_0^\infty e^{-\alpha xy}\coth(\beta xy)f^p(x)\frac{y^\beta}{x^{p\beta/q}}dx\,dy\right]^{1/p}$$
$$\times\left[\iint_0^\infty e^{-\alpha xy}\coth(\beta xy)g^q(y)\frac{x^\beta}{y^{q\beta/p}}dx\,dy\right]^{1/q} \tag{21}$$
$$= \left\{\int_0^\infty \omega(\alpha,\beta,x)f^p(x)dx\right\}^{1/p}$$
$$\times\left\{\int_0^\infty \omega(\alpha,\beta,y)g^q(y)dy\right\}^{1/q}$$
$$= C(\alpha,\beta)\,\|f\|_{p,\varphi}\|g\|_{q,\varphi}.$$

If inequality (21) keeps the form of equality, then there exist constants A and B [14], which are not all zeroes, such that $A(y^\beta/x^{p\beta/q})f^p(x) = B(x^\beta/y^{q\beta/p})g^q(y)$ a.e. in $(0,\infty)\times(0,\infty)$. It follows that $Ax^{-p\beta}f^p(x) = By^{-q\beta}g^q(y)$ a.e. on $(0,\infty)\times(0,\infty)$. Assuming that $A \ne 0$, there exists $y > 0$, such that $x^{-p\beta-1}f^p(x) = [By^{-q\beta}g^q(y)](1/Ax)$ a.e. in $x \in (0,\infty)$, which contradicts the fact that $0 < \|f\|_{p,\varphi} < \infty$. Then, inequality (21) keeps the strict form.

If the constant factor $C(\alpha,\beta)$ of (20) is not the best possible, then there exists a positive$K < C(\alpha,\beta)$, such that inequality (20) is still valid if we replace $C(\alpha,\beta)$ by K; then, by (16) and (17), we have

$$C(\alpha,\beta)(1-o(1)) < K. \tag{22}$$

Letting $\varepsilon \to 0^+$, we get $K \geq C(\alpha,\beta)$, which contradicts the fact that $K < C(\alpha,\beta)$, so the constant factor $C(\alpha,\beta)$ of (20) is the best possible. □

Theorem 6. *If $p > 1$, $1/p + 1/q = 1$, $\alpha, \beta > 0$, $\varphi(x) = x^{-p\beta-1} f \in L^p_\varphi(0,\infty)$, $\|f\|_{p,\varphi} > 0$, then one has*

$$\int_0^\infty y^{(q\beta+1)/(q-1)} dy \left[\int_0^\infty e^{-\alpha xy} \coth(\beta xy) f(x) dx\right]^p < C^p(\alpha,\beta) \|f\|^p_{p,\varphi}, \tag{23}$$

where the constant factor $C^p(\alpha,\beta)$ is the best possible, and inequality (23) is equivalent to inequality (20).

Proof. Setting a bounded measurable function as

$$[f(x)]_n := \min\{n, f(x)\} = \begin{cases} f(x), & \text{for } f(x) < n, \\ n, & \text{for } f(x) \geq n, \end{cases} \tag{24}$$

since $0 < \|f\|_{p,\varphi} < \infty$, there exists $n_0 \in \mathbf{N}$, such that $0 < \int_{1/n}^n \varphi_p(x)[f(x)]_n^p dx < \infty$ $(n \geq n_0)$. Setting

$$g_n(y) := y^{(p\beta+1)/(q-1)} \left[\int_{1/n}^n e^{-\alpha xy} \coth(\beta xy) [f(x)]_n dx\right]^{p/q}, \quad \left(\frac{1}{n} < y < n, n \geq n_0\right), \tag{25}$$

when $n \geq n_0$, from (20), we find that

$$\begin{aligned} 0 &< \int_{1/n}^n \psi(y) g_n^q(y) dy \\ &= \int_{1/n}^n y^{(q\beta+1)/(q-1)} \left[\int_{1/n}^n e^{-\alpha xy} \coth(\beta xy) [f(x)]_n dx\right]^p dy \\ &= \iint_{1/n}^n e^{-\alpha xy} \coth(\beta xy) [f(x)]_n g_n(y) dx\, dy \\ &< C(\alpha,\beta) \left\{\int_{1/n}^n \varphi(x) [f(x)]_n^p dx\right\}^{1/p} \\ &\quad \times \left\{\int_{1/n}^n \psi(y) g_n^q(y) dy\right\}^{1/q}, \\ 0 &< \int_{1/n}^n \psi(y) g_n^q(y) dy < C^p(\alpha,\beta) \int_0^\infty \varphi(x) f^p(x) dx \\ &= C^p(\alpha,\beta) \|f\|^p_{p,\varphi} < \infty. \end{aligned} \tag{26}$$

It follows that $0 < \|f\|_{p,\varphi} < \infty$. For $n \to \infty$, by (20), (26) still keep the forms of strict inequality. Hence, we have (23).

