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*Research Article*

# Upper Bound of Second Hankel Determinant for Certain Subclasses of Analytic Functions

**Ming-Sheng Liu,[1] Jun-Feng Xu,[2] and Ming Yang[1]**

[1] *School of Mathematical Sciences, South China Normal University, Guangzhou, Guangdong 510631, China*
[2] *Department of Mathematics, Wuyi University, Jiangmen, Guangdong 529020, China*

Correspondence should be addressed to Ming-Sheng Liu; liumsh@scnu.edu.cn

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In this present investigation, we first give a survey of the work done so far in this area of Hankel determinant for univalent functions. Then the upper bounds of the second Hankel determinant $|a_2a_4 - a_3^2|$ for functions belonging to the subclasses $S(\alpha, \beta)$, $K(\alpha, \beta)$, $S_s^*(\alpha, \beta)$, and $K_s(\alpha, \beta)$ of analytic functions are studied. Some of the results, presented in this paper, would extend the corresponding results of earlier authors.

## 1. Introduction

Let $\mathscr{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the unit disc $\mathbb{U} = \{z : |z| < 1\}$, and let $S$ denote the subclass of $\mathscr{A}$ that is univalent in $\mathbb{U}$. Suppose that $f$ and $g$ are analytic functions in $\mathbb{U}$; we say that $f$ is subordinate to $g$, written $f \prec g$, if there exists a Schwarz function $\omega$, which is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(\omega(z))$, $z \in \mathbb{U}$. In particular, if $g$ is univalent in $\mathbb{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Let $\mathscr{P}$ be the family of all functions $p$ analytic in $\mathbb{U}$ for which $\Re\{p(z)\} > 0$ and

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots \tag{2}$$

for $z \in \mathbb{U}$.

It is well known that the following correspondence between the class $\mathscr{P}$ and the class of Schwarz functions $\omega$ exists [1]:

$$p \in \mathscr{P} \Longleftrightarrow p = \frac{1 + \omega}{1 - \omega}. \tag{3}$$

Let $S^*$ denote the starlike subclass of $S$. It is well known that $f \in S^*$ if and only if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \mathbb{U}). \tag{4}$$

Let $K$ denote the class of all functions $f \in \mathscr{A}$ that are convex. Further, $f$ is convex if and only if $zf'$ is starlike. Also we know that $K \subset S^* \subset S$.

In 1959, Sakaguchi [2] introduced the class $S_s^*$ of functions starlike with respect to symmetric points, consisting of functions $f \in S$ satisfying

$$\Re\left\{\frac{2zf'(z)}{f(z) - f(-z)}\right\} > 0 \quad (z \in \mathbb{U}). \tag{5}$$

In 1977, Das and Singh [3] introduced the class $K_s$ of functions convex with respect to symmetric points, which consists of functions $f \in S$ satisfying

$$\Re\left\{\frac{2(zf'(z))'}{(f(z) - f(-z))'}\right\} > 0 \quad (z \in \mathbb{U}). \tag{6}$$

It is evident that $f \in K_s$ if and only if $zf' \in S_s^*$.

In 2007, Wang and Jiang [4] introduced the following subclass.

*Definition 1* (see [4]). Suppose that $0 \leq \alpha \leq 1$ and $0 < \beta \leq 1$. Let $S(\alpha, \beta)$ denote the class of functions $f$ in $\mathscr{A}$ satisfying the following inequality:

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < \beta \left| \frac{\alpha z f'(z)}{f(z)} + 1 \right| \quad (z \in \mathbb{U}). \tag{7}$$

From [4], one knows that the above condition is equivalent to

$$\frac{zf'(z)}{f(z)} \prec \frac{1+\beta z}{1-\alpha\beta z} \quad (z \in \mathbb{U}), \tag{8}$$

which implies that

$$S(\alpha, \beta) \subset S^* \subset S. \tag{9}$$

If $\alpha = \beta = 1$, then the class $S(\alpha, \beta)$ reduces to the class $S^*$. In the similar way, one can easily get the following definitions.

*Definition 2.* Suppose that $0 \leq \alpha \leq 1$ and $0 < \beta \leq 1$. Let $K(\alpha, \beta)$ denote the class of functions $f$ in $\mathscr{A}$ satisfying the following inequality:

$$\left| \frac{(zf'(z))'}{f'(z)} - 1 \right| < \beta \left| \frac{\alpha (zf'(z))'}{f'(z)} + 1 \right| \quad (z \in \mathbb{U}). \tag{10}$$

It is evident that the above condition is equivalent to

$$\frac{(zf'(z))'}{f'(z)} \prec \frac{1+\beta z}{1-\alpha\beta z} \quad (z \in \mathbb{U}), \tag{11}$$

which implies that

$$K(\alpha, \beta) \subset K \subset S. \tag{12}$$

If $\alpha = 1$ and $\beta = 1$, then the class $K(\alpha, \beta)$ reduces to the class $K$.

*Definition 3.* Suppose that $0 \leq \alpha \leq 1$ and $0 < \beta \leq 1$. Let $S_s^*(\alpha, \beta)$ denote the class of functions $f$ in $\mathscr{A}$ satisfying the following inequality:

$$\left| \frac{2zf'(z)}{f(z) - f(-z)} - 1 \right| < \beta \left| \frac{2\alpha z f'(z)}{f(z) - f(-z)} + 1 \right| \quad (z \in \mathbb{U}). \tag{13}$$

From [5], one knows that the above condition is equivalent to

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \frac{1+\beta z}{1-\alpha\beta z} \quad (z \in \mathbb{U}). \tag{14}$$

The function class $S_s^*(\alpha, \beta)$ was introduced and investigated by Sudharsan et al. [6]. If $\alpha = 1$ and $\beta = 1$, then the class $S_s^*(\alpha, \beta)$ reduces to the class $S_s^*$.

*Definition 4.* Suppose that $0 \leq \alpha \leq 1$ and $0 < \beta \leq 1$. Let $K_s(\alpha, \beta)$ denote the class of functions $f$ in $\mathscr{A}$ satisfying the following inequality:

$$\left| \frac{2(zf'(z))'}{(f(z) - f(-z))'} - 1 \right| < \beta \left| \frac{2\alpha (zf'(z))'}{(f(z) - f(-z))'} + 1 \right| \quad (z \in \mathbb{U}). \tag{15}$$

It is evident that the above condition is equivalent to

$$\frac{2(zf'(z))'}{(f(z) - f(-z))'} \prec \frac{1+\beta z}{1-\alpha\beta z} \quad (z \in \mathbb{U}). \tag{16}$$

If $\alpha = 1$ and $\beta = 1$, then the class $K_s(\alpha, \beta)$ reduces to the class $K_s$.

In 1966, Pommerenke [7] stated the $q$th Hankel determinant for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, \quad (a_1 = 1). \tag{17}$$

This Hankel determinant is useful and has also been considered by several authors. The growth rate of Hankel determinant $H_q(n)$ as $n \to \infty$ was investigated, respectively, when $f$ is a member of certain subclass of analytic functions, such as the class of $p$-valent functions [7, 8], the class of starlike functions [7], the class of univalent functions [9], the class of close-to-convex functions [10], the class of strong close-to-convex functions [11], a new class $V_k$ [12], and a new class $\widetilde{N}_k(\eta, \rho, \beta)$ [13]. Similar to the above discussions, we can also refer to [14, 15]. Ehrenborg [16] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence was defined and some of its properties were discussed by Layman [17]. Pommerenke [9] proved that the Hankel determinants of univalent function satisfy

$$\left| H_q(n) \right| \leq K n^{-(1/2+\beta)q+3/2}. \tag{18}$$

Later, $|H_2(n)| \leq A n^{1/2}$ was also proved by Hayman [18]. One can easily observe that the Fekete and Szegö functional is $H_2(1) = a_3 - a_2^2$. For results related to the functional, see [19, 20]. Fekete and Szegö further generalized the estimate $|a_3 - \mu a_2^2|$, where $\mu$ is real and $f \in S$. For results related to the functional, see [21, 22]. In 2010, Hayami and Owa [21, 22] also generalized the estimate $|a_n a_{n+2} - \mu a_n^2|$ for analytic function. Later, in 2012, Krishna and Ramreddy [23] also generalized the estimate $|a_{p+1} a_{p+3} - \mu a_{p+2}^2|$ for $p$-valent analytic function; see also [24, 25].

For our discussion in this paper, we consider the second Hankel determinant in the case of $q = 2$ and $n = 2$, namely,

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2. \tag{19}$$

Janteng et al. [26] have considered the functional $|H_2(2)|$ and found a sharp bound, the subclass of $S$ denoted by $R$, defined as $\Re\{f'(z)\} > 0$. In their work, they have shown that if $f \in R$, then $|H_2(2)| \le 4/9$. These authors [27, 28] also studied the second Hankel determinant and sharp bound for the classes of starlike and convex functions, close-to-starlike and close-to-convex functions with respect to symmetric points denoted by $S^*$, $K$, $S_c^*$, and $K_c$ and have shown that $|H_2(2)| \le 1$, $|H_2(2)| \le 1/8$, $|H_2(2)| \le 1$, and $|H_2(2)| \le 1/9$, respectively.

Singh [29] established the second Hankel determinant and sharp bound for the classes of close-to-starlike and close-to-convex functions with respect to conjugate and symmetric conjugate points denoted by $S_c^*$, $S_{sc}^*$, $K_c$, and $K_{sc}$ and has shown that $|H_2(2)| \le 1$, $|H_2(2)| \le 1$, $|H_2(2)| \le 1/8$, and $|H_2(2)| \le 1/9$, respectively.