On the other hand, from Hölder inequality, we find that

$$\begin{aligned} I &= \iint_0^\infty e^{-\alpha xy} \coth(\beta xy) f(x) g(y) dx\, dy \\ &= \int_0^\infty \left[y^{(q\beta+1)/p(q-1)} \int_0^\infty e^{-\alpha xy} \coth(\beta xy) f(x) dx\right] \\ &\quad \times \left[y^{(-q\beta-1)/p(q-1)} g(y)\right] dy \\ &\leq \left\{\int_0^\infty y^{(q\beta+1)/(q-1)} dy \left[\int_0^\infty e^{-\alpha xy} \coth(\beta xy) f(x) dx\right]^p\right\}^{1/p} \\ &\quad \times \|g\|_{q,\varphi} \\ &< C(\alpha,\beta) \|f\|_{p,\varphi} \|g\|_{q,\varphi}. \end{aligned} \tag{27}$$

The inequality is (20), which is equivalent to (23).

If the constant factor of (23) is not the best, by (23), then we obtain that the constant factor of (20) is not the best too, which contradicts Theorem 5. Thus, the constant factor $C^p(\alpha,\beta)$ in (23) is the best possible. □

By taking the special parameter values in (20) and (23), some meaningful inequalities are obtained as follows.

Example 7. Let $\alpha = \beta = 1$, $p = q = 2$; by (8) and (11), we get $C(1,1) = 2\sum_{k=0}^\infty (1/(2k+1)^2) - 1 = \pi^2/4 - 1$. If $\varphi(x) = x^{-3}$, $\psi(y) = y^{-3}$, $f \in L^2_\varphi(0,\infty)$, $g \in L^2_\psi(0,\infty)$, $\|f\|_{2,\varphi}, \|g\|_{2,\psi} > 0$, then we have (5) and its equivalent form as

$$\int_0^\infty y^3 dy \left[\int_0^\infty e^{-xy} \coth(xy) f(x) dx\right]^2 < \left(\frac{\pi^2}{4} - 1\right)^2 \|f\|^2_{2,\varphi}, \tag{28}$$

where the constant factor $(\pi^2/4 - 1)^2$ is the best possible.

Example 8. Let $\alpha = 1$, $\beta = 1/2$, $p = q = 2$; by (12), we get $C(1, 1/2) = \sqrt{\pi}[\zeta(3/2) - 1/2] = 1.87204391^+$. If $\varphi(x) = x^{-2}$, $\psi(y) = y^{-2}$, $f \in L^2_\varphi(0,\infty)$, $g \in L^2_\psi(0,\infty)$, $\|f\|_{2,\varphi}, \|g\|_{2,\psi} > 0$, then we have the following equivalent inequalities:

$$\begin{aligned} &\iint_0^\infty e^{-xy} \coth\left(\frac{1}{2}xy\right) f(x) g(y) dx\, dy \\ &\quad < \sqrt{\pi}\left[\zeta\left(\frac{3}{2}\right) - \frac{1}{2}\right] \|f\|_{2,\varphi} \|g\|_{2,\psi}, \\ &\int_0^\infty y^2 dy \left[\int_0^\infty e^{-xy} \coth\left(\frac{1}{2}xy\right) f(x) dx\right]^2 \\ &\quad < \pi\left[\zeta\left(\frac{3}{2}\right) - \frac{1}{2}\right]^2 \|f\|^2_{2,\varphi}, \end{aligned} \tag{29}$$

where the constant factors $\sqrt{\pi}[\zeta(3/2) - 1/2]$, $\pi[\zeta(3/2) - 1/2]^2$ in (29) are the best possible.

Example 9. Let $\alpha = 2$, $\beta = 1$, $p = q = 2$; by (13), we get $C(2,1) = (1/4)(\pi^2/3 - 1)$. If $\varphi(x) = x^{-3}$, $\psi(y) = y^{-3}$,

$f \in L^2_{\varphi}(0,\infty)$, $g \in L^2_{\psi}(0,\infty)$, $\|f\|_{2,\varphi}$, $\|g\|_{2,\psi} > 0$, then we have the following equivalent inequalities:

$$\begin{aligned} &\iint_0^{\infty} e^{-2xy} \coth(xy) f(x) g(y)\,dx\,dy \\ &\qquad < \frac{1}{4}\left(\frac{\pi^2}{3}-1\right)\|f\|_{2,\varphi}\|g\|_{2,\psi}, \\ &\int_0^{\infty} y^3 dy \left[\int_0^{\infty} e^{-2xy} \coth(xy) f(x)\,dx\right]^2 \\ &\qquad < \frac{1}{16}\left(\frac{\pi^2}{3}-1\right)^2 \|f\|^2_{2,\varphi}, \end{aligned} \tag{30}$$

where the constant factors $(1/4)(\pi^2/3-1)$, $(1/16)(\pi^2/3-1)^2$ in (30) are the best possible.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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