Mishra and Gochhayat [30] obtained the sharp bound to $|H_2(2)|$ for the functions in the class denoted by $R_\lambda(\alpha,\rho)$, $(0 \le \lambda < 1, |\alpha| < \pi/2, 0 \le \rho \le 1)$ and defined as $\Re\{e^{i\alpha}(\Omega_z^\lambda f(z)/z)\} > \rho\cos\alpha$, using the fractional differential operator denoted by $\Omega_z^\lambda f(z)$ and defined by Owa and Srivastava [31]. These authors have shown that if $f \in R_\lambda(\alpha,\rho)$, then $|H_2(2)| \le \{((1-\rho)^2(2-\lambda)^2(3-\lambda)^2\cos^2\alpha)/9\}$.

Mohammed and Darus [32] have obtained a sharp upper bound to $|H_2(2)|$ for the functions in the class denoted by $S_m^{\lambda,n}(\alpha,\sigma)$, $(|\alpha| < \pi/2, 0 \le \sigma < 1)$ and defined as $\Re\{e^{i\alpha}(\Theta_m^{\lambda,n} f(z)/z)\} > \sigma\cos\alpha$. These authors have proved that if $f \in S_m^{\lambda,n}(\alpha,\sigma)$, then $|H_2(2)| \le \{(4m^2(1-\sigma)^2(1+m)^2\cos^2\alpha)/(3^{2n}(\lambda+1)^2(\lambda+2)^2)\}$.

Similar to the above discussions in a new subclass of analytic function with different operators, we can also refer to [33, 34]. Singh [35] also obtained a sharp upper bound for the functional $|H_2(2)|$ for the function $f \in M(\alpha)$, where

$$M(\alpha) = \left\{ f \in \mathcal{A} : \Re\left[\frac{zf'(z) + \alpha z^2 f''(z)}{(1-\alpha) f(z) + \alpha z f'(z)}\right] > 0, \quad 0 \le \alpha \le 1, z \in \mathbb{U} \right\}, \tag{20}$$

and showed that if $f \in M(\alpha)$, then $|H_2(2)| \le 1/((1+\alpha)(1+3\alpha))$.

Mehrok and Singh [36] have obtained a sharp upper bound to $|H_2(2)|$ for the function in the classes denoted by $M^\alpha$ and $C_s^{*(\alpha)}$ and defined as, respectively,

$$M^\alpha = \left\{ f \in \mathcal{A} : \Re\left[\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha}\left(\frac{(zf'(z))'}{f'(z)}\right)^{\alpha}\right] > 0, \quad 0 \le \alpha \le 1, z \in \mathbb{U} \right\},$$

$$C_s^{*(\alpha)} = \left\{ f \in \mathcal{A} : \Re\left[\left(\frac{2zf'(z)}{f(z) - f(-z)}\right)^{1-\alpha} \times \left(\frac{2(zf'(z))'}{(f(z) - f(-z))'}\right)^{\alpha}\right] > 0, \quad 0 \le \alpha \le 1, z \in \mathbb{U} \right\}. \tag{21}$$

In their work, they proved that if $f \in M^\alpha$, then

$$\begin{aligned} |H_2(2)| &\le \frac{1}{(1+2\alpha)^2} \\ &\quad \times \left[\alpha\left(11 + 36\alpha + 38\alpha^2 + 12\alpha^3 - \alpha^4\right)\right. \\ &\quad \times \left((1+3\alpha)\left(-4 + 263\alpha + 603\alpha^2 + 253\alpha^3 + 37\alpha^4\right)\right. \\ &\quad \left.\left. \times (1+\alpha)^4\right)^{-1} + 1\right], \end{aligned} \tag{22}$$

and if $f \in C_s^{*(\alpha)}$, then $|H_2(2)| \le 1/(1+2\alpha)^2$.

Shanmugam et al. [37] established the sharp upper bound of the second Hankel determinant for the classes of $S_\alpha^*$ and $C_\alpha$, defined as, respectively,

$$S_\alpha^* = \left\{ f \in \mathcal{A} : \Re\left[\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)}\right] > 0, z \in \mathbb{U} \right\},$$

$$C_\alpha = \left\{ f \in \mathcal{A} : \Re\left[\frac{\left(zf'(z) + \alpha z^2 f''(z')\right)'}{f'(z)}\right] > 0, z \in \mathbb{U} \right\}. \tag{23}$$

These authors proved that if $f \in S_\alpha^*$, then $|H_2(2)| \le 1/(1+3\alpha)^2$ and if $f \in C_\alpha$, then

$$|H_2(2)| \le \frac{1}{144}\left|\frac{280\alpha^3 + 340\alpha^2 + 138\alpha + 18}{(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)}\right|. \tag{24}$$

Krishna and Ramreddy [38] obtained a sharp upper bound to the nonlinear functional $|H_2(2)|$ for a new subclass of analytic functions $Q(\alpha,\beta,\gamma)$, $(\alpha,\beta > 0, 0 \le \gamma < \alpha + \beta \le 1)$, defined by

$$Q(\alpha,\beta,\gamma) = \left\{ f \in \mathcal{A} : \Re\left[\alpha\frac{f(z)}{z} + \beta f'(z)\right] \ge \gamma, z \in \mathbb{U} \right\}. \tag{25}$$

These authors proved that if $f \in Q(\alpha,\beta,\gamma)$, then $|H_2(2)| \le [4(\alpha+\beta-\gamma)^2/(\alpha+3\beta)^2]$.

Similar to the above discussions defined as different classes of analytic functions, we can also refer to [39–49].

Raza and Malik [50] studied the third Hankel determinant $H_3(1)$ of analytic functions related with lemniscate of Bernoulli; see also [51].

Motivated by the above-mentioned results obtained by different authors in this direction, in this present investigation, we determine the upper bounds of the second Hankel determinant $H_2(2)$ for functions belonging to these classes $S(\alpha,\beta)$, $K(\alpha,\beta)$, $S_s^*(\alpha,\beta)$, and $K_s(\alpha,\beta)$.

## 2. Preliminary Results

In order to prove our main results, we need the following lemmas.

**Lemma 5** (see [52]). *If the function $p \in \mathscr{P}$ is given by the power series* (2)*, then* $|c_k| \le 2$ $(k = 1, 2, \ldots)$.

**Lemma 6** (see [53, 54]). *If the function $p \in \mathscr{P}$ is given by the power series* (2)*, then*

$$2c_2 = c_1^2 + \left(4 - c_1^2\right)x \tag{26}$$

*for some $x$ with $|x| \le 1$ and*

$$\begin{aligned}4c_3 = c_1^3 &+ 2c_1\left(4 - c_1^2\right)x - c_1\left(4 - c_1^2\right)x^2 \\ &+ 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)z\end{aligned} \tag{27}$$

*for some $z$ with $|z| \le 1$.*

## 3. Main Results

**Theorem 7.** *Let $0 \le \alpha \le 1$ and $0 < \beta \le 1$. Suppose that the function $f$ given by* (1) *is in the class $S(\alpha,\beta)$. Then*

$$\left|a_2a_4 - a_3^2\right| \le \frac{1}{4}\beta^2(1+\alpha)^2. \tag{28}$$

*The result is sharp, with the extremal function*

$$f_1(z) = \begin{cases} z\left(1 - \alpha\beta z^2\right)^{-(1+\alpha)/2\alpha}, & 0 < \alpha \le 1, \\ ze^{\beta z^2/2}, & \alpha = 0. \end{cases} \tag{29}$$

*Proof.* Since $f \in S(\alpha,\beta)$, it follows from (8) that there exists a Schwarz function $\omega$, which is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{U}$, such that

$$\frac{zf'(z)}{f(z)} = \phi(\omega(z)) \quad (z \in \mathbb{U}), \tag{30}$$

where

$$\begin{aligned}\phi(z) = \frac{1+\beta z}{1-\alpha\beta z} &= 1 + \beta(1+\alpha)z + \alpha\beta^2(1+\alpha)z^2 \\ &+ \alpha^2\beta^3(1+\alpha)z^3 + \cdots.\end{aligned} \tag{31}$$

Define the function $p$ by

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + c_1z + c_2z^2 + \cdots. \tag{32}$$

From (3), we get $p \in \mathscr{P}$ and

$$\begin{aligned}\omega(z) = \frac{p(z)-1}{p(z)+1} &= \frac{1}{2}c_1z + \frac{1}{2}\left(c_2 - \frac{1}{2}c_1^2\right)z^2 \\ &+ \frac{1}{2}\left(c_3 - c_1c_2 + \frac{1}{4}c_1^3\right)z^3 + \cdots.\end{aligned} \tag{33}$$

In view of (30), (31), and (33), we have

$$\begin{aligned}\frac{zf'(z)}{f(z)} &= \phi(\omega(z)) \\ &= \phi\left(\frac{1}{2}c_1z + \frac{1}{2}\left(c_2 - \frac{1}{2}c_1^2\right)z^2 + \frac{1}{2}\left(c_3 - c_1c_2 + \frac{1}{4}c_1^3\right)z^3 + \cdots\right) \\ &= 1 + \frac{1}{2}\beta(1+\alpha)c_1z \\ &\quad + \left[\frac{1}{2}\beta(1+\alpha)\left(c_2 - \frac{1}{2}c_1^2\right) + \frac{1}{4}\alpha\beta^2(1+\alpha)c_1^2\right]z^2 \\ &\quad + \left[\frac{1}{2}\beta(1+\alpha)\left(c_3 - c_1c_2 + \frac{1}{4}c_1^3\right) + \frac{1}{2}\alpha\beta^2(1+\alpha)\left(c_2 - \frac{1}{2}c_1^2\right)c_1 + \frac{1}{8}\alpha^2\beta^3(1+\alpha)c_1^3\right]z^3 + \cdots.\end{aligned} \tag{34}$$

Similarly,

$$\begin{aligned}\frac{zf'(z)}{f(z)} &= 1 + a_2z + \left(2a_3 - a_2^2\right)z^2 \\ &+ \left(3a_4 - 3a_2a_3 + a_2^2\right)z^3 + \cdots.\end{aligned} \tag{35}$$

Comparing the coefficients of $z$, $z^2$, and $z^3$ in (34) and (35), we obtain

$$\begin{aligned}a_2 &= \frac{1}{2}\beta(1+\alpha)c_1, \\ a_3 &= \frac{1}{8}\beta(1+\alpha)\left[2c_2 + (\beta + 2\alpha\beta - 1)c_1^2\right], \\ a_4 &= \frac{1}{8}\beta(1+\alpha) \\ &\quad \times \left(\frac{1}{3} - \frac{1}{2}\beta - \frac{7}{6}\alpha\beta + \frac{5}{6}\alpha\beta^2 + \alpha^2\beta^2 + \frac{1}{6}\beta^2\right)c_1^3 \\ &\quad - \frac{1}{2}\beta(1+\alpha)\left(\frac{1}{3} - \frac{1}{4}\beta - \frac{7}{12}\alpha\beta\right)c_1c_2 + \frac{1}{6}\beta(1+\alpha)c_3.\end{aligned} \tag{36}$$

Thus we have

$$a_2a_4 - a_3^2 = -\frac{1}{192}\beta^2(1+\alpha)^2 \times \left[\left(2\alpha\beta^2 + 2\alpha\beta + \beta^2 - 1\right)c_1^4 - 4\left(\alpha\beta - 1\right)c_1^2c_2 - 16c_1c_3 + 12c_2^2\right], \quad (37)$$

$$\left|a_2a_4 - a_3^2\right| = \frac{1}{192}\beta^2(1+\alpha)^2 \times \left|\left(2\alpha\beta^2 + 2\alpha\beta + \beta^2 - 1\right)c_1^4 - 4\left(\alpha\beta - 1\right)c_1^2c_2 - 16c_1c_3 + 12c_2^2\right|. \quad (38)$$

Since the functions $p(z)$ and $p(e^{i\theta}z)$ $(\theta \in \mathbb{R})$ are members of the class $\mathcal{P}$ simultaneously, we assume without loss of generality that $c_1 > 0$. For convenience of notation, we take $c_1 = c$ $(c \in [0,2])$. By substituting the values of $c_2$ and $c_3$, respectively, from (26) and (27) in (38), we have

$$\left|a_2a_4 - a_3^2\right| = \frac{1}{192}\beta^2(1+\alpha)^2 \times \left|(2\alpha+1)\beta^2c^4 - 2\alpha\beta c^2\left(4-c^2\right)x + \left(12+c^2\right)\left(4-c^2\right)x^2 - 8c\left(4-c^2\right)\left(1-|x|^2\right)z\right|. \quad (39)$$

Using the triangle inequality and $|z| \le 1$, we have

$$\begin{aligned}\left|a_2a_4 - a_3^2\right| &\le \frac{1}{192}\beta^2(1+\alpha)^2 \times \left[(2\alpha+1)\beta^2c^4 + 2\alpha\beta c^2\left(4-c^2\right)|x| + \left(12+c^2\right)\left(4-c^2\right)|x|^2 + 8c\left(4-c^2\right)\left(1-|x|^2\right)\right] \\ &= \frac{1}{192}\beta^2(1+\alpha)^2 \times \left[8c\left(4-c^2\right) + (2\alpha+1)\beta^2c^4 + 2\alpha\beta c^2\left(4-c^2\right)|x| + (c-2)(c-6)\left(4-c^2\right)|x|^2\right] \\ &= F(c,\mu), \quad (say),\end{aligned} \quad (40)$$

where $\mu = |x| \le 1$.

We next maximize the function $F(c,\mu)$ on the closed square $[0,2] \times [0,1]$. Differentiating $F(c,\mu)$ in (40) partially with respect to $\mu$, we get

$$\frac{\partial F(c,\mu)}{\partial \mu} = \frac{1}{96}\beta^2(1+\alpha)^2 \times \left[\alpha\beta c^2\left(4-c^2\right) + (c-2)(c-6)\left(4-c^2\right)\mu\right]. \quad (41)$$

For $0 < \mu < 1$ and for any fixed $c$ with $0 < c < 2$, from (41), we observe that $\partial F(c,\mu)/\partial\mu > 0$. Consequently, $F(c,\mu)$ is an increasing function of $\mu$ and hence it cannot have a maximum value at any point in the interior of the closed square $[0,2] \times [0,1]$. Moreover, for fixed $c \in [0,2]$, we have

$$\max_{0\le\mu\le1} F(c,\mu) = F(c,1) = G(c) \quad (say). \quad (42)$$

From the relations (40) and (42), upon simplification, we obtain

$$G(c) = F(c,1) = \frac{1}{192}\beta^2(1+\alpha)^2 \times \left[(2\alpha\beta+\beta+1)(\beta-1)c^4 + 8(\alpha\beta-1)c^2 + 48\right]. \quad (43)$$

Next, since

$$G'(c) = \frac{1}{48}\beta^2(1+\alpha)^2c \times \left[(2\alpha\beta+\beta+1)(\beta-1)c^2 + 4(\alpha\beta-1)\right], \quad (44)$$

we get that $G'(c) \le 0$ for $0 < c \le 2$ and $G(c)$ has real critical point at $c = 0$. Therefore, the maximum of $G(c)$ occurs at $c = 0$. Thus, the upper bound of $F(c,\mu)$ corresponds to $\mu = 1$ and $c = 0$. Hence,

$$\left|a_2a_4 - a_3^2\right| \le \frac{1}{4}\beta^2(1+\alpha)^2. \quad (45)$$

Equality holds for the function

$$f_1(z) = \begin{cases} z\left(1-\alpha\beta z^2\right)^{-(1+\alpha)/2\alpha}, & 0 < \alpha \le 1, \\ ze^{\beta z^2/2}, & \alpha = 0. \end{cases} \quad (46)$$

By calculating, we have

$$\frac{zf_1'(z)}{f_1(z)} = \frac{1+\beta z^2}{1-\alpha\beta z^2} \prec \frac{1+\beta z}{1-\alpha\beta z} \tag{47}$$

and $a_2 = 0$, $a_3 = (1/2)\beta(1+\alpha)$, and $a_4 = 0$. So $f_1(z) \in S(\alpha, \beta)$ and equality holds. This shows that the result is sharp, and the proof of Theorem 7 is complete. □

Setting $\alpha = \beta = 1$ in Theorem 7, we obtain the following result due to Janteng et al. [27].

**Corollary 8.** *If $f(z) \in S^*$, then*

$$\left|a_2 a_4 - a_3^2\right| \le 1. \tag{48}$$

*The result is sharp, with the extremal function*

$$f_2(z) = \frac{z}{1-z^2}. \tag{49}$$

By using the similar method as in the proof of Theorem 7, one can similarly prove Theorem 9.

**Theorem 9.** *Let $0 \le \alpha \le 1$ and $0 < \beta \le 1$. Suppose that the function $f$ given by* (1) *is in the class $K(\alpha, \beta)$. Then*

$$\left|a_2 a_4 - a_3^2\right| \le \begin{cases} \dfrac{1}{36}\beta^2(1+\alpha)^2, & 5\alpha\beta + \beta - 2 \le 0, \\ \dfrac{1}{576}\beta^2(1+\alpha)^2\left[\dfrac{(5\alpha\beta+\beta-2)^2}{2+\beta(5\alpha+1)-\beta^2(1-\alpha)(2\alpha+1)} + 16\right], & 5\alpha\beta+\beta-2 > 0. \end{cases} \tag{50}$$

*The results are sharp, with the extremal function*

$$f_3(z) = \begin{cases} \displaystyle\int_0^z \left(1-\alpha\beta\mu^2\right)^{-(1+\alpha)/2\alpha} d\mu, & 0 < \alpha \le 1, \\ \displaystyle\int_0^z e^{\beta\mu^2/2} d\mu, & \alpha = 0 \end{cases} \tag{51}$$

*for the case $5\alpha\beta + \beta - 2 \le 0$, and there is no extremal function for the case $5\alpha\beta + \beta - 2 > 0$.*

Setting $\alpha = \beta = 1$ in Theorem 9, one obtains the following result due to Janteng et al. [27].

**Corollary 10.** *If $f(z) \in K$, then*

$$\left|a_2 a_4 - a_3^2\right| \le \frac{1}{8}. \tag{52}$$

*The result is sharp.*

**Theorem 11.** *Let $0 \le \alpha \le 1$ and $0 < \beta \le 1$. Suppose that the function $f$ given by* (1) *is in the class $S_s^*(\alpha, \beta)$. Then*

$$\left|a_2 a_4 - a_3^2\right| \le \frac{1}{4}\beta^2(1+\alpha)^2. \tag{53}$$

*The result is sharp, with the extremal function*

$$f_4(z) = \begin{cases} \displaystyle\int_0^z \left(1-\alpha\beta\mu^2\right)^{-(1+\alpha)/2\alpha} \times \left(\frac{1+\beta\mu^2}{1-\alpha\beta\mu^2}\right) d\mu, & 0 < \alpha \le 1, \\ \displaystyle\int_0^z e^{\beta\mu^2/2}\left(1+\beta\mu^2\right) d\mu, & \alpha = 0. \end{cases} \tag{54}$$

*Proof.* Since $f \in S_s^*(\alpha, \beta)$, it follows from (14) that there exists a Schwarz function $\omega$, which is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{U}$, such that

$$\frac{2zf'(z)}{f(z) - f(-z)} = \phi(\omega(z)) \quad (z \in \mathbb{U}), \tag{55}$$

where $\phi$ was defined by (31).

In view of (31), (33), and (55), we have

$$\begin{aligned} \frac{2zf'(z)}{f(z)-f(-z)} &= \phi(\omega(z)) \\ &= \phi\left(\frac{1}{2}c_1 z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + \frac{1}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)z^3 + \cdots\right) \\ &= 1 + \frac{1}{2}\beta(1+\alpha)c_1 z \\ &\quad + \left[\frac{1}{2}\beta(1+\alpha)\left(c_2 - \frac{1}{2}c_1^2\right) + \frac{1}{4}\alpha\beta^2(1+\alpha)c_1^2\right]z^2 \\ &\quad + \left[\frac{1}{2}\beta(1+\alpha)\left(c_3 - c_1c_2 + \frac{1}{4}c_1^3\right) + \frac{1}{2}\alpha\beta^2(1+\alpha)\left(c_2 - \frac{1}{2}c_1^2\right)c_1 + \frac{1}{8}\alpha^2\beta^3(1+\alpha)c_1^3\right]z^3 + \cdots. \end{aligned} \tag{56}$$

Similarly,

$$\frac{2zf'(z)}{f(z)-f(-z)} = 2a_2 z + 2a_3 z^2 + 2(2a_4 - a_2 a_3) z^3 + \cdots . \tag{57}$$

Comparing the coefficients of $z$, $z^2$, and $z^3$ in (56) and (57), we obtain

$$\begin{aligned} a_2 &= \frac{1}{4}\beta(1+\alpha)c_1, \\ a_3 &= \frac{1}{4}\beta(1+\alpha)\left[(\alpha\beta-1)c_1^2 + 2c_2\right], \\ a_4 &= \frac{1}{64}\beta(1+\alpha) \\ &\quad \times \left(2 - 4\alpha\beta + 3\alpha^2\beta^2 + \alpha\beta^2\right)c_1^3 \\ &\quad + \frac{1}{32}\beta(1+\alpha)(5\alpha\beta+\beta-4)c_1c_2 \\ &\quad + \frac{1}{8}\beta(1+\alpha)c_3. \end{aligned} \tag{58}$$

Thus we have

$$\begin{aligned} a_2a_4 - a_3^2 = -\frac{1}{256}\beta^2(1+\alpha)^2 & \\ \times \Big[\left(\alpha^2\beta^2 - \alpha\beta^2 - 4\alpha\beta + 2\right)c_1^4 & \\ + (6\alpha\beta - 2\beta - 8)c_1^2c_2 - 8c_1c_3 + 16c_2^2\Big], & \end{aligned} \tag{59}$$

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| = \frac{1}{256}\beta^2(1+\alpha)^2 & \\ \times \Big|\left(\alpha^2\beta^2 - \alpha\beta^2 - 4\alpha\beta + 2\right)c_1^4 & \\ + (6\alpha\beta - 2\beta - 8)c_1^2c_2 - 8c_1c_3 + 16c_2^2\Big|. & \end{aligned} \tag{60}$$

Since the functions $p(z)$ and $p(e^{i\theta}z)$ ($\theta \in \mathbb{R}$) are members of the class $\mathscr{P}$ simultaneously, we assume without loss of generality that $c_1 > 0$. For convenience of notation, we take $c_1 = c$ ($c \in [0,2]$). By substituting the values of $c_2$ and $c_3$, respectively, from (26) and (27) in (60), we have

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| = \frac{1}{256}\beta^2(1+\alpha)^2 & \\ \times \Big|\left(\alpha^2\beta^2 - \alpha\beta^2 - \alpha\beta - \beta\right)c^4 & \\ + (3\alpha\beta - \beta + 4)c^2\left(4-c^2\right)x + 2\left(4-c^2\right) & \\ \times \left(8-c^2\right)x^2 - 4c\left(4-c^2\right)\left(1-|x|^2\right)z\Big|. & \end{aligned} \tag{61}$$

Using the triangle inequality and $|z| < 1$, we have

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| &\le \frac{1}{256}\beta^2(1+\alpha)^2 \\ &\quad \times \Big[\left(\beta + \alpha\beta + \alpha\beta^2 - \alpha^2\beta^2\right)c^4 \\ &\qquad + (3\alpha\beta - \beta + 4)c^2\left(4-c^2\right)|x| + 2\left(4-c^2\right) \\ &\qquad \times \left(8-c^2\right)|x|^2 + 4c\left(4-c^2\right)\left(1-|x|^2\right)\Big] \\ &= \frac{1}{256}\beta^2(1+\alpha)^2 \\ &\quad \times \Big[\left(\beta + \alpha\beta + \alpha\beta^2 - \alpha^2\beta^2\right)c^4 + 4c\left(4-c^2\right) \\ &\qquad + (4 + 3\alpha\beta - \beta)c^2\left(4-c^2\right)|x| \\ &\qquad + 2(2-c)(4+c)\left(4-c^2\right)|x|^2\Big] \\ &= F(c,\mu), \quad (say), \end{aligned} \tag{62}$$

where $\mu = |x| \le 1$.

We next maximize the function $F(c,\mu)$ on the closed square $[0,2] \times [0,1]$. Differentiating $F(c,\mu)$ in (62) partially with respect to $\mu$, we get

$$\begin{aligned} \frac{\partial F(c,\mu)}{\partial \mu} &= \frac{1}{256}\beta^2(1+\alpha)^2 \\ &\quad \times \Big[(4 + 3\alpha\beta - \beta)c^2\left(4-c^2\right) \\ &\qquad + 4(2-c)(4+c)\left(4-c^2\right)\mu\Big]. \end{aligned} \tag{63}$$

For $0 < \mu < 1$ and for any fixed $c$ with $0 < c < 2$, from (63), we observe that $\partial F(c,\mu)/\partial\mu > 0$. Consequently, $F(c,\mu)$ is an increasing function of $\mu$ and hence it cannot have a maximum value at any point in the interior of the closed square $[0,2] \times [0,1]$. Moreover, for fixed $c \in [0,2]$, we have

$$\max_{0\le\mu\le1} F(c,\mu) = F(c,1) = G(c) \quad (say). \tag{64}$$

From the relations (62) and (64), upon simplification, we obtain

$$\begin{aligned} G(c) &= F(c,1) \\ &= \frac{1}{256}\beta^2(1+\alpha)^2 \\ &\quad \times \Big[\left(2\beta - 2\alpha\beta + \alpha\beta^2 - \alpha^2\beta^2 - 2\right)c^4 \\ &\qquad + 4(3\alpha\beta - \beta - 2)c^2 + 64\Big]. \end{aligned} \tag{65}$$

Next, since

$$\begin{aligned} G'(c) &= \frac{1}{64}\beta^2(1+\alpha)^2 c \\ &\quad \times \Big[\left(2\beta - 2\alpha\beta + \alpha\beta^2 - \alpha^2\beta^2 - 2\right)c^2 \\ &\qquad + 2(3\alpha\beta - \beta - 2)\Big], \end{aligned} \tag{66}$$

we get that $G'(c) \leq 0$ for $0 < c \leq 2$ and $G(c)$ has real critical point at $c = 0$. Therefore, the maximum of $G(c)$ occurs at $c = 0$. Thus, the upper bound of $F(c, \mu)$ corresponds to $\mu = 1$ and $c = 0$. Hence,

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{4}\beta^2(1+\alpha)^2. \tag{67}$$

Equality holds for the function

$$f_4(z) = \begin{cases} \displaystyle\int_0^z \left(1-\alpha\beta\mu^2\right)^{-(1+\alpha)/2\alpha} \times \left(\frac{1+\beta\mu^2}{1-\alpha\beta\mu^2}\right) d\mu, & 0 < \alpha \leq 1, \\ \displaystyle\int_0^z e^{\beta\mu^2/2}\left(1+\beta\mu^2\right) d\mu, & \alpha = 0. \end{cases} \tag{68}$$

By calculating, we have

$$\frac{2zf_4'(z)}{f_4(z) - f_4(-z)} = \frac{1+\beta z^2}{1-\alpha\beta z^2} \prec \frac{1+\beta z}{1-\alpha\beta z} \tag{69}$$

and $a_2 = 0$, $a_3 = -(1/2)\beta(1+\alpha)$, and $a_4 = 0$. So $f_4(z) \in S(\alpha, \beta)$ and equality holds. This shows that the result is sharp, and the proof of Theorem 11 is complete. □

Setting $\alpha = \beta = 1$ in Theorem 11, we obtain the following result due to Janteng et al. [28].

**Corollary 12.** *If $f(z) \in S_s^*$, then*

$$\left|a_2a_4 - a_3^2\right| \leq 1. \tag{70}$$

*The result is sharp, with the extremal function*

$$f_5(z) = \int_0^z \frac{1+\mu^2}{\left(1-\mu^2\right)^2} d\mu. \tag{71}$$

By using the similar method as in the proof of Theorem 11, one can similarly prove Theorem 13.

**Theorem 13.** *Let $0 \leq \alpha \leq 1$ and $0 < \beta \leq 1$. Suppose that the function $f(z)$ given by (1) is in the class $K_s(\alpha, \beta)$. Then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{36}\beta^2(1+\alpha)^2. \tag{72}$$

*The result is sharp, with the extremal function*

$$f_6(z) = \begin{cases} \displaystyle\int_0^z \frac{1}{\omega}\left\{\int_0^\omega \left(\frac{2}{2-\alpha\beta\mu^2}\right)^{(1+\alpha)/2\alpha} \times \left(\frac{2+\beta\mu^2}{2-\alpha\beta\mu^2}\right) d\mu\right\} d\omega, & 0 < \alpha \leq 1, \\ \displaystyle\int_0^z \frac{1}{\omega}\left\{\int_0^\omega e^{\beta\mu^2/2}\left(1+\frac{\beta\mu^2}{2}\right) d\mu\right\} d\omega, & \alpha = 0. \end{cases} \tag{73}$$

Setting $\alpha = \beta = 1$ in Theorem 13, one obtains the following result due to Janteng et al. [28].

**Corollary 14.** *If $f(z) \in K_s$, then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{9}. \tag{74}$$

*The result is sharp, with the extremal function*

$$f_7(z) = 2\int_0^z \frac{1}{\omega}\left\{\int_0^\omega \frac{2+\mu^2}{\left(2-\mu^2\right)^2} d\mu\right\} d\omega. \tag{75}$$

## Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.


